TRICHOTOMIES FOR IDEALS OF COMPACT SETS

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Abstract. We prove several trichotomy results for ideals of compact sets. Typically, we show that a "sufficiently rich" universally Baire ideal is either $\mathbf{\Pi}^0_3$-hard, or $\mathbf{\Sigma}^0_3$-hard, or else a σ-ideal.

1. Introduction

Throughout this paper, E will be a (nonempty) Polish space. Let $\mathcal{K}(E)$ be the hyperspace of all compact subsets of E equipped with the Vietoris topology. This topology is Polish. A family of compact subsets of E is said to be *hereditary* if it is downward closed under inclusion. An *ideal* of $\mathcal{K}(E)$ is a family of compact subsets of E which is hereditary and stable under finite unions. A *σ-ideal* of $\mathcal{K}(E)$ is a family of compact subsets of E which is hereditary and stable under countable (compact) unions.

It is well known that there exist connections between the descriptive complexity of an ideal and its structural properties (see e.g. [C], [E], [KLW], [K$_2$], [DSR]). For example, a $\mathbf{\Pi}^1_1$ σ-ideal of compact sets is either $\mathbf{\Pi}^1_1$-complete or G_δ; a $\mathbf{\Sigma}^1_1$ σ-ideal is necessarily G_δ; and a G_δ ideal is necessarily a σ-ideal. In a similar vein, it was proved in [M] that if an ideal $\mathcal{I} \subset \mathcal{K}(E)$ is a countable union of G_δ hereditary sets and contains a dense G_δ hereditary subset of $\mathcal{K}(E)$, then either $\mathcal{I}$ is $\mathbf{\Sigma}^0_3$-complete, or there exists a nonempty open set $V \subset E$ such that $\mathcal{I} \cap \mathcal{K}(V)$ is a G_δ σ-ideal. This result was used in [M] to show that many natural families of thin sets from Harmonic Analysis are $\mathbf{\Sigma}^0_3$-complete. Some other aspects of these matters were considered in [BD].

In this note, we shall proceed further by proving some "trichotomy" results for ideals of compact sets. To formulate them, we need to introduce the following notion, which is crucial for our paper.

Definition. We say that a family $\mathcal{A} \subset \mathcal{K}(E)$ is *rich in sequences at $x \in E$* if there exists a dense set $\mathcal{D} \subset \mathcal{K}(E)$ such that $\{x\} \cup \bigcup_{n\in\omega} K_n \in \mathcal{A}$ for each sequence

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$(K_n) \subset \mathcal{D}$ converging to $\{x\}$. The family $\mathcal{A}$ is said to be *rich in sequences* if $\bigcup \mathcal{A} \neq \emptyset$ and $\mathcal{A}$ is rich in sequences at each point $x \in \bigcup \mathcal{A}$.

The following remarks may help to justify this terminology.

Remark. If there exists a dense set $D \subset E$ such that $\{x\} \cup \{x_n;\ n \in \omega\} \in \mathcal{A}$ for each sequence $(x_n) \subset D$ converging to x, then $\mathcal{A}$ is rich in sequences at x: one may take for $\mathcal{D}$ the family of all finite subsets of D. The converse is true if $\mathcal{A}$ is hereditary. Indeed, if $\mathcal{A}$ is hereditary and rich in sequences at x with witness $\mathcal{D} \subset \mathcal{K}(E)$, it is easy to check that the set

$$D := \{z \in E;\ \exists K \in \mathcal{D}:\ z \in K \text{ and } \operatorname{diam}(K) \leq d(z,x)\}$$

is dense in E and satisfies the above property.

Recall that a set A in some Polish space X is said to be *universally Baire* if, for every Polish space Z and every continuous map $f : Z \to X$, the set $f^{-1}(A)$ has the Baire property in Z. Our main result reads as follows.

Theorem 1.1. *Let $\mathcal{I}$ be a universally Baire ideal of $\mathcal{K}(E)$. Assume that $\mathcal{I}$ is rich in sequences at some point $x \in E$. Then one of the following holds.*

(i) *There exists some neighborhood V of x such that $\mathcal{I} \cap \mathcal{K}(V)$ is a σ-ideal.*
(ii) *$\mathcal{I}$ is $\mathbf{\Pi}^0_3$-hard.*
(iii) *$\mathcal{I}$ is $\mathbf{\Sigma}^0_3$-hard.*

This implies the following "global" trichotomy result.

Corollary 1.2. *Let $\mathcal{I}$ be a universally Baire ideal of $\mathcal{K}(E)$ which is rich in sequences. Then one of the following holds.*

(i) *$\mathcal{I}$ is a σ-ideal.*
(ii) *$\mathcal{I}$ is $\mathbf{\Pi}^0_3$-hard.*
(iii) *$\mathcal{I}$ is $\mathbf{\Sigma}^0_3$-hard.*

Proof. Observe that if $\mathcal{V}$ is a family of open subsets of E, then any compact set $K \subset \bigcup \mathcal{V}$ can be written as $K = K_1 \cup \cdots \cup K_n$, where for all $i \leq n$, $K_i \in \mathcal{K}(V_i)$ for some $V_i \in \mathcal{V}$. It follows easily that if we denote by U the union of all open sets $V \subset E$ such that $\mathcal{I} \cap \mathcal{K}(V)$ is a σ-ideal, then $\mathcal{I} \cap \mathcal{K}(U)$ is again a σ-ideal. Thus, 1.2 follows at once from 1.1. □

From 1.2, we deduce the following result, which highlights the interplay between various levels of definability and structural properties of $\mathcal{I}$. The implication (i)⇒(iii) was proved by Dougherty and Kechris and, independently, Louveau, see [K_2]. The implication (iii)⇒(i) is due to Kechris, Louveau, and Woodin ([KLW]).

Corollary 1.3. *Let $\mathcal{I}$ be a dense subset of $\mathcal{K}(E)$. The following conditions are equivalent.*

(i) *$\mathcal{I}$ is G_δ and an ideal.*

(ii) *$\mathcal{I}$ is $\mathbf{\Delta}^0_3$ and a rich in sequences ideal.*
(iii) *$\mathcal{I}$ is $\mathbf{\Sigma}^1_1$ and a σ-ideal.*

The following well known lemma (see [K_2]) is used in our derivation of 1.3 and also in subsequent arguments. We give a proof for completeness. Here and afterwards, we say that a sequence of compact sets $(L_n) \subset \mathcal{K}(E)$ *accumulates* to some compact set $L \subset E$ if every open set containing L contains all but finitely many sets L_n.

Lemma 1.4. *Let $\mathcal{G}$ be a G_δ subset of $\mathcal{K}(E)$. Let (K_n) be a sequence of compact sets accumulating to some compact set K. Assume that for each finite set $b \subset \omega$, the set $K \cup \bigcup_{n\in b} K_n$ belongs to $\mathcal{G}$. Then $K \cup \bigcup_{n\in\omega} K_n$ is the union of two elements of $\mathcal{G}$.*

Proof. We identify $\mathcal{P}(\omega)$ with 2^ω. Since (K_n) accumulates to K, the map $\Phi : \mathcal{P}(\omega) \to \mathcal{K}(E)$ defined by $\Phi(b) = K \cup \bigcup_{n\in b} K_n$ is continuous, so the sets $\mathcal{G}_1 := \{b \subset \omega;\ \Phi(b) \in \mathcal{G}\}$ and $\mathcal{G}_2 := \{b \subset \omega;\ \Phi(\omega \setminus b) \in \mathcal{G}\}$ are both G_δ subsets of $\mathcal{P}(\omega)$. Moreover, they are both dense in $\mathcal{P}(\omega)$ because $\mathcal{G}_1$ contains all finite sets and $\mathcal{G}_2$ contains all cofinite sets. By the Baire Category Theorem, $\mathcal{G}_1 \cap \mathcal{G}_2$ is nonempty, which proves the lemma. □

Proof of Corollary 1.3. The implication (ii)⇒(iii) is obvious from 1.2, and (iii)⇒(i) follows from [KLW]. (For a direct proof of this implication the reader may consult [ST, Theorem 4.1].) Finally, to prove the implication (i)⇒(ii), it is enough to check that any dense G_δ ideal $\mathcal{I}$ is rich in sequences. This follows from Lemma 1.4: for each point $x \in \bigcup \mathcal{I}$, one may take $\mathcal{D} = \mathcal{I}$. □

Our next result uncovers a connection between richness in sequences and comeagerness of an ideal. It will allow us to transfer the conclusion of Theorem 1.1 to ideals which are nonmeager or comeager. Note however that richness in sequences and comeagerness do not coincide. For example, the ideal of all compact sets $K \subset E$ having finitely many limit points is clearly rich in sequences, but it is meager in $\mathcal{K}(E) \setminus \{\emptyset\}$ if E is perfect. On the other hand, if $E = \omega + 1$, then the ideal of all finite subsets of E is comeager in $\mathcal{K}(E)$ because it contains $\mathcal{K}(\omega)$, but it is not rich in sequences.

Theorem 1.5. *Let $\mathcal{I}$ be an ideal of $\mathcal{K}(E)$.*

(i) *If $\mathcal{I}$ is comeager in $\mathcal{K}(E)$, then $\mathcal{I}$ is rich in sequences at comeagerly many points of E.*
(ii) *If $\mathcal{I}$ is $\mathbf{\Sigma}^0_3$ and rich in sequences at comeagerly many points of E, then $\mathcal{I}$ is comeager in $\mathcal{K}(E)$.*

A sharper version of point (ii) in the theorem above is proved in Proposition 3.1. For importance of assuming that $\mathcal{I}$ be $\mathbf{\Sigma}^0_3$, see the remarks in Section 3.

Note that in certain circumstances (i) from the theorem above suffices to get richness in sequences at all points of E. For example, if E is a topological group and $\mathcal{I} \subset \mathcal{K}(E)$ is a comeager, translation-invariant ideal, then $\mathcal{I}$ is rich in sequences.

Indeed, by Theorem 1.5(i), $\mathcal{I}$ is rich in sequences at some point $x \in E$, hence at all points of E by translation invariance.

The following corollary is a "nonmeager" version of Theorem 1.5(i). The "nonmeager" analogue of Theorem 1.5(ii) follows immediately from Proposition 3.1.

Corollary 1.6. *If $\mathcal{I} \subset \mathcal{K}(E)$ is an ideal with the property of Baire such that $\mathcal{I} \setminus \{\emptyset\}$ is nonmeager in $\mathcal{K}(E)$, then $\mathcal{I}$ is rich in sequences at nonmeagerly many points of E.*

To derive this corollary, we will need a lemma from [MZ]. We reproduce its proof here for completeness.

Lemma 1.7. *Let $\mathcal{A}$ be a hereditary subset of $\mathcal{K}(E)$ with the Baire property. If $\mathcal{A} \setminus \{\emptyset\}$ is nonmeager in $\mathcal{K}(E)$, then there exists a nonempty open set $U \subset E$ such that $\mathcal{A} \cap \mathcal{K}(U)$ is comeager in $\mathcal{K}(U)$.*

Proof. Since $\mathcal{A} \setminus \{\emptyset\}$ is nonmeager and has the Baire property, it is comeager in some nonempty open set of the form

$$\mathcal{U} = \{K \in \mathcal{K}(E);\ K \subset U, K \cap U_i \neq \emptyset \text{ for } i = 1, \dots, n\},$$

where U is a nonempty open subset of E and $U_1, \dots, U_n$ are open subsets of U. Since $\mathcal{A}$ is hereditary, $\mathcal{K}(U) \setminus \mathcal{A}$ is contained in

$$\mathcal{M} = \{K \in \mathcal{K}(E);\ \forall L \in \mathcal{U} :\ K \cup L \in \mathcal{U} \setminus \mathcal{A}\}.$$

By the Kuratowski–Ulam Theorem, the latter set is meager in $\mathcal{K}(E)$, because $\mathcal{U} \setminus \mathcal{A}$ is meager and the map $(K, L) \mapsto K \cup L$ is continuous and open. Thus, $\mathcal{A} \cap \mathcal{K}(U)$ is comeager in $\mathcal{K}(U)$. □

Proof of Corollary 1.6. Using Lemma 1.7, the corollary follows immediately from Theorem 1.5(i). □

Using 1.1, 1.5(i), and 1.6, we immediately get the following sharpening of the result from [M] mentioned above. Notice that one cannot get a global trichotomy as in 1.2, even if $\mathcal{I}$ is assumed to be comeager in $\mathcal{K}(E)$: consider again the ideal of finite subsets of $\omega + 1$.

Corollary 1.8. *Let $\mathcal{I} \subset \mathcal{K}(E)$ be a universally Baire ideal. If $\mathcal{I} \setminus \{\emptyset\}$ is nonmeager in $\mathcal{K}(E)$, then one of the following holds.*

(i) *There exists a nonempty open set $U \subset E$ such that $\mathcal{I} \cap \mathcal{K}(U)$ is a σ-ideal.*
(ii) *$\mathcal{I}$ is $\mathbf{\Pi}^0_3$-hard.*
(iii) *$\mathcal{I}$ is $\mathbf{\Sigma}^0_3$-hard.*

If $\mathcal{I}$ is comeager, then in (i) *we can take U to be dense in E.*

Proof. If $\mathcal{I} \setminus \{\emptyset\}$ is nonmeager, (i)–(iii) follow from 1.1 and 1.6. For the last part, we define U to be the union of all open sets $W \subset E$ such that $\mathcal{I} \cap \mathcal{K}(W)$ is a σ-ideal. An easy compactness argument shows that $\mathcal{I} \cap \mathcal{K}(U)$ is a σ-ideal (see the proof of

Corollary 1.2 above), and if $\mathcal{I}$ is comeager, it follows from 1.5(i) that U is dense in E. □

Let us also point out the following easy consequence of 1.8 and the Kechris–Louveau–Woodin dichotomy ([KLW]) saying that a $\mathbf{\Pi}^1_1$ σ-ideal of compact sets is either $\mathbf{\Pi}^1_1$-complete or a G_δ.

Corollary 1.9. *Let $\mathcal{I} \subset \mathcal{K}(E)$ be a $\mathbf{\Pi}^1_1$ ideal. If $\mathcal{I} \setminus \{\emptyset\}$ is nonmeager in $\mathcal{K}(E)$, then either $\mathcal{I}$ is $\mathbf{\Sigma}^0_3$-hard, or $\mathcal{I}$ is $\mathbf{\Pi}^0_3$-hard, or else $\mathcal{I} \cap \mathcal{K}(U)$ is a G_δ σ-ideal, for some nonempty open set $U \subset E$. If $\mathcal{I}$ is comeager, the open set U can be taken to be dense in E.*

For $\mathbf{\Pi}^1_1$ ideals, comeagerness and nonmeagerness are particularly meaningful. By results of [MZ] a $\mathbf{\Pi}^1_1$ family $\mathcal{I}$ is comeager if and only if it contains G_δ set which is both hereditary and dense in $\mathcal{K}(E)$; and $\mathcal{I} \setminus \{\emptyset\}$ is nonmeager if and only if it contains a hereditary G_δ set which is dense in $\mathcal{K}(U)$ for some open nonempty set U. Hence the following corollary to 1.5.

Corollary 1.10. *Let $\mathcal{I} \subset \mathcal{K}(E)$ be a $\mathbf{\Sigma}^0_3$ ideal. The following are equivalent.*

(i) *$\mathcal{I}$ contains a dense G_δ hereditary set.*
(ii) *$\mathcal{I}$ is comeager.*
(iii) *$\mathcal{I}$ is rich in sequences at comeagerly many points of E.*

We shall also give a criterion (Proposition 3.2) for non-meagerness of not necessarily hereditary families of compact sets, in terms of a condition close to richness in sequences. This generalizes a result of Balcerzak and Darji ([BD]).

Finally, we must point out that the alternatives exhibited in 1.2 and 1.8 do not exclude each other. Indeed, any comeager, $\mathbf{\Pi}^1_1$-complete σ-ideal $\mathcal{I}$ satisfies (i), (ii), and (iii) together (for 1.2, notice that a σ-ideal is rich in sequences if and only if it is dense in $\mathcal{K}(E)$). For example, one may take $\mathcal{I} = \mathcal{K}(A)$, where $A \subset E$ is any Borel comeager set which is not G_δ. However, note that for each one of the three properties involved in 1.2 or 1.8, there exist ideals satisfying this property but neither of the other two. This should be clear for 1.2, but perhaps less obvious for 1.8 since condition (i) there is only local. Let us give some examples.

(1) Any dense G_δ σ-ideal of $\mathcal{K}(E)$ satifies (i) but neither (ii) nor (iii) in 1.2 or 1.8. For example, one may take the σ-ideal of compact nowhere dense sets if the Polish space E is perfect.

(2) Let E be any perfect compact metric space. Let (E_n) be a sequence of perfect compact subsets of E with empty interior such that each nonempty open subset of E contains one of the sets E_n. For each $n \in \omega$, write $E \setminus E_n = \bigcup_{p\in\omega} C_{n,p}$, where the $C_{n,p}$'s are closed and $C_{n,p}$ is contained in the interior of $C_{n,p+1}$ for every $n, p \in \omega$. Finally, set

$$\mathcal{I} := \{K \in \mathcal{K}(E);\ \forall n \in \omega\ \exists p \in \omega :\ K \setminus C_{n,p} \text{ is finite}\}.$$

Clearly $\mathcal{I}$ is an ideal of $\mathcal{K}(E)$, which is comeager since it contains $\mathcal{K}(G)$, where G is the dense G_δ set $E \setminus \bigcup_n E_n$. It is easy to check that $\mathcal{I}$ is also rich in sequences. Moreover for each $n \in \omega$, the set $\{K \in \mathcal{K}(E);\ K \setminus C_{n,p} \text{ is finite}\}$ is a σ-compact subset of $\mathcal{K}(E)$, because E is compact. Thus, $\mathcal{I}$ is $\mathbf{\Pi}^0_3$. It is clear that $\mathcal{I}$ is not a σ-ideal. Finally, if V is a nonempty open subset of E, then V contains some set E_n, so $\mathcal{I} \cap \mathcal{K}(V)$ contains $\mathcal{I} \cap \mathcal{K}(E_n)$ as a closed subset; but $\mathcal{I} \cap \mathcal{K}(E_n)$ is just the family of finite subsets of E_n, so it is not G_δ. Thus, $\mathcal{I}$ is nowhere G_δ (i.e., $\mathcal{I} \cap \mathcal{K}(U)$ is G_δ for no nonempty open set $U \subset E$). It follows from 1.2 or 1.8 that $\mathcal{I}$ is $\mathbf{\Pi}^0_3$-complete, but this is also easy to check directly.

(3) Let $\mathcal{I} = \{K \in \mathcal{K}(\mathbb{R});\ K \cap \mathbb{Q} \text{ is finite}\}$ is an ideal of $\mathcal{K}(\mathbb{R})$ which is comeager, rich in sequences, $\mathbf{\Sigma}^0_3$, not a σ-ideal, and nowhere G_δ. Hence, it follows from 1.8 that $\mathcal{I}$ is $\mathbf{\Sigma}^0_3$-complete. Of course, this could also be proved directly, though perhaps not so easily.

2. Trichotomy

We shall write r.i.s. for "rich in sequences".

The following lemma shows that if an ideal $\mathcal{I}$ fails to be a σ-ideal, then it does so in a very special way. This observation is implicit in [K_2]. We give a simple proof here.

Lemma 2.1. *Let $\mathcal{I}$ be an ideal of $\mathcal{K}(E)$. Assume that $\mathcal{I}$ is not a σ-ideal. Then there exists a sequence $(K_n) \subset \mathcal{I}$ and a compact set $K \in \mathcal{I}$ such that $K_n \to K$ but $K \cup \bigcup_n K_n \notin \mathcal{I}$.*

Proof. Fix some compatible metric d on E with $\operatorname{diam}(E) \le 1$. It is enough to prove the following claim.

Claim. *There exist nonempty $K, L \in \mathcal{K}(E)$ such that*

(a) *$K \subset L$;*
(b) *$K \in \mathcal{I}$, $L \notin \mathcal{I}$;*
(c) *$\{x \in L;\ d(x, K) \ge \varepsilon\} \in \mathcal{I}$ for all $\varepsilon > 0$.*

Indeed, once this is done, one just has to put

$$K_n = K \cup \{x \in L;\ 2^{-n-1} \le d(x, K) \le 2^{-n}\}.$$

□

Proof of the claim. By contradiction, assume that the claim fails. Since $\mathcal{I}$ is not a σ-ideal, there exists a sequence $(C_n) \subset \mathcal{I}$ such that $C = \bigcup_n C_n$ is compact but $C \notin \mathcal{I}$. Since $\mathcal{I}$ is an ideal, we may assume that the sequence (C_n) is increasing, with $C_0 \neq \emptyset$. Since the claim does not hold with $K = C_0$ and $L = C$, one can find $\varepsilon_0 > 0$ such that $V_0 = \{x \in E;\ d(x, C_0) < \varepsilon_0\}$ satisfies $C \setminus V_0 \notin \mathcal{I}$. Choose an integer $n_1 \ge 1$ such that $C_{n_1} \setminus V_0 \neq \emptyset$. Since the claim does not hold with $K = C_{n_1} \setminus V_0$ and $L = C \setminus V_0$, one can find an open set $V_1 \subset E$ such that $C_1 \subset C_{n_1} \subset V_0 \cup V_1$ and $C \setminus (V_0 \cup V_1) \notin \mathcal{I}$.

By induction, one constructs a sequence of open sets (V_n) such that $C_n \subset \bigcup_{i\leq n} V_i$ and $C\setminus \bigcup_{i\leq n} V_i\notin \mathcal{I}$ for all n. By compactness, one gets $C\setminus\bigcup_{n\in\omega} V_n\neq\emptyset$, which is a contradiction since $C=\bigcup_n C_n$. □

Remark. Combined with 1.4, Lemma 2.1 implies that a G_δ ideal is necessarily a σ-ideal.

Lemma 2.2. *Let $\mathcal{I}$ be a universally Baire ideal of $\mathcal{K}(E)$, and let $K\in\mathcal{I}$. Assume there exists a sequence $(K_n)\subset\mathcal{I}$ such that (K_n) accumulates to K and $K\cup\bigcup_n K_n\notin\mathcal{I}$. Then one can find a sequence $(F_n)\subset\mathcal{I}$ such that (F_n) accumulates to K and, for each $b\subset\omega$, the following property holds true: b is infinite if and only if $K\cup\bigcup_{n\in b} F_n\notin\mathcal{I}$.*

Proof. The map $b\mapsto K\cup\bigcup_{n\in b} K_n$ is continuous from $\mathcal{P}(\omega)$ into $\mathcal{K}(E)$. Since $\mathcal{I}$ is a universally Baire ideal, it follows that the family

$$\tilde{\mathcal{I}}=\left\{b\subset\omega;\ K\cup\bigcup_{n\in b} K_n\in\mathcal{I}\right\}$$

is an ideal of $\mathcal{P}(\omega)$ with the Baire property in $\mathcal{P}(\omega)$. Moreover, $\omega\notin\tilde{\mathcal{I}}$ by assumption. By the so-called Jalali-Naini–Mathias–Talagrand Theorem (see [T]), it follows that there exists an increasing sequence of integers $p_0<p_1<\dots$ such that no member of $\tilde{\mathcal{I}}$ contains infinitely many intervals $a_n=[p_n,p_{n+1}[$. So it is enough to put $F_n=\bigcup_{j\in a_n} K_j$. □

Remark. In particular, it follows from 2.1 and 2.2 that if $\mathcal{I}$ is a universally Baire ideal of $\mathcal{K}(E)$ which is not a σ-ideal, then $\mathcal{I}$ reduces the ideal of finite subsets of ω (in a very particular way), so it is F_σ-hard.

Proof of Theorem 1.1. Let $\mathcal{I}$ be a universally Baire ideal of $\mathcal{K}(E)$ which is r.i.s. at some point $x\in\bigcup\mathcal{I}$. Assume that $\mathcal{I}\cap\mathcal{K}(V)$ is a σ-ideal for no open neighborhood V of x.

Case 1. *For each sequence $(K_n)\subset\mathcal{I}$ such that $K_n\to\{x\}$, the set $\{x\}\cup\bigcup_n K_n$ belongs to $\mathcal{I}$.*

We show that $\mathcal{I}$ is $\mathbf{\Pi}^0_3$-hard. Fix some compatible metric on E, and put $V_n=B(x,2^{-n})$, $n\in\omega$. As observed above, $\mathcal{I}\cap\mathcal{K}(V_n)$ is F_σ-hard for each n, so one can choose a continuous map $\Phi_n:2^\omega\to\mathcal{K}(V_n)$ such that $\Phi_n^{-1}(\mathcal{I})=\mathbf{Q}$, where $\mathbf{Q}=\{(\alpha_n)\in 2^\omega;\ (\alpha_n)\text{ is eventually }0\}$. By assumption, if we define $\Phi:(2^\omega)^\omega\to\mathcal{K}(E)$ by

$$\Phi((\xi_n))=\{x\}\cup\bigcup_{n\in\omega}\Phi_n(\xi_n)\,,$$

then $\Phi^{-1}(\mathcal{I})=\mathbf{W}$, where $\mathbf{W}:=\{(\xi_n)\in(2^\omega)^\omega;\ \forall n\in\omega:\ \xi_n\in\mathbf{Q}\}$. Since the map Φ is obviously continuous and $\mathbf{W}$ is $\mathbf{\Pi}^0_3$-complete, this shows that $\mathcal{I}$ is $\mathbf{\Pi}^0_3$-hard.

Case 2. *There exists a sequence $(K_n)\subset\mathcal{I}$ such that $K_n\to\{x\}$ but $\{x\}\cup\bigcup_n K_n\notin\mathcal{I}$.*

We show that $\mathcal{I}$ is $\mathbf{\Sigma}^0_3$-hard. By Lemma 2.2, one can find a sequence $(F_n)\subset\mathcal{I}$ such that $F_n\to\{x\}$ and, for each $b\subset\omega$, b is finite if and only if $\{x\}\cup\bigcup_{n\in b}F_n\in\mathcal{I}$. Since $\mathcal{I}$ is r.i.s. at x, we can choose a dense set $\mathcal{D}\subset\mathcal{K}(E)$ so that for any sequence (L_n) of members of $\mathcal{D}$ converging to $\{x\}$, we have $\{x\}\cup\bigcup_n L_n\in\mathcal{I}$. By density of $\mathcal{D}$, we can find sets $L_n^k\in\mathcal{D}$, $n,k\in\omega$, such that $L_n^k\subset\{x;\ d(x,F_n)\le 2^{-n}\}$ and $L_n^k\to F_n$ as $k\to\infty$. Put $L_n^\omega=F_n$. Now, for $(\alpha_n)\in(\omega+1)^\omega$ define

$$\Phi((\alpha_n))=\{x\}\cup\bigcup_n L_n^{\alpha_n}.$$

It is not hard to check that the map $\Phi:(\omega+1)^\omega\to\mathcal{K}(E)$ is continuous; here of course, we put the usual order topology on $\omega+1$ and the product topology on $(\omega+1)^\omega$. Moreover, if we set

$$\mathbf{A}:=\{\alpha\in(\omega+1)^\omega;\ \exists m\in\omega\ \forall n\ge m:\ \alpha_n<\omega\},$$

then $\Phi^{-1}(\mathcal{I})=\mathbf{A}$. Indeed, if $\alpha\notin\mathbf{A}$, then $\alpha_n=\omega$ for infinitely many n's, so that $\Phi(\alpha)$ contains $\{x\}\cup\bigcup_{n\in b}F_n$, for some infinite set $b\subset\omega$. Since $\mathcal{I}$ is hereditary, this implies that $\Phi(\alpha)\notin\mathcal{I}$. On the other hand, if $\alpha\in\mathbf{A}$, then there exists $m\in\omega$ such that $\alpha_n<\omega$ for all $n\ge m$. By definition of $\mathcal{D}$, this implies that $\{x\}\cup\bigcup_{n\ge m}L_n^{\alpha_n}\in\mathcal{I}$. Finally, all sets L_n^k, $k\in\omega+1$, are in $\mathcal{I}$, so $\bigcup_{n<m}L_n^{\alpha_n}\in\mathcal{I}$, and we get $\Phi(\alpha)=\{x\}\cup\bigcup_{n<m}L_n^{\alpha_n}\cup\bigcup_{n\ge m}L_n^{\alpha_n}\in\mathcal{I}$.

Thus, to prove that $\mathcal{I}$ is $\mathbf{\Sigma}^0_3$-hard, it suffices to show that $\mathbf{A}$ is $\mathbf{\Sigma}^0_3$-complete. Now, since ω is a $\mathbf{\Sigma}^0_1$-complete subset of $\omega+1$, one checks immediately that $\mathbf{A}$ reduces the set

$$\mathbf{S}:=\{(\xi_n)\in(2^\omega)^\omega;\ \exists m\in\omega\ \forall n\ge m:\ \exists j\in\omega\ \ \xi_n(j)=0\},$$

which is known to be $\mathbf{\Sigma}^0_3$-complete (see [K$_1$, 23.A]). This concludes the proof. □

3. Nonmeagerness and richness in sequences

In this section, we discuss the connections between "nonmeagerness", "comeagerness", and "richness in sequences".

Proof of Theorem 1.5(i). Assume that $\mathcal{I}$ is comeager, and choose a dense G_δ set $\mathcal{G}\subset\mathcal{I}$. Let $(\mathcal{V}_j)_{j\ge 0}$ be a countable basis of $\mathcal{K}(E)$ consisting of nonempty open sets with $\mathcal{V}_0=\mathcal{K}(E)$. Put $\mathcal{X}=\prod_{j\in\omega}\mathcal{V}_j$. For each $j_0<\cdots<j_p$, the map $(K_0,\dots,K_p)\mapsto K_0\cup\cdots\cup K_p$ is continuous and open from $\mathcal{V}_{j_0}\times\cdots\times\mathcal{V}_{j_p}$ into $\mathcal{K}(E)$, so the set

$$\big\{(K_0,\dots,K_p)\in\mathcal{V}_{j_0}\times\cdots\times\mathcal{V}_{j_p};\ K_0\cup\cdots\cup K_p\in\mathcal{G}\big\}$$

is a dense G_δ subset of $\mathcal{V}_{j_0}\times\dots\mathcal{V}_{j_p}$. This implies that

$$\mathcal{Q}:=\left\{(K_j)\in\mathcal{X};\ \bigcup_{j\in b}K_j\in\mathcal{G}\text{ for every finite }b\subset\omega\right\}$$

is a dense G_δ subset of $\mathcal{X}$. Let $\mathcal{H}$ be the projection of $\mathcal{Q}$ to the first coordinate of $\mathcal{X}$. By the Kuratowski–Ulam theorem the set $\mathcal{H}$ is comeager in $\mathcal{K}(E)$, hence $H := \bigcup \mathcal{H}$ is comeager in E. Indeed, since $\mathcal{H}$ is a $\mathbf{\Sigma}^1_1$ subset of $\mathcal{K}(E)$, the set H is $\mathbf{\Sigma}^1_1$ in E, so it has the Baire property in E. If H were not comeager, it would be meager in some nonempty open set $V \subset E$; thus, one could find a G_δ set $G \subset V$ dense in V such that $H \cap G = \emptyset$. Then $\mathcal{H} \cap \mathcal{K}(G) = \{\emptyset\}$, which is a contradiction since $\mathcal{H} \cap \mathcal{K}(V)$ and $\mathcal{K}(G)$ are both comeager in $\mathcal{K}(V)$.

We now show that $\mathcal{I}$ is rich in sequences at each point of H. Let x be any point of H. Then there exists a sequence $(K_j) \in \mathcal{Q}$ with $x \in K_0$; we set $\mathcal{D} = \{K_j;\ j \geq 1\}$. The set $\mathcal{D}$ is obviously dense in $\mathcal{K}(E)$. Observe that if a sequence $(T_n) \subset \mathcal{D}$ converges to $\{x\}$, then $K_0 \cup \bigcup_{n\in b} T_n \in \mathcal{G}$ for every finite $b \subset \omega$. Since $\mathcal{I}$ is an ideal and $\mathcal{G} \subset \mathcal{I}$, it follows from Lemma 1.4 that $K_0 \cup \bigcup_{n\in\omega} T_n \in \mathcal{I}$, hence $\{x\} \cup \bigcup_{n\in\omega} T_n \in \mathcal{I}$ because $\mathcal{I}$ is hereditary. Thus $\mathcal{D}$ witnesses that $\mathcal{I}$ is r.i.s. at x. □

We will derive Theorem 1.5(ii) from the next result. We employ the following notation. Let $\mathcal{N} = \omega^\omega$. Given $s \in \omega^{<\omega}$ and $\nu \in \omega^{<\omega} \cup \mathcal{N}$, we write $s \prec \nu$ if ν is an extension of s. For each finite sequence $s \in \omega^{<\omega}$, we put $\mathcal{N}(s) = \{\nu \in \mathcal{N};\ s \prec \nu\}$. If $A \subset E$, then the set of all *nonempty* compact subset of A is denoted by $\mathcal{K}^*(A)$. If $O_1, \dots, O_p$ are open subsets of E, we put

$$\mathcal{U}(O_1, \dots, O_p) = \{K \in \mathcal{K}(E);\ K \subset O_1 \cup \dots \cup O_p \text{ and } K \cap O_j \neq \emptyset,\ j = 1, \dots, p\}.$$

Proposition 3.1. *Let $\mathcal{A} \subset \mathcal{K}(E)$ be $\mathbf{\Sigma}^0_3$ and hereditary. If $\mathcal{A}$ is rich in sequences at nonmeagerly many points, then $\mathcal{A} \setminus \{\emptyset\}$ is nonmeager in $\mathcal{K}(E)$.*

Proof. 1. Towards a contradiction, assume that $\mathcal{A}$ is r.i.s. at nonmeagerly many points of E, and yet $\mathcal{A} \setminus \{\emptyset\}$ is meager in $\mathcal{K}(E)$. By richness in sequences and the fact that $\mathcal{A}$ is hereditary, the set $\{x \in E;\ \{x\} \in \mathcal{A}\}$ is dense in E, and by the meagerness assumption, it follows that E has no isolated point.

Write $\mathcal{A} = \bigcup_n \mathcal{A}_n$, where the $\mathcal{A}_n$ are G_δ. Then all sets $\mathcal{A}_n \setminus \{\emptyset\}$ are nowhere dense, by the Baire Category Theorem. We assume $\mathcal{A}_0 = \emptyset$. Finally, we fix some compatible Polish metric on E.

Claim. *There exist a nonempty open set $V \subset E$, a G_δ set $H \subset V$ dense in V and a continuous map $\varphi : H \to \mathcal{K}(E)^{\omega\times\omega}$ such that for each $x \in H$, the following properties hold:*

- (i) *the set $\{\varphi(x)(n, m);\ m \in \omega\}$ is a dense subset of $\mathcal{K}(B(x, 2^{-n}))$ for every $n \in \omega$;*
- (ii) *$\{x\} \cup \bigcup_{n=0}^\infty \varphi(x)(n, \nu(n)) \in \mathcal{A}$ for comeagerly many $\nu \in \mathcal{N}$.*

Proof of the claim. Define $\mathfrak{S} \subset E \times \mathcal{K}(E)^{\omega\times\omega}$ by letting $(x, (K(n, m))_{(n,m)\in\omega\times\omega}) \in \mathfrak{S}$ if and only if the following two conditions hold:

- $\{K(n, m);\ m \in \omega\}$ is a dense subset of $\mathcal{K}(B(x, 2^{-n}))$ for every $n \in \omega$,
- $\forall^*\nu \in \mathcal{N} :\ \{x\} \cup \bigcup_{n=0}^\infty K(n, \nu(n)) \in \mathcal{A}$.

Since the "comeagerly many" quantifier $\forall^*$ preserves Borel measurability (see [K$_1$, 16.A]), the set $\mathfrak{S}$ is Borel in $E \times \mathcal{K}(E)^{\omega\times\omega}$, so the projection $\pi(\mathfrak{S})$ of $\mathfrak{S}$ into E is $\mathbf{\Sigma}^1_1$. Moreover, if $\mathcal{A}$ is r.i.s. at some point $x \in E$ then $x \in \pi(\mathfrak{S})$. Consequently, $\pi(\mathfrak{S})$ is a nonmeager set with the Baire property, so there exist a nonempty open set $V \subset E$ and a G_δ set H dense in V such that $H \subset \pi(\mathfrak{S})$.

By the Jankov–von Neumann theorem (see [K$_1$, 18.A]) $\mathfrak{S}$ has a $\sigma(\mathbf{\Sigma}^1_1)$-measurable uniformizing function $\varphi : H \to \mathcal{K}(E)^{\omega\times\omega}$. In particular, φ is Baire measurable, so, replacing H by a smaller G_δ set dense in V, we may in fact assume that $\varphi : H \to \mathcal{K}(E)^{\omega\times\omega}$ is continuous, which proves the claim. □

Let $\mathbf{G}_\varphi$ be the graph of φ from the claim. By the Kuratowski–Ulam theorem, we can find dense G_δ sets $\mathfrak{D} \subset \mathbf{G}_\varphi \times \mathcal{N}$ and $G \subset H$ such that for each point $x \in G$, the $(x, \varphi(x))$ - section of $\mathfrak{D}$ is comeager in $\mathcal{N}$ and for each of its elements ν we have $\{x\} \cup \bigcup_{n=0}^{\infty} \varphi(x)(n, \nu(n)) \in \mathcal{A}$. We write $\mathfrak{D} = \bigcap_{n=0}^{\infty} \mathfrak{D}_n$, where the sets $\mathfrak{D}_n$ are dense open subsets of $\mathbf{G}_\varphi \times \mathcal{N}$, $\mathfrak{D}_0 = \mathbf{G}_\varphi \times \mathcal{N}$, and $G = \bigcap_{k=0}^{\infty} G_k$, where the G_k's are dense open subsets of V.

2. We are going to construct

- a sequence $(s^k) \subset \omega^{<\omega}$, increasing with respect to the ordering $\prec$,
- a decreasing sequence (B_k) of open balls from V,

such that for every $k \in \omega$ we have

(a) for all $x \in B_k \cap G$ and all $K \in \mathcal{K}(B_k)$ containing x:
$$\bigcup_{p<|s^k|} \varphi(x)(p, s^k(p)) \cup K \notin \mathcal{A}_k;$$
(b) $\forall x \in B_k \cap G : \ \{(x, \varphi(x))\} \times \mathcal{N}(s^k) \subset \mathfrak{D}_k$;
(c) $\overline{B_{k+1}} \subset B_k \subset G_k$, $\operatorname{diam} B_k \le 2^{-k}$;
(d) $\forall x \in B_{k+1} \cap G : \ \bigcup_{|s^k|\le p<|s^{k+1}|} \varphi(x)(p, s^{k+1}(p)) \subset B_k$.

3. Assume the construction has been carried out. Let $\nu \in \mathcal{N}$ be such that $s^k \prec \nu$ for every $k \in \omega$. Then the balls B_k shrink to some point $x \in G$ by (c). Thus by (b) $K_\infty := \{x\} \cup \bigcup_{p=0}^{\infty} \varphi(x)(p, \nu(p)) \in \mathcal{A}$. But on the other hand, for arbitrary $k \in \omega$, we have
$$K_\infty = \bigcup_{p<|s^k|} \varphi(x)(p, \nu(p)) \cup \left(\{x\} \cup \bigcup_{p\ge|s^k|} \varphi(x)(p, \nu(p)) \right) \notin \mathcal{A}_k$$
by (a), since by (c) and (d) the set $\{x\} \cup \bigcup_{p\ge|s^k|} \varphi(x)(p, \nu(p))$ is a compact subset of B_k containing x. Consequently, $K_\infty \notin \mathcal{A}$, a contradiction.

4. Now we perform the construction.

Step 0. Set $s^0 = \emptyset$ and let B_0 be any ball of diameter less than 1 with $B_0 \subset G_0$.

Step $k+1$**.** Assume $s^0,\dots,s^k$ and $B_0,\dots,B_k$ have been constructed. Let $x_0\in G\cap B_k$, and choose $p_0>|s^k|$ such that $B(x_0,2^{-p_0})\subset B_k$. By definition of the map φ, one can find a finite sequence $t_0\succ s^k$ of length p_0 such that $\varphi(x_0)(p,t_0(p))\subset B_k$ for all $p\in\{|s^k|,\dots,p_0-1\}$. Then, by the choice of p_0, we have $\varphi(x_0)(p,\nu(p))\subset B_k$ for all $\nu\in\mathcal{N}(t_0)$ and all $p\geq|s^k|$. Since the $(x_0,\varphi(x_0))$-section of $\mathfrak{O}$ is dense in $\mathcal{N}$, one can find $\nu\in\mathcal{N}(t_0)$ such that $(x_0,\varphi(p_0),\nu)\in\mathfrak{O}$. By continuity of φ, it follows that one can find some finite sequence $t\succ t_0\succ s^k$ and some nonempty open set U with $\overline{U}\subset B_k\cap G_{k+1}$, $\operatorname{diam}(U)<2^{-|t|-1}$, such that for all $x\in U$, we have

- $\{(x,\varphi(x))\}\times\mathcal{N}(t)\subset\mathfrak{O}_{k+1}$;
- $\bigcup_{|s^k|\leq p<|t|}\varphi(x)(p,t(p))\subset B_k$;
- $B(x,2^{-|t|})\subset B_k$.

We put $\mathcal{Z}:=\{(x,L)\in U\times\mathcal{K}(U);\ x\in L\}$. Notice that $\mathcal{Z}$ is a closed subset of $U\times\mathcal{K}(U)$, so it is a Polish space.

Since $\mathcal{A}$ is hereditary, the set

$$\mathcal{F}=\left\{(M,(x,L))\in\mathcal{K}^*(U)\times\mathcal{Z};\ \bigcup_{p<|t|}\varphi(x)(p,t(p))\cup M\cup L\in\mathcal{A}\right\}$$

has all its (x,L)-sections contained in $\mathcal{A}$, hence it is meager in $\mathcal{K}^*(U)\times\mathcal{Z}$, by the Kuratowski–Ulam Theorem. It follows that the G_δ set

$$\mathcal{G}=\left\{(M,(x,L))\in\mathcal{K}^*(U)\times\mathcal{Z};\ \bigcup_{p<|t|}\varphi(x)(p,t(p))\cup M\cup L\in\mathcal{A}_{k+1}\right\}$$

is nowhere dense in $\mathcal{K}^*(U)\times\mathcal{Z}$. Thus, one can find nonempty open sets $V_1,\dots,V_q$, $W_1,\dots,W_r$, B all contained in U such that $\mathcal{Z}\cap(B\times\mathcal{U}(W_1,\dots,W_r))\neq\emptyset$ and, for each $(x,L)\in\mathcal{Z}\cap(B\times\mathcal{U}(W_1,\dots,W_r))$, the set $\mathcal{U}(V_1,\dots,V_q)$ is disjoint from $\mathcal{G}_{(x,L)}$. Of course, we may assume that B is a ball and $B\subset\bigcup_{j=1}^r W_j$. Moreover, since E has no isolated point, we may also assume that all V_i's and W_j's are pairwise disjoint.

Put $\mathcal{V}:=\mathcal{U}(V_1,\dots,V_q,W_1,\dots,W_r)$ and $n:=|t|$; let also z_0 be any point of $B\cap G$. Since the set $\{\varphi(z_0)(n,m);\ m\in\omega\}$ is dense in $\mathcal{K}(B(z_0,2^{-n}))$ and $U\subset B(z_0,2^{-n})$, we can find $m\in\omega$ with $\varphi(z_0)(n,m)\in\mathcal{V}$. Using continuity of φ and taking B smaller if necessary we get $\varphi(x)(n,m)\in\mathcal{V}$ for every $x\in B\cap G$. Put $s^{k+1}:=(t(0),\dots,t(n-1),m)$ and $B_{k+1}:=B$. Then conditions (b), (c), and (d) are obviously satisfied. We check that (a) holds. Take $K\in\mathcal{K}(B_{k+1})$ and $x\in K\cap G$. The sets

$$M=\varphi(x)(n,s^{k+1}(n))\cap\bigcup_{i=1}^q V_i\qquad\text{and}$$

$$L=\left(\varphi(x)(n,s^{k+1}(n))\cap\bigcup_{j=1}^r W_j\right)\cup K$$

are both compact and we have $(x, L) \in \mathcal{Z} \cap (B \times \mathcal{U}(W_1, \dots, W_r))$, $M \in \mathcal{U}(V_1, \dots, V_q)$, and

$$\bigcup_{p<|s^{k+1}|} \varphi(x)(p, s^{k+1}(p)) \cup K = \bigcup_{p<|t|} \varphi(x)(p, s^k(p)) \cup M \cup L.$$

Thus, we get

$$\bigcup_{p<|s^{k+1}|} \varphi(x)(p, s^{k+1}(p)) \cup K \notin \mathcal{A}_{k+1},$$

as required. This concludes the proof. □

Proof of Theorem 1.5(ii). Assume that $\mathcal{I}$ is r.i.s. at comeagerly many points. Since $\mathcal{I}$ has the Baire property in $\mathcal{K}(E)$, it is enough to show that $\mathcal{I}$ is nonmeager in any nonempty open set $\mathcal{U} \subset \mathcal{K}(E)$. We may assume $\mathcal{U}$ has the form $\mathcal{U}(V_1, \dots, V_p)$, where $V_1, \dots, V_p$ are nonempty open subsets of E with pairwise disjoint closures. The map $\Phi : \prod_{i=1}^p \mathcal{K}^*(V_i) \to \mathcal{U}$ defined by $\Phi(K_1, \dots, K_p) = \bigcup_{i=1}^p K_i$ is a homeomorphism from $\prod_{i=1}^p \mathcal{K}^*(V_i)$ onto $\mathcal{U}$, and since $\mathcal{I}$ is an ideal, we have $\Phi^{-1}(\mathcal{I} \cap \mathcal{U}) = \prod_{i=1}^p \mathcal{I} \cap \mathcal{K}^*(V_i)$. By the preceding proposition applied to $\mathcal{K}(V_i)$, we know that $\mathcal{I} \cap \mathcal{K}^*(V_i)$ is nonmeager in $\mathcal{K}^*(V_i)$, for each $i \in \{1, \dots, p\}$. By the Kuratowski–Ulam Theorem, it follows that $\Phi^{-1}(\mathcal{I} \cap \mathcal{U})$ is nonmeager in $\prod_{i=1}^p \mathcal{K}^*(V_i)$, so $\mathcal{I} \cap \mathcal{U}$ is nonmeager in $\mathcal{U}$. This concludes the proof. □

Remark. Even if the $\mathbf{\Sigma}_3^0$ hereditary set $\mathcal{A}$ in 3.1 is r.i.s. at all points of E, one cannot hope to conclude that $\mathcal{A}$ is comeager in $\mathcal{K}(E)$, as shown by the following example. Let V_0, V_1 be pairwise disjoint nonempty clopen subsets of 2^ω such that $V_0 \cup V_1 = 2^\omega$, and let

$$\mathcal{A} := \{K \in \mathcal{K}(2^\omega);\ K \cap V_0 \text{ or } K \cap V_1 \text{ is finite}\}.$$

The set $\mathcal{A}$ is hereditary and F_σ in $\mathcal{K}(2^\omega)$. Moreover, $\mathcal{A}$ is clearly r.i.s. at all points of 2^ω: one may take $\mathcal{D}$ as the set of all finite subsets of 2^ω for each $x \in 2^\omega$. But $\mathcal{A}$ is not comeager because no perfect set $K \in \mathcal{A}$ intersects both V_0 and V_1.

Remark. The assumption that the family $\mathcal{A}$ in 3.1 be $\mathbf{\Sigma}_3^0$ is essential. For example, the hereditary family consisting of compact sets with at most one limit point is $\mathbf{\Pi}_3^0$, rich in sequences, and yet meager $\mathcal{K}(E) \setminus \{\emptyset\}$ if E is perfect. Similarly, in 1.5(ii) the assumption that $\mathcal{I}$ be $\mathbf{\Sigma}_3^0$ cannot be removed, as the ideal consisting of all compact subsets of E with finitely many limit points is $\mathbf{\Sigma}_4^0$ and rich in sequences, but it is meager if E is perfect.

To conclude the paper, we prove the following result, which is clearly similar to 3.1 but, more importantly, might be of independent interest. For that reason, and since the proof is simpler, we chose to state it separately. As explained below, it generalizes the main result of [BD].

Proposition 3.2. *Let $\mathcal{M} \subset \mathcal{K}(E)$. Assume that*

(a) *if $K \in \mathcal{M}$ and $L \subset K$ is relatively clopen in K, then $L \in \mathcal{M}$;*

(b) *for some dense set* $\mathcal{D} \subset \mathcal{K}(E)$*, if* $K_n \in \mathcal{D}$*,* $K_n \to \{x\}$ *for some* $x \in E$*, then* $\{x\} \cup \bigcup_n K_n \in \mathcal{M}$*;*
(c) $\mathcal{M}$ *is* $\mathbf{\Sigma}^0_3$*.*

Then $\mathcal{M} \setminus \{\emptyset\}$ *is nonmeager.*

Proof. By (b), $\{x\} \in \mathcal{M}$ for every isolated point $x \in E$, so we may assume that E is perfect. Write $\mathcal{M} = \bigcup_n \mathcal{M}_n$ where each $\mathcal{M}_n$ is a G_δ subset of $\mathcal{K}(E)$ and $\mathcal{M}_0 = \emptyset$. Assume towards a contradiction that $\mathcal{M} \setminus \{\emptyset\}$ is meager in $\mathcal{K}(E)$. Let us fix some Polish metric on E. We construct by induction nonempty open sets $V_n \subset E$ and nonempty compact sets $K_n \in \mathcal{D}$ such that the following properties hold:

(i) $\operatorname{diam} V_n \le 2^{-n}$, $\overline{V_{n+1}} \subset V_n$;
(ii) $K_{n+1} \subset V_n$;
(iii) $V_n \cap \bigcup_{i \le n} K_i = \emptyset$;
(iv) for each compact set $K \subset V_n$, one has $K \cup \bigcup_{i \le n} K_i \notin \mathcal{M}_n$.

If the construction is carried out, then $\bigcap_n \overline{V_n}$ is a one point set $\{x\}$ with $K_n \to \{x\}$ as $n \to \infty$. By (b) and (iv), we have

$$\{x\} \cup \bigcup_n K_n \in \mathcal{M} \setminus \bigcup_n \mathcal{M}_n,$$

which is a contradiction.

Step 0**.** Let V_0 be any nonempty open set of diameter less than 1 with $\overline{V_0} \neq E$ and choose $\emptyset \neq K_0 \in \mathcal{D}$ such that $V_0 \cap K_0 = \emptyset$. This can be done because $\mathcal{D}$ is dense in $\mathcal{K}(E)$ and E is perfect.

Step $n+1$**.** Assume $V_0, \dots, V_n$ and $K_0, \dots, K_n$ have been constructed. Let

$$\mathcal{A} := \left\{ K \in \mathcal{K}^*(V_n);\ K \cup \bigcup_{i \le n} K_i \in \mathcal{M}_{n+1} \right\}.$$

The set $\mathcal{A}$ is G_δ and, by (a), we have that $\mathcal{A}$ is a subset of $\mathcal{M}$. Hence $\mathcal{A}$ is nowhere dense in $\mathcal{K}^*(V_n)$. Thus, one can choose nonempty open sets $U_1, \dots, U_p$, U with $U_j \subset U \subset V_n$ such that, whenever a compact set L satisfies $L \subset U$ and $L \cap U_j \neq \emptyset$ for $j = 1, \dots, p$, it follows that $L \cup \bigcup_{i \le n} K_i \notin \mathcal{M}_{n+1}$. Moreover, since E is perfect, we may also assume that $U \setminus \bigcup_j \overline{U_j} \neq \emptyset$. By density of $\mathcal{D}$, one can pick $K_{n+1} \in \mathcal{D} \cap \mathcal{U}(U_1, \dots, U_p)$. Finally, let V_{n+1} be any nonempty open set of diameter less than 2^{-n-1} such that $\overline{V_{n+1}} \subset U$ and $V_{n+1} \cap \bigcup_{i \le n+1} K_i = \emptyset$. It is easily checked that properties (i)–(iv) are satisfied. This concludes the inductive step and the proof of 3.2. □

Remark. Let $\mathcal{I} \subset \mathcal{K}(E)$ be hereditary. A compact set $K \subset E$ is said to be $\mathcal{I}$*-perfect* if the intersection of K with any element of $\mathcal{I}$ is nowhere dense in K. By $\mathcal{I}^{\text{perf}}$ we denote the family of all compact $\mathcal{I}$-perfect sets. Clearly, $\mathcal{M} = \mathcal{I}^{\text{perf}}$ satisfies condition (a) of the above proposition. Moreover, $\mathcal{I}^{\text{perf}}$ also satisfies (b) (with $\mathcal{D} = \mathcal{I}^{\text{perf}}$) if it is assumed to be dense in $\mathcal{K}(E)$. Now, assume that $\mathcal{I}$ has the Baire property in

$\mathcal{K}(E)$ and that $\mathcal{I} \setminus \{\emptyset\}$ is nonmeager. By Lemma 1.7, there exists a nonempty open set $U \subset E$ such that $\mathcal{I} \cap \mathcal{K}(U)$ is comeager in $\mathcal{K}(U)$. Then $(\mathcal{M} \setminus \{\emptyset\}) \cap \mathcal{K}(U)$ is meager in $\mathcal{K}(U)$, and applying 3.2 in $\mathcal{K}(U)$, we get the following result:

If $\mathcal{I} \subset \mathcal{K}(E)$ is a hereditary family with the Baire property in $\mathcal{K}(E)$, such that $\mathcal{I} \setminus \{\emptyset\}$ is nonmeager and $\mathcal{I}^{\mathrm{perf}}$ is dense in $\mathcal{K}(E)$, then $\mathcal{I}^{\mathrm{perf}}$ is not $\mathbf{\Sigma}^0_3$.

This statement generalizes the main result of [BD], where the authors obtain the same conclusion under the following assumptions: E is compact, all sets in $\mathcal{I}$ are nowhere dense, and the hereditary family $\mathcal{I}$ is G_δ and contains all finite sets. (Note that if E is compact and all sets in $\mathcal{I}$ are nowhere dense, then $\mathcal{I}^{\mathrm{perf}}$ is dense in $\mathcal{K}(E)$ as closures of non-empty open sets are elements of $\mathcal{I}^{\mathrm{perf}}$ and they form a dense set.) This generalization may be of some relevance since there exist important nonmeager ideals which are not G_δ. For example, the above statement can be used to show the the family $M_0^p := U_0^{\mathrm{perf}}$ is a $\mathbf{\Pi}^0_3$-complete subset of $\mathcal{K}(\mathbb{T})$, where U_0 is the $\mathbf{\Pi}^1_1$-complete σ-ideal of extended uniqueness sets in $\mathbb{T}$ (see [KL]). Since U_0 is comeager in $\mathcal{K}(\mathbb{T})$ and all compact sets in U_0 are nowhere dense, M_0^p is not $\mathbf{\Sigma}^0_3$. Since U_0 has a $\mathbf{\Sigma}^0_3$ hereditary basis U_0', $M_0^p = (U_0')^{\mathrm{perf}}$ is $\mathbf{\Pi}^0_3$.

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