THREE PROOFS OF TYCHONOFF'S THEOREM

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Abstract. We give three proofs of Tychonoff's theorem on the compactness of a product of compact topological spaces. The first one proceeds "from scratch." The second one relies on the characterization of compactness in terms of convergent subnets. The third one has a "categorical" flavor.

1. Introduction.

The very famous *Tychonoff product theorem*, "probably the most important single theorem in general topology" according to [5], states that the product of any family of compact topological spaces is compact when endowed with the product topology.

This result is of course quite well understood, and several extremely nice proofs are available in the literature. In particular, we mention Tychonoff's original proof as presented in [7], the proof using Alexander's subbasis theorem (see, e.g., [5]), Cartan's proof using ultrafilters [1], Kelley's dual version using ultranets [4], and Chernoff's proof using nets but not ultranets [2]. See also the discussion on MathOverflow about people's "favorite proofs" of Tychonoff's theorem.

In this note, we present three more proofs of Tychonoff's theorem (and no excuse for that, except perhaps a pedagogical one). The first one relies only on the definition of compactness in terms of open covers. This proof is not new; indeed, it comes from Wright's paper [7], and it might be called the *Wisconsin proof* since it has been known and used at the University of Wisconsin for many years. However, in view of its straightforward nature and of its rather amazing simplicity, it deserves to be better known than it apparently is. (One may also add that the proof in [7] is fully detailed for a product of 2 spaces, but quite briefly sketched for an arbitrary product.) The second proof relies on the characterization of compactness in terms of nets: given a net $(x_d)_{d\in D}$ in the product space, we directly "construct" a convergent subnet. This proof seems to be new at least as written; and we find it quite natural because it is completely naive: a convergent subnet is needed, so we just try to find one by successive extractions. The third proof is an elementary version of the proof of a more general category-theoretical result due to Clementino and Tholen [3]. We find it interesting because it relies on a perhaps not so well-known characterization of compactness; namely, that a topological space X is compact if and only if, for any topological space Z, the canonical projection $\pi_Z : X \times Z \to Z$ is a closed map.

2. A straightforward proof "from scratch."

Let $(X_i)_{i\in I}$ be a family of compact spaces, and let $X := \prod_{i\in I} X_i$. To show that X is compact, we proceed as follows. We start with a family $\mathcal{U}$ of open subsets of X, and we

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assume that no finite subfamily of $\mathcal{U}$ covers X. Our goal is to show that $\mathcal{U}$ is not a cover of X, i.e., there exists a point $x \in X$ which does not belong to any $U \in \mathcal{U}$.

We first fix some notations. For any $J \subseteq I$, we set $X_J := \prod_{i\in J} X_i$. In particular, $X_I = X$ and $X_\varnothing = \{\varnothing\}$. If $J \subseteq J' \subseteq I$, we denote by $\pi_{J',J} : X_{J'} \to X_J$ the canonical projection map; and when $J' = I$ we write $\pi_J : X \to X_J$ instead of $\pi_{I,J}$. So we see that if $A \subseteq X_J$ and if $B = \pi_{J',J}^{-1}(A)$, then $\pi_{J'}^{-1}(B) = \pi_J^{-1}(A)$.

We denote by $\mathbf{P}$ the union of all the spaces X_J, $J \subseteq I$. An element of $\mathbf{P}$ will be called a *partial point*. If p is a partial point, the *domain* of p is the unique $J \subseteq I$ such that $p \in X_J$. If p, q are partial points with disjoint domains, we denote by $p \vee q$ their concatenation, which is a well defined partial point. Finally, we denote by $\preceq$ the natural extension ordering on $\mathbf{P}$.

A partial point p with domain J will be said to be *bad* if the following holds true: for any neighborhood V of p in X_J, the set $\pi_J^{-1}(V)$ cannot be covered by finitely many $U \in \mathcal{U}$. (In particular, $\varnothing$ is a bad partial point by our standing assumption.) With this terminology, it is enough to find a bad partial point p *whose domain J is equal to I*. In what follows, we denote by $\mathbf{B}$ the set of all bad partial points.

Remark. Since this will be needed later, we point out that $\mathbf{B}$ is downward closed for $\preceq$. In other words, if q_0 is a bad partial point, then any partial point $q \preceq q_0$ is also bad. Indeed, let us denote by J_0 the domain of q_0 and by $J \subseteq J_0$ the domain of q. If V is a neighborhood of q in X_J, then $V_0 := \pi_{J_0,J}^{-1}(V)$ is a neighborhood of q_0 in X_{J_0}. So $\pi_{J_0}^{-1}(V_0)$ cannot be covered by finitely many $U \in \mathcal{U}$, which shows that q is bad since $\pi_{J_0}^{-1}(V_0) = \pi_J^{-1}(V)$.

Fact 2.1. *Let p be a bad partial point with domain $J \neq I$. For any $i_0 \in I\backslash J$, one can find a point $a \in X_{i_0}$ such that $p \vee a \in \mathbf{B}$.*

Proof. For notational simplicity, we have written X_{i_0} instead of $X_{\{i_0\}}$. Assume that $p \vee a \notin \mathbf{B}$ for all $a \in X_{i_0}$. Then, for each $a \in X_{i_0}$, one can find an open neighborhood V_a of $p \vee a$ in $X_{J\cup\{i_0\}}$ such that $\pi_{J\cup\{i_0\}}^{-1}(V_a)$ can be covered by finitely many $U \in \mathcal{U}$. Moreover, we may assume that V_a has the form $V_a = O_{p,a} \times W_a$, where $O_{p,a}$ is an open neighborhood of p in X_J and W_a is an open neighborhood of a in X_{i_0}. Since X_{i_0} is compact, one can find $a_1, \dots, a_N \in X_{i_0}$ such that $X_{i_0} = W_{a_1} \cup \cdots \cup W_{a_N}$. Then $O_p := O_{p,a_1} \cap \cdots \cap O_{p,a_N}$ is a neighborhood of p in X_J, and $\pi_J^{-1}(O_p) = \pi_{J\cup\{i_0\}}^{-1}(O_p \times W_{a_1}) \cup \cdots \cup \pi_{J\cup\{i_0\}}^{-1}(O_p \times W_{a_N})$ is contained in $\pi_{J\cup\{i_0\}}^{-1}(V_{a_1}) \cup \cdots \cup \pi_{J\cup\{i_0\}}^{-1}(V_{a_N})$. So $\pi_J^{-1}(O_p)$ can be covered by finitely many $U \in \mathcal{U}$, a contradiction since p is assumed to be a bad partial point. □

Fact 2.2. *There is a $\preceq$-maximal bad partial point.*

Proof. By Zorn's lemma, it is enough to show that any chain $\mathbf{C}$ in $(\mathbf{B}, \preceq)$ has an upper bound in $\mathbf{B}$. Since $\mathbf{C}$ is a chain in $(\mathbf{P}, \preceq)$, the "union" of all $q \in \mathbf{C}$ is a well-defined partial point p; and of course it is an upper bound for $\mathbf{C}$ in $\mathbf{P}$. Let us check that $p \in \mathbf{B}$. Let J be the domain of p, and let V be any neighborhood of p in X_J. We have to show that $\pi_J^{-1}(V)$ cannot be covered by finitely many $U \in \mathcal{U}$. By definition of the product topology on X_J, we may assume that V has the form $V = \pi_{J,F}^{-1}(W)$, where $F \subseteq J$ is finite and W is an open set in X_F with $p_{|F} := \pi_{J,F}(p) \in W$. Since F is a finite subset of the domain of $p = \bigcup\{q;\ q \in \mathbf{C}\}$ and since $\mathbf{C}$ is a chain, one can find $q_0 \in \mathbf{C}$ such that $q_0 \succeq p_{|F}$. Since q_0 is a bad partial point and since $\mathbf{B}$ is downward closed for $\preceq$, it follows that $p_{|F}$ is a bad partial point. Since W is a neighborhood of $p_{|F}$ in X_F, this implies that $\pi_F^{-1}(W)$ cannot be covered by finitely many $U \in \mathcal{U}$; and since $\pi_F^{-1}(W) = \pi_J^{-1}(V)$, this shows that p is indeed a bad partial point. □

The proof of Tychonoff's theorem is now complete: by Fact 2.2, there is a maximal bad partial point p; and by Fact 2.1, the domain of p must be equal to I.

3. A proof by extracting a convergent subnet.

We first recall a few definitions concerning directed sets and nets.

A *directed set* is a non-empty set D endowed with a preordering relation $\leqslant$ (i.e., $\leqslant$ is reflexive and transitive) such that

$$\forall d_0, d_1 \in D\ \exists d \in D\ :\ d \geqslant d_0 \text{ and } d \geqslant d_1.$$

A *net* in a space X is a family $(x_d)_{d\in D}$ of points of X indexed by some directed set D. If X is a topological space, a net $(x_d)_{d\in D}$ in X is said to *converge* to some point $a \in X$ if for every neighborhood V of a, one can find $d_V \in D$ such that $x_d \in V$ for all $d \geqslant d_V$. The well known "subnet characterization" of compactness reads as follows: *A topological space X is compact if and only if every net $(x_d)_{d\in D}$ has a convergent subnet.* (The definition of a subnet is recalled in the next few lines.)

If D is a directed set, a *cofinal map into* D is a map $\varphi : S \to D$, where S is a directed set, which has the following property:

$$\forall d_0 \in D\ \exists s_0 \in S\ \forall s \geqslant s_0\ :\ \varphi(s) \geqslant d_0.$$

With this terminology, we can state the definition of a subnet as follows: If $(x_d)_{d\in D}$ is a net in some space X, then a subnet of $(x_d)_{d\in D}$ is a net of the form $\big(x_{\varphi(s)}\big)_{s\in S}$, where $\varphi : S \to D$ is a cofinal map.

The following notation will be ueful for the proof. If $\varphi : S \to D$ and $\varphi' : S' \to D$ are cofinal maps into the same directed set D, we write

$$\varphi' \succeq \varphi$$

if the following holds true:

$$\forall s_0 \in S\ \exists s_0' \in S'\ \forall s' \geqslant s_0'\ \exists s \geqslant s_0\ :\ \varphi'(s') = \varphi(s).$$

For example, it follows directly from the definition of a cofinal map that we have $\varphi' \succeq \varphi$ if $\varphi' = \varphi \circ \psi$ for some cofinal map $\psi : S' \to S$. But the converse need not be true, since it may well happen that $\varphi' \succeq \varphi$ but the range of φ' is not even contained in the range of φ. (Think about the difference between being a subsequence of some given sequence and being a "subsequence except for finitely many terms.")

It is easy to check that the relation $\succeq$ is reflexive and transitive on the class of all cofinal maps into D. Moreover, we have the following almost trivial but essential fact. (To prove it, just write down the definitions.)

Remark. Let $(z_d)_{d\in D}$ be a net in some topological space Z, and let $\varphi : S \to D$ and $\varphi' : S' \to D$ be cofinal maps with $\varphi' \succeq \varphi$. If the net $\big(z_{\varphi(s)}\big)_{s\in S}$ is convergent, then so is the net $\big(z_{\varphi'(s')}\big)_{s'\in S'}$, to the same limit.

Now, let $(X_i)_{i\in I}$ be a family of compact spaces, and let $X := \prod_{i\in I} X_i$. Let us show that any net $(x_d)_{d\in D}$ in X has a convergent subnet.

In what follows, a point $x \in X$ will be written as $x = (x(i))_{i\in I}$. So we are looking for a cofinal map $\varphi : S \to D$ such that for every $i \in I$, the net $\big(x_{\varphi(s)}(i)\big)_{s\in S}$ is convergent in X_i.

By Zermelo's theorem, we can well-order the set I. Hence, without loss of generality I is an ordinal α, so that $X = \prod_{i<\alpha} X_i$.

By transfinite induction, we construct a family $(\varphi_j)_{j\leqslant\alpha}$ of cofinal maps into D such that the following properties hold true (where $S_j := \mathrm{dom}(\varphi_j)$):

(1) the net $(x_{\varphi_j(s)}(i))_{s\in S_j}$ is convergent in X_i for every $i < j$;
(2) $\varphi_j \succeq \varphi_i$ for all $i < j$.

To start the construction, we set $S_0 := D$ and we define $\varphi_0 : S_0 \to D$ by $\varphi_0(s) = s$. Then (1) and (2) are vacuously satisfied.

Let $j \leqslant \alpha$, and assume that φ_i has been constructed for every $i < j$. We have to define the map φ_j.

If j is a successor ordinal, $j = i_0 + 1$, we choose a cofinal map $\psi : S_{i_0+1} \to S_{i_0}$ such that the net $(x_{\varphi_{i_0}(\psi(s))})_{s\in S_{i_0+1}}$ is convergent in X_{i_0} (which is possible by compactness of X_{i_0}), and we set $\varphi_{i_0+1} := \varphi_{i_0} \circ \psi : S_{i_0+1} \to D$. Then (1) holds for $j = i_0 + 1$. Indeed, it holds with $i = i_0$ by the choice of ψ, and it holds as well with $i < i_0$ by the induction hypothesis because $(x_{\varphi_{i_0+1}(s)})$ is a subnet of $(x_{\varphi_{i_0}(s)})$. Moreover, we have $\varphi_{i_0+1} \succeq \varphi_{i_0}$ because ψ is a cofinal map into S_{i_0}; so (2) holds for $j = i_0 + 1$ by the induction hypothesis.

If j is a limit ordinal, we proceed as follows. Set

$$S_j := \{(i, s);\ i < j\ ,\ s \in S_i\},$$

and define a preordering relation $\leqslant$ on S_j by declaring that

$$(i', s') \geqslant (i, s) \qquad \text{if and only if} \qquad \forall t' \geqslant s'\ \exists t \geqslant s\ :\ \varphi_{i'}(t') = \varphi_i(t).$$

We claim that S_j is directed by $\leqslant$. Indeed, let $(i_0, s_0), (i_1, s_1) \in S_j$, and assume that $i_1 \geqslant i_0$. Then $\varphi_{i_1} \succeq \varphi_{i_0}$ by the induction hypothesis. So one can find $s_1' \in S_{i_1}$ such that

$$\forall t' \geqslant s_1'\ \exists t \geqslant s_0\ :\ \varphi_{i_1}(t') = \varphi_{i_0}(t).$$

Next, since S_{i_1} is directed one can find $\widetilde{s}_1 \in S_{i_1}$ such that $\widetilde{s}_1 \geqslant s_1, s_1'$. Then $(i_1, \widetilde{s}_1) \geqslant (i_0, s_0)$ by the choice of s_1' since $\widetilde{s}_1 \geqslant s_1'$; and we also have obviously $(i_1, \widetilde{s}_1) \geqslant (i_1, s_1)$ since $\widetilde{s}_1 \geqslant s_1$. This shows that S_j is indeed a directed set. Moreover, the map $\varphi_j : S_j \to D$ defined by

$$\varphi_j(i, s) := \varphi_i(s)$$

is easily seen to be cofinal. To conclude the inductive step, we have to check that for every $i < j$, it holds that $\varphi_j \succeq \varphi_i$ and that the net $(x_{\varphi_j(s)}(i))_{s\in S_j}$ is convergent in X_i. Given $s_0 \in S_i$, consider $\widetilde{s}_0 := (i, s_0) \in S_j$. For any $(i', s') \geqslant \widetilde{s}_0 = (i, s_0)$, one can find $t \geqslant s_0$ such that $\varphi_{i'}(s') = \varphi_i(t)$, i.e., $\varphi_j(i', s') = \varphi_i(t)$; so we see that $\varphi_j \succeq \varphi_i$. To prove the second part, choose j' such that $i < j' < j$ (this is possible since j is a limit ordinal). Then the net $(x_{\varphi_{j'}(s)}(i))_{s\in S_{j'}}$ is convergent by the induction hypothesis, and hence the net $(x_{\varphi_j(s)}(i))_{s\in S_j}$ is convergent as well because $\varphi_j \succeq \varphi_{j'}$.

To conclude the proof of Tychonoff's theorem, we now just have to set $S := S_\alpha$ and $\varphi := \varphi_\alpha : S \to D$. By (1) for $j := \alpha$, the subnet $\big(x_{\varphi(s)}\big)_{s\in S}$ of (x_d) is convergent in $X = \prod_{i<\alpha} X_i$.

4. A "categorical" proof.

In [3], Clementino and Tholen give an interesting proof of Tychonoff's theorem based on ideas from category theory. Their main result is rather general, and Tychonoff's theorem follows as a special case. However, if one is interested in Tychonoff's theorem only, it seems desirable to write down carefully what their proof gives in this special case, without any categorical apparatus. This is what we do in this section.

The proof relies on the following characterization of compactness, which can be found, for example, in [6]. We include a proof for convenience of the reader. Let us first introduce a notation: for any spaces X, Z and any subset F of $X \times Z$, we denote by $\exists^X F$ the projection of F into Z:

$$\exists^X F := \{z \in Z\,;\ \exists x \in X\ :\ (x,z) \in F\}.$$

Lemma 4.1. *A topological space X is compact if and only it has the following property: For any topological space Z and any closed set $F \subseteq X \times Z$, the set $\exists^X F$ is closed in Z. (In other words, the canonical projection $\pi_Z : X \times Z \to Z$ is a closed map.)*

Proof. The "only if" part is a simple exercise (the easiest way to do it is probably by using nets). Conversely, assume that X is not compact. Then there exists a net $(x_d)_{d\in D}$ in X without any cluster point. Choose some "point" ∞ not belonging to D (for example $\infty := D$) and let $Z := D \cup \{\infty\}$, topologized as follows: On D the topology is the discrete topology; and a neighborhood basis for ∞ consists of all sets of the form $\{\infty\} \cup [d, \infty)$, $d \in D$, where $[d, \infty) = \{d' \in D\ :\ d' \geqslant d\}$. Now consider the set $A := \{(x_d, d)\,;\ d \in D\} \subseteq X \times Z$. Since the net (x_d) has no cluster point in X, it is not hard to check that $F := \overline{A}$ does not contain any point of the form (x, ∞). So the set $\exists^X F$ is contained in D. Since $\exists^X A = D$, it follows that $\exists^X F = D$, which is not closed in $Z = D \cup \{\infty\}$. □

We will also need the following technical-looking but rather simple fact.

Fact 4.2. *Let β be a limit ordinal, and let $(X_s)_{s<\beta}$ be a family of topological spaces. For any $J \subseteq [0, \beta)$, set $X_J := \prod_{s\in J} X_s$. Also let Z be a topological space, and let $A \subseteq X_{[0,\beta)} \times Z$. For any $1 \leqslant t < \beta$, define*

$$A^t := \exists^{X_{[t,\beta)}} A = \Big\{(u, z) \in X_{[0,t)} \times Z\,;\ \exists v \in X_{[t,\beta)}\ :\ (u, v, z) \in A\Big\}.$$

Finally, let $x = (x_s)_{s<\beta} \in X_{[0,\beta)}$, and let $a \in Z$. Assume that for any $t < \beta$, it holds that $\big((x_s)_{s\leqslant t}, z\big) \in \overline{A^{t+1}}$. Then $(x, a) \in \overline{A}$.

Proof. Let U be a neighborhood of (x, a) in $X_{[0,\beta)} \times Z$; we have to show that $U \cap A \neq \varnothing$. Since β is a limit ordinal, we may suppose that U has the form

$$U = \pi^{-1}_{[0,t]}(O) \times V,$$

where V is a neighborhood of a in Z and O is an open set in $X_{[0,t]}$ for some $t < \beta$. Then $O \times V$ is a neighborhood of $(\pi_{[0,t]}(x), a) = \big((x_s)_{s\leqslant t}, a\big)$ in $X_{[0,t]} \times Z$, so $(O \times V) \cap A^{t+1} \neq \varnothing$ by assumption. By the definition of A^{t+1}, this means that one can find $v \in X_{[t+1,\beta)}$ such that $\big((\pi_{[0,t]}(x), v), a\big) \in F$. Then $\big((\pi_{[0,t]}(x), v), a\big) \in U \cap F$, which concludes the proof. □

Now, let let $(X_i)_{i\in I}$ be a family of compact spaces. We use Lemma 4.1 to show that $X := \prod_{i\in I} X_i$ is compact. Again using Zermelo's theorem, we may assume that I is an ordinal α, so that $X = \prod_{i<\alpha} X_i$.

Let Z be an arbitrary topological space, and let F be a closed subset of $X \times Z$. To show that $\exists^X Z$ is closed in Z, we fix a point $a \in \overline{\exists^X F}$. Our aim is to show that in fact $a \in \exists^X F$, i.e., to find a point $x \in X$ such that $(x, a) \in F$.

Following the notations of Fact 4.2, for any $J \subseteq [0, \alpha)$ we set $X_J := \prod_{i\in J} X_i$. If $1 \leqslant j < \alpha$, we define

$$F^j := \exists^{X_{[j,\alpha)}} F = \Big\{(u, z) \in X_{[0,j)} \times Z;\ \exists v \in X_{[j,\alpha)}\ :\ (u, v, z) \in F\Big\}.$$

We also put $F^\alpha := F$. Observe that with these notations, we have

$$\exists^X F = \exists^{X_0} F^1 \qquad \text{and} \qquad F^j = \exists^{X_j} F^{j+1} \quad \text{for all } 1 \leqslant j < \alpha.$$

By transfinite induction, we construct for $0 \leqslant i < \alpha$ a point $x_i \in X_i$ such that the following holds true for all $0 \leqslant i < \alpha$:

$$(*_i) \qquad \text{the point } \big((x_s)_{s\leqslant i}, a\big) \text{ belongs to } \overline{F^{i+1}}.$$

By assumption, we know that $a \in \overline{\exists^X F} = \overline{\exists^{X_0} F^1} \subseteq \overline{\exists^{X_0} \overline{F^1}}$. Since X_0 is compact, the set $\exists^{X_0}\overline{F^1}$ is closed in Z by Lemma 4.1. So we have $a \in \exists^{X_0}\overline{F^1}$; in other words, one can find $x_0 \in X_0$ such that $(*_0)$ holds true.

Let $1 \leqslant i_0 < \alpha$, and assume that the points x_s have been found for all $s < i_0$. Then, we have

$$\big((x_s)_{s<i_0}, a\big) \in \overline{F^{i_0}}.$$

Indeed, this is just $(*_{i_0-1})$ if i_0 is a successor ordinal; and if i_0 is a limit ordinal this follows from the induction hypothesis and Fact 4.2 applied with $\beta := i_0$ and $A := F^{i_0}$, because $(F^{i_0})^i = F^i$ for all $i < i_0$. Since $F^{i_0} = \exists^{X_{i_0}} F^{i_0+1}$ and since X_{i_0} is compact, it follows that $\big((x_i)_{i<i_0}, a\big) \in \exists^{X_{i_0}}\overline{F^{i_0+1}}$ by Lemma 4.1. So we can find $x_{i_0} \in X_{i_0}$ such that $(*_{i_0})$ holds true.

Now let $x := (x_i)_{i<\alpha}$, which is a point of X. Then $(x, a) \in F$. Indeed, if α is a successor ordinal, $\alpha = i_0 + 1$, then $(x, a) = \big((x_i)_{i\leqslant i_0}, a\big) \in \overline{F^{i_0+1}} = \overline{F^{\alpha}} = F$; and if α is a limit ordinal, then $(x, a) \in \overline{F} = F$ by Fact 4.2. This concludes the proof.

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