Dynamics of linear operators

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Introduction

Linear dynamics is a young and rapidly evolving branch of functional analysis, which was probably "born" in 1982 with the Toronto PhD thesis of C. Kitai ([156]). It has become quite popular by now, thanks to the efforts of many mathematicians. In particular, the seminal paper [120] by G. Godefroy and J. H. Shapiro, the authoritative survey [130] by K.-G. Grosse-Erdmann and the beautiful notes [217] by J. H. Shapiro had considerable influence on both its internal development and its diffusion within the mathematical community. After more than two decades of active research, this seems to be the proper time to write a book about it.

As the name indicates, linear dynamics is mainly concerned with the behaviour of iterates of linear transformations. On finite-dimensional spaces, things are rather well understood since linear transformations are completely described by their Jordan canonical form. On the other hand, a new phenomenon appears in the infinite-dimensional setting: linear operators may have dense orbits. In fact, quite a lot of natural operators have this property.

To settle some terminology, let us recall that if T is a continuous linear operator acting on some topological vector space X, the T**- orbit** of a vector $x \in X$ is the set $O(x,T) := \{x, T(x), T^2(x), \dots\}$. The operator T is said to be **hypercyclic** if there exists some vector $x \in X$ whose T- orbit is dense in X. Such a vector x is said to be hypercyclic for T. Hypercyclicity will be the main topic of the present book.

From the very definition of hypercyclicity, it is immediately apparent that linear dynamics lies at the intersection of at least three different domains of mathematics.

1. **Topological dynamics.** The definition of hypercyclicity does not require any linear structure. It makes sense for an arbitrary continuous map $T : X \to X$ acting on a topological space X; and in fact, continuous maps with dense orbits are in some sense the main objects of study in topological dynamics.

On the other hand, the usual setting of topological dynamics is that of *compact* topological spaces, and compactness is really essential at many places. In linear dynamics, the underlying space is never compact, not even locally compact because hypercyclicity turns out to be a purely infinite-dimensional property. Thus, it seems hard to use sophisticated tools from topological dynamics. However, when the linear structure is added, interesting phenomena appear.

2. **Operator theory.** The word "hypercyclic" comes from the much older notion of a **cyclic** operator. An operator $T \in \mathfrak{L}(X)$ is said to be cyclic if there exists a vector $x \in X$ such that the *linear span* of $O(x, T)$ is dense in X. This notion is of course related to the famous **invariant subspace problem**: given an operator $T \in \mathfrak{L}(X)$, is it possible to find a non-trivial closed subspace $F \subset X$ which is T- invariant (i.e. $T(F) \subset F$)? Here, non-trivial means that $F \neq \{0\}$ and $F \neq X$. Clearly, the closed linear span of any T- orbit is an invariant subspace for T; hence, T lacks non-trivial invariant closed subspaces iff every non-zero vector $x \in X$ is cyclic for T.

Similarly, the **invariant subset problem** asks whether any operator $T \in \mathfrak{L}(X)$ has a non-trivial closed invariant subset. Since the closure of any T- orbit is a T- invariant closed set, an operator T lacks non-trivial invariant closed sets iff all non-zero vectors $x \in X$ are hypercyclic for T. In the language of topological dynamics, this means that $(X \setminus \{0\}, T)$ is a *minimal* dynamical system.

Despite considerable efforts, the invariant subspace problem remains largely open, most notably for Hilbert space operators. Since P. Enflo's negative solution on a rather peculiar Banach space ([101]), the most spectacular achievement is C. J. Read's construction of an operator T on $\ell^1(\mathbb{N})$ for which every non-zero vector $x \in \ell^1(\mathbb{N})$ is hypercyclic ([144]). This means that the invariant *subset* problem has a negative solution on the space ℓ^1.

3. **Universality.** Let $(T_i)_{i \in I}$ be a family of continuous maps $T_i : X \to Y$ between two fixed topological spaces X and Y. The family (T_i) is said to be **universal** if there exists $x \in X$ such that the set $\{T_i(x);\ i \in I\}$ is dense in Y. The first example of universality seems to go back to M. Fekete in 1914 (quoted in [188]) who discovered the existence of a *universal Taylor series* $\sum_{n \geq 1} a_n t^n$: for any continuous function g on $[-1\,, 1]$ with $g(0) = 0$, there exists an increasing sequence of integers (n_k) such that $\sum_{n=1}^{n_k} a_n t^n \to g(t)$ uniformly as $k \to \infty$. (Here, $X = \mathbb{C}^{\mathbb{N}}$, Y is the space of all continuous functions on $[-1, 1]$ vanishing at 0, and $T_i((a_n)) = \sum_{n=1}^{i} a_n t^n$, $i \geq 1$). Since then, universal families have been exhibited in a huge number of situations; see [130].

Hypercyclicity is of course a very particular instance of universality, where $X = Y$ is a topological vector space and $(T_i)_{i \in \mathbb{N}}$ is the sequence of iterates of a single linear operator $T \in \mathfrak{L}(X)$. Nevertheless, it is worth keeping in mind that a number of results pertaining to hypercyclic operators can be formulated (and proved!) in the more general setting of universal families. On the other hand, when working with the iterates of an operator, more specific tools can be used. In particular, spectral theory is often helpful.

One particularly seductive feature of linear dynamics is the diversity of ideas and techniques that are involved in its study, due to its strong connections with a number of distinct branches of mathematics. For some of them, e.g. topology, operator theory or approximation theory, this is rather obvious. More unexpectedly, Banach space geometry and probability theory also play quite an important role. Even number theory may be useful at times! For that reason, we believe that linear dynamics is an extremely attractive area, where many beautiful results are still to be discovered. Hopefully, the present book will give some substance to this affirmation.

It is now time to describe the contents of the book in some more details.

Chapter 1 contains the "basics" of linear dynamics. We introduce hypercyclicity and the weaker (typically linear) notion of supercyclicity. An operator $T \in \mathfrak{L}(X)$ is said to be **supercyclic** if there is some vector $x \in X$ such that the *cone* generated by $O(x, T)$ is dense in X. Our approach is based on the Baire category theorem. We start with the well-known equivalence between hypercyclicity and **topological transitivity**: an operator T acting on some separable, completely metrizable space X is hypercyclic iff for each pair of non-empty open sets (U, V) in X, one can find $n \in \mathbb{N}$ such that $T^n(U) \cap V \neq \varnothing$; and in that case, there is in fact a residual set of hypercyclic vectors. From that, one gets immediately the so-called **Hypercyclicity Criterion**, a sufficient set of conditions for hypercyclicity with a remarkably wide range of application. The analogous Supercyclicity Criterion is proved along the same lines. Next, we show that hypercyclicity and supercyclicity induce noteworthy spectral properties. Then we discuss the algebraic and topological properties of $HC(T)$, the set of all hypercyclic vectors for a given hypercyclic operator $T \in \mathfrak{L}(X)$. We show that $HC(T)$ always contains a dense linear subspace of X (except 0); and that $HC(T)$ is *homeomorphic* to X when X is a Fréchet space. Finally, several fundamental examples are treated in detail: weighted shifts on ℓ^p spaces, operators commuting with translations on the space of entire functions $H(\mathbb{C})$, and composition operators acting on the Hardy space $H^2(\mathbb{D})$. These examples will come back several times in the book.

Chapter 2 contains some rather spectacular showing that hypercyclicity is definitely not a mere curiosity. We first prove that hypercyclic operators can be found in any infinite-dimensional separable Fréchet space. The key point here is that operators of the form "Identity plus a backward shift" are always hypercyclic, and even **topologically mixing**; that is, for each pair of non-empty open sets (U, V), all but finitely many $n \in \mathbb{N}$ satisfy $T^n(U) \cap V \neq \varnothing$. Then, we show that any countable, dense, linearly independent set in a separable infinite-dimensional Banach space is an orbit of some hypercyclic operator. Next, we discuss the size of the set of all hypercyclic operators on some given infinite-dimensional separable Banach space X. This set is always dense in $\mathfrak{L}(X)$ with respect to the strong operator topology, but nowhere dense with respect to the norm topology (at least when X is a Hilbert space). Then, we show that linear dynamics provides a universal model for topological (non-linear) dynamics: there exists a single hypercyclic operator T acting on the separable Hilbert space H such that any continuous self-map of a compact metric space is topologically conjugate to the restriction of T to some invariant compact set $K \subset H$. We conclude the chapter by showing that any Hilbert space operator is the sum of two hypercyclic operators.

In Chapter 3, we present several elegant and useful results pointing out a kind of "rigidity" in linear dynamics: powers and rotations of hypercyclic operators remain hypercyclic; every single operator in a hypercyclic C_0 - semigroup is already hypercyclic; and any orbit of an arbitrary operator is either nowhere dense, or everywhere dense in the underlying topological vector space. Besides obvious formal similarities, these results have another interesting common feature: the proof of each of them ultimately relies on some suitable connectedness argument.

Chapter 4 is devoted to the Hypercyclicity Criterion. It turns out that a linear operator $T \in \mathfrak{L}(X)$ satisfies the Hypercyclicity Criterion iff it is topologically **weakly mixing**, which means that the product operator $T \times T$ is hypercyclic on $X \times X$. The **Hypercyclicity Criterion Problem** asks whether any hypercyclic operator has to be weakly mixing. In a non-linear context, very simple examples show that the answer is negative. However, the linear problem proved to be much more difficult. It was solved only recently by C. J. Read and M. De La Rosa, who showed that a counterexample exists in some suitably manufactured Banach space. We present here a variant of their construction, which allows us to exhibit counterexamples in a large class of separable Banach spaces, including the separable Hilbert space. The chapter also contains various characterizations of the weak mixing property involving the sets of natural numbers $\mathbf{N}(U, V) := \{n \in \mathbb{N};\ T^n(U) \cap V \neq \varnothing\}$, where U, V are non-empty open sets in X. These characterizations are undoubtly quite well-known to people working in topological dynamics, but perhaps less so to the operator theory community.

In Chapter 5, we give a rather detailed account of the connections between linear dynamics and **measurable dynamics**, i.e. ergodic theory. The basic idea is the following: if an operator T turns out to be ergodic with respect to some measure with full support, then T is hypercyclic by Birkhoff's ergodic theorem. Accordingly, it is desirable to find conditions ensuring the existence of such an ergodic measure. We concentrate on **Gaussian** measures only, since they are by far the best understood infinite-dimensional measures. We start with a general and essentially self-contained discussion of Gaussian measures and covariance operators on Banach spaces. Then we show how one can construct an ergodic Gaussian measure for an operator T provided T has "sufficiently many" eigenvectors associated with unimodular eigenvalues. The geometry of the underlying Banach space turns out to be quite important here, which should not look too surprising to anyone who has heard about probability in Banach spaces.

In Chapter 6, we discuss some variants or strengthenings of hypercyclicity. We first show that an operator is hypercyclic as soon as it has an orbit passing "not too far" from any point of the underlying space. Then, we consider **chaotic** and **frequently hypercyclic** operators (the latter being implicitly present in Chapter 5). Chaoticity and frequent hypercyclicity are qualitative strengthenings of hypercyclicity, both strictly stronger because operators with one or the other property are shown to be weakly mixing. There are interesting similarities and differences between hypercyclicity and these two variants. For example, any rotation of a chaotic or frequently hypercyclic operator has the same property; but on the other hand, some separable Banach spaces do not support any chaotic or frequently hypercyclic operator. Moreover, we show that there exist frequently hypercyclic operators which are not chaotic; and Hilbert space operators which are both chaotic and frequently hypercyclic, but not topologically mixing

In Chapter 7, we discuss in some detail the problem of the existence of **common hypercyclic vectors** for uncountable families of operators. By the Baire category theorem, any *countable* family of hypercyclic operators has a residual set of common hypercyclic vectors; but there is no obvious result of that kind for uncountable families. We present several positive criteria which prove to be efficient in various situations. These criteria may be viewed as kinds of "uncountable Baire category theorems" applying, of course, to very special families of open sets. Then we consider the particular case of weighted shifts and show that continuous paths of weighted shifts may or may not admit common hypercyclic vectors.

Chapter 8 is centered around the following question: when does a given hypercyclic operator admit a **hypercyclic subspace**, i.e. when is it possible to find an infinite-dimensional *closed* subspace of the underlying space consisting entirely of hypercyclic vectors (except 0)? If the operator T acts on a complex Banach space and satisfies the Hypercyclicity Criterion, there is a complete and very simple characterization: this holds iff the essential spectrum of T intersects the closed unit disk. We prove this result in two different ways, and then give several natural examples. We also prove some results related to the existence of non-trivial *algebras* of hypercyclic vectors.

Chapter 9 is the only one entirely devoted to supercyclicity. We prove the so-called **Angle Criterion**, a geometrical result which is often useful for showing that a given operator is *not* supercyclic. Then, we illustrate this criterion with two nice examples: composition operators on $H^2(\mathbb{D})$ associated with parabolic non-automorphisms of the disk, and the classical Volterra operator acting on $L^2([0,1])$.

In Chapter 10, we consider hypercyclicity or supercyclicity with respect to the weak topology of a given Banach space X. We start with a detailed discussion of weakly dense sequences which are not dense with respect to the norm topology. Then we concentrate on weak hypercyclicity or supercyclicity for bilateral weighted shifts acting on $\ell^p(\mathbb{Z})$. In particular, we show that there exist bilateral weighted shifts which are weakly hypercyclic but not hypercyclic; that weak hypercyclicity or supercyclicity of a weighted shift really depends on the exponent p (unlike norm hypercyclicity and supercyclicity); and that the unweighted shift is weakly supercyclic on $\ell^p(\mathbb{Z})$ iff $p > 2$. Then, we consider *unitary* operators. We show that, surprisingly enough, there exist Borel probability measures μ on $\mathbb{T}$ for which the "multiplication by the variable" operator M_z is weakly supercyclic on $L^2(\mu)$. This holds if the support of μ is "very small", but it is also possible to require that the Fourier coefficients of μ vanish at infinity, in which case the support of μ is rather "big". We conclude the chapter by discussing the still not well understood notions of weak *sequential* hypercyclicity and supercyclicity.

Chapter 11 is devoted to the universality properties of the Riemann zeta function. We show that any holomorphic function in the strip $\{1/2 < \mathrm{Re}(s) < 1\}$ and without zeros can be uniformly approximated on compact sets by imaginary translates of the zeta function. The proof is very much in the spirit of the whole book, being a mixture of analytic number theory, function theory, Hilbert space geometry and ergodic theory.

In Chapter 12, we try to give a user-friendly description of one of the many operators constructed by C. J. Read in connection with the invariant subspace problem. We concentrate on the simplest example: an operator without non-trivial invariant subspaces on the space $X = \ell^1(\mathbb{N})$. The construction is already quite involved, but can be presented at a relatively slow pace in a reasonable number of pages. When working on that chapter, our dream was, of course, to be able to solve the problem on the separable Hilbert space! The final result is much less impressive; but still, we hope that the chapter will be useful to some people. Typically, a non-expert interested in the invariant subspace problem (just like us) may find our exposition convenient.

From this outline, it should be clear that we have not written an encyclopaedic treatise on linear dynamics. This book is rather a selection of results and ideas that we made mostly according to our personal tastes, but also because they fit together to give a reasonably accurate global picture of the subject. As a result, the chapters have few overlaps and can be read more or less independently.

At the end of the book, we have added four appendices on complex analysis, function spaces, Banach space theory and spectral theory. The reader will find there only definitions and results that are explicitely needed in the main body of the book. Several proofs are given. One reason is that the reader may find it more comfortable to have grouped together the proofs of important results that are used several times, rather than to look for them in various sources. Another reason is that some results definitely needed to be proved, but it seemed better to postpone the proofs to the appendices in order to keep the reading of the book reasonably fluent. Concerning the appendix on spectral theory, we must confess that the first reason for including proofs was in fact that it was useful for *us*, since we are very far from being experts in that area.

As a rule, we have tried to give the simplest and most natural proofs we were able to produce. However, this does not mean that we prevented ourselves from stating a result in great generality whenever this seemed to be both possible and desirable. We hope that various kinds of readers will find our book useful as written, e.g. PhD students, specialists of the area, or non-specialists willing to have a flavour of the subject. We do think that it should be accessible to a rather large audience, including graduate students with an interest in functional analysis. Much more ambitiously, it may also happen that some people simply *enjoy* reading the book!

Each chapter ends up with some comments and a set of exercises. Some of the exercises are quite easy, but these are not necessarily the less interesting ones. Some others outline proofs of nice published results that could have been included in the main body of the book, but have been relegated into the exercises due to the lack of space. A few exercises aim at proving new results which will probably not be published elsewhere. Finally, some exercises are devoted to results which are used in the text, but for which we did not give full proofs in order to make the cooking more digestible. We have worked out each exercise rather carefully, and included a number of very explicit hints. In that way, we believe that any motivated reader will succeed in finding solutions without excessive effort.

Just like any other, this book cannot pretend to be perfect and certainly suffers from obvious flaws. All of them are the authors' sole responsibility, including mathematical mistakes and Frenglish sentences. In particular, that some result has not been included by no means indicates that it did not deserve to be mentionned. Most likely, the omission was due to a lack of space; and if the result does not even appear in the comments, this simply means that the authors were not aware of it. We would be extremely grateful to anyone pointing out to us unfortunate absences, mathematical inaccuracies or troublesome typos.

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