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STRONGLY SEQUENTIALLY CONTINUOUS FUNCTIONS

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Abstract. *Given a topological abelian group G, we study the class of strongly sequentially continuous functions on G. Strong sequential continuity is a property intermediate between sequential continuity and uniform sequential continuity, which appeared naturally in the study of smooth functions on Banach spaces. In this paper, we shall mainly concentrate on the "gap" between strong sequential continuity and uniform sequential continuity. It turns out that if G has some "completeness" property - for example, if it is completely metrizable - then all strongly sequentially continuous functions on G are uniformly sequentially continuous. On the other hand, we exhibit a large and natural class of groups for which the two notions differ. This class is defined by a property reminiscent of the classical Dirichlet theorem; it includes all dense sugroups of $\mathbb{R}$ generated by an increasing sequence of Dirichlet sets, and groups of the form (X, w), where X is a separable Banach space failing the Schur property. Finally, we show that the family of bounded, real-valued strongly sequentially continuous functions on G is a closed subalgebra of $l_\infty(G)$.*

Introduction

This paper is in some sense a continuation of **[DM]**, which was itself motivated by recent results of P. Hàjek.

In **[H]**, Hàjek proves that if $f : c_0 \to \mathbb{R}$ is a C^1-smooth function whose derivative is uniformly continuous on the unit ball B_{c_0}, then f is uniformly continuous on B_{c_0} for the *weak* topology.

One purpose of **[DM]** was to give a short proof of this rather intriguing result. This proof proceeds as follows. First, it is shown that, when considered as functions on the topological group $G = (c_0, w)$, smooth functions on c_0 satisfy a property formally intermediate between sequential continuity and uniform sequential continuity, which has been called *strong sequential continuity*. Then, it is proved that strongly sequentially continuous functions turn Cauchy sequences into Cauchy sequences; since each bounded sequence in c_0 has a weak-Cauchy subsequence, this implies that every strongly sequentially continuous function on (c_0, w) is actually uniformly continuous on bounded sets, and Hàjek result follows.

Strong sequential continuity was instrumental in [**DM**], but in view of its natural appearance and of its usefulness in seemingly unrelated matters, we believe that this notion deserves a more detailed study.

Throughout this paper, the letter G will always designate a Hausdorff topological abelian group.

Definition 1. Let A be a subset of G. If (Y, δ) is a metric space, we say that a function $f : G \to Y$ is *strongly sequentially continuous on A* - in short, *SSC on A* - if, for every sequence $(x_n) \subseteq A$ and for every sequence $(h_i) \subseteq G$ converging to 0, one has

$$\lim_{i\to\infty} \left[\liminf_{n\to\infty} \delta\big(f(x_n + h_i), f(x_n)\big) \right] = 0\,.$$

If $A = G$, we just say that f is strongly sequentially continuous.

Remark. Clearly, SSC functions are sequentially continuous, and *uniformly* sequentially continuous functions on G are SSC.

As explained above, the following result was the main reason for introducing SSC functions in [**DM**].

Proposition 1. *If $f : G \to (Y, \delta)$ is SSC on some set $A \subseteq G$, then f turns Cauchy sequences in A into Cauchy sequences in Y.*

Corollary. *If every sequence in A admits a Cauchy subsequence, then all SSC functions on A are uniformly sequentially continuous on A.*

In view of Proposition 1, it should be clear that strong sequential continuity is "in general" strictly stronger than sequential continuity.

For example, it is not difficult to show that if G is metrizable, noncompact and nondiscrete, then there exist continuous real-valued functions on G which are not SSC. Indeed, if G is totally bounded, then each sequence in G has a Cauchy subsequence, hence SSC functions on G are in fact uniformly continuous by Proposition 1. If G is not totally bounded, then one can find a sequence $(x_n) \subseteq G$ and a neighbourhood of 0, $V \subseteq G$, such that $(x_n + V) \cap (x_m + V) = \emptyset$ if $n \neq m$. If (h_i) is a null sequence in G such that $h_i \in V$ and $h_i \neq 0$ for all i, then $F = \{x_n + h_i;\ i < n\}$ is a closed, discrete subset of G. By the Tietze extension Theorem, there exists a continuous function $f : G \to \mathbb{R}$ such that $f(x_n) = 0$ and $f(x_n + h_i) = 1$ whenever $0 \leq i < n$; this function f is not SSC.

On the other hand, it was observed in **[DM]** that even on very simple groups, SSC functions need not be uniformly continuous.

Example. *Let $\mathbb{D}$ be the group of dyadic real numbers, and let $f : \mathbb{D} \to \mathbb{R}$ be the even function defined by $f(x) = \sin(\pi x)\sin(2^n \pi x)$ if $x \in \mathbb{D} \cap [n; n+1]$, $n \in \mathbb{N}$. Then f is is not uniformly continuous, but it is SSC.*

Proof.

Clearly, f is not uniformly continuous. To show that f is SSC, fix a null sequence $(h_i) \subseteq \mathbb{D}$ and an arbitrary sequence $(x_n) \subseteq \mathbb{D}$.

For each $h \in \mathbb{D}$, $2^n h$ is an even integer if n is large enough, hence $f(x+h) - f(x) \to 0$ as $|x| \to \infty$. Consequently, $\liminf_n |f(x_n + h_i) - f(x_n)| = 0$ for all $i \in \mathbb{N}$ if the sequence (x_n) is unbounded. If (x_n) is bounded, we still get $\lim_i \big(\liminf_n |f(x_n + h_i) - f(x_n)|\big) = 0$ because f is uniformly continuous on bounded sets.

In this paper, we shall be mainly interested in the "gap" between strong sequential continuity and uniform sequential continuity.

We shall say that G is a *nice group* if, for every metric space Y, every SSC function $f : G \to Y$ is uniformly sequentially continuous. On the the other hand, we say that G is *nasty* for some metric space Y if there exist SSC functions from G into Y which are not uniformly sequentially continuous, and we say that G is a *nasty group* if it is nasty for $Y = \mathbb{R}$.

In section I, we show that "completeness" assumptions ensure niceness of the group. The two main results imply that completely metrizable groups are nice; in particular, $\mathbb{R}$ is a nice group. We also show that nonmeasurable subgroups of $\mathbb{R}$ are nice. All the proofs depend on a general lemma that may be useful in other situations.

In section II, we introduce the class of *Dirichlet-like* groups, which is defined by a property reminiscent of the classical Dirichlet theorem, and we show that Dirichlet-like groups are nasty provided they are not totally bounded. It follows for example that countable dense subgroups of $\mathbb{R}$ are nasty, as well as infinite-dimensional normed spaces with a countable algebraic basis.

In section III, we consider the particular case of a Banach space endowed with its weak topology. We show that if X is a separable Banach space, then $G = (X, w)$ is a nice group if and only if X satisfies the so-called *Schur property*, which means that every weakly convergent sequence in X is actually norm-convergent. We also give an example of a function $f : c_0 \times l_1 \to \mathbb{R}$ which is SSC for the weak topology, but not uniformly sequentially continuous on the unit ball of $c_0 \times l_1$. Such a function can be produced neither in c_0, nor in l_1.

We conclude this paper with two additional results. We show that SSC functions have the same stability properties under algebraic operations as uniformly continuous functions, and we prove that SSC functions defined on a dense subgroup of $\mathbb{R}$ are either uniformly continuous, or highly oscillating.

The following easy lemma will be used repeatedly.

Lemma 0. *A function $f : G \to (Y, \delta)$ is SSC on $A \subseteq G$ if and only if, for every null sequence $(h_i) \subseteq G$,*

$$\inf_{i<n} \delta\big(f(x+h_i), f(x)\big) \to 0 \quad as \quad n \to \infty\,,$$

uniformly for $x \in A$.

In other words, f is not SSC on A if and only if one can find a positive number ε, a sequence $(x_n) \subseteq A$ and a null sequence $(h_i) \subseteq G$ such that

$$\delta\big(f(x_n+h_i), f(x_n)\big) \geq \varepsilon \quad if \quad n > i.$$

I - Nice groups.

Our first result follows at once from Proposition 1.

Proposition 2. *Metrizable, totally bounded groups are nice.*

Proof.

In a metrizable, totally bounded group, every sequence has a Cauchy subsequence.

We now turn to more interesting examples. As already said, the main idea is that "completeness" properties ensure niceness of the group.

Theorem 1. *Completely metrizable groups are nice.*

Corollary. *Every Banach space is a nice group. In particular, $\mathbb{R}$ is a nice group.*

We shall in fact prove two extensions of Theorem 1. Both results, as well as Proposition 3 below, depend on the following lemma.

Recall that a subset A of some topogical space X is said to have the *Baire property* in X if one can write $A = U \triangle M$, where U is open and M is meager in X; and that a series $\sum k_n$ in G is said to be *unconditionally convergent* if $\sum \alpha_n k_n$ converges in G for any choice of the sequence $\alpha = (\alpha_n) \in \{0,1\}^{\mathbb{N}}$.

Lemma 1. *Let (A_n), (B_n) be two sequences of subsets of G, and let (k_n) be a null sequence in G. Assume that $A_n \cup (k_n + B_n) = G$ for each n, and put*

$$Z = \overline{\lim}_n (A_n \cup B_n) = \bigcap_{n\geq 0} \bigcup_{p \geq n} \big(A_p \cup B_p\big)\,.$$

(a) If G is a Baire space and if all the sets A_n, B_n have the Baire property in G, then Z is comeager in G.

(b) If the series $\sum k_n$ is unconditionally convergent and if all the sets A_n, B_n are sequentially open - which means that their complements are sequentially closed - then Z contains a point of the form $\sum_0^\infty \alpha_n k_n$, where $\alpha_n \in \{0;1\}$.

(c) If G is locally compact and if all the sets A_n, B_n are Haar-measurable, then $G \setminus Z$ is inner-negligible.

Proof.

(a) If all the sets A_n, B_n are open - which will precisely happen in the proof of Theorem $1'$ below - the proof is immediate: the sets $Z_n = \bigcup_{p\geq n}(A_p \cup B_p)$ are open in G, and they are also dense because $k_n \to 0$; hence $Z = \cap_n Z_n$ is comeager.

In the general case, we only need to prove that all the Z_n's are comeager in G, and since the Z_n's have the Baire property, it is enough to check that if W is a nonempty open subset of G, then each $Z_n \cap W$ is nonmeager in G. Let us fix n and W.

Pick $a \in W$ together with a neighbourhood of 0, $U \subseteq G$, such that $U + a + U \subseteq W$, and choose $p \geq n$ such that $k_p \in U$. If $A_p \cap W$ and $B_p \cap W$ were both meager, then so would be $A_p \cap (k_p + a + U)$ (because $k_p + a + U \subseteq W$) and $(k_p + B_p) \cap (k_p + a + U)$ (by translation); this is impossible because $k_p + a + U \subseteq V \subseteq A_p \cup (k_p + B_p)$ and, since G is a Baire space, $k_p + a + U$ is nonmeager. It follows that $(A_p \cup B_p) \cap W$ is nonmeager, and the proof is complete.

(b) By the unconditional convergence of $\sum k_n$, the series $\Sigma(\alpha) = \sum \alpha_n k_n$ is uniformly convergent for $\alpha \in \{0,1\}^{\mathbb{N}}$. Identifying $\{0;1\}^{\mathbb{N}}$ with the compact abelian group $\Delta = (\mathbb{Z}/2\mathbb{Z})^{\mathbb{N}}$, it follows that the map $\varphi : \Delta \to G$ defined by $\varphi(\alpha) = \sum_{n\geq 0} \alpha_n k_n$ is continuous, hence all the sets $A'_n = \varphi^{-1}(A_n)$, $B'_n = \varphi^{-1}(B_n)$ are open in Δ.

For each $n \in \mathbb{N}$, put $W_0^{(n)} = \{\alpha \in \Delta;\ \alpha_n = 0\}$ and $W_1^{(n)} = \{\alpha \in \Delta;\ \alpha_n = 1\}$.

Observe that $\varphi^{-1}(B_n + k_n) \cap W_1^{(n)} \subseteq B'_n + \delta^{(n)}$ for all n, where $\delta^{(n)} \in \Delta$ is defined by $\delta_n^{(n)} = 1$ and $\delta_i^{(n)} = 0$ if $i \neq n$. Since $\Delta = A'_n \cup \varphi^{-1}(B_n + k_n)$, it follows that $W_1^{(n)} \subseteq A'_n \cup (B'_n + \delta^{(n)})$ for each $n \in \mathbb{N}$.

Put $Z'_n = \bigcup_{p\geq n}(A'_n \cup B'_n)$, $n \in \mathbb{N}$. The sets Z'_n are open in Δ; moreover, since $\delta^{(n)} \to 0$ as $n \to \infty$, the closure of each Z'_n contains $M = \{\alpha \in \Delta;\ \alpha_n = 1 \text{ for infinitely many } n\}$, a dense subset of Δ. By the Baire category theorem, $Z' = \cap_n Z'_n$ is nonempty (actually comeager), and the proof is complete.

(c) Let V be any open subset of G. If L is a compact subset of V, then $m(L) \leq m(A_p \cap L) + m((k_p + B_p) \cap L) = m(A_p \cap L) + m(B_p \cap (-k_p + L))$ for all p, where m is the Haar measure on G. If p is large enough, then $-k_p + L \subseteq V$, and we get $m(L) \leq m(A_p \cap V) + m(B_p \cap V)$. Since L is an arbitrary compact subset of V, it follows that $m\big(V \cap \bigcup_{p\geq n}(A_p \cup B_p)\big) \geq m(V)/2$ for all n. Consequently, $m(Z \cap V) \geq m(V)/2$ for each open set $V \subseteq G$.

Now, assume that $m(K) > 0$ for some compact set $K \subseteq G \setminus Z$, and choose an open set V such that $K \subseteq V$ and $m(V \setminus K) \leq m(K)/3$. Then $m(V \cap Z) \leq m(V \setminus K) \leq m(V)/3$, which contradicts the preceding remark.

Theorem $1'$. *If G is metrizable and is a Baire space, then G is a nice group.*

Proof.

Assume that G is metrizable and is a Baire space.

Since SSC functions on G are continuous by metrizability of G, we have to show that if $f : G \to (Y, \delta)$ is a continuous function which is not uniformly continuous, then f is not SSC; let us fix such an f.

By assumption there exist $\varepsilon > 0$ and two sequences $(u_n), (v_n) \subseteq G$ such that $(u_n - v_n) \to 0$ and

$$\delta\big(f(u_n), f(v_n)\big) > 2\varepsilon, \qquad n \geq 0.$$

We want to construct two sequences (x_n) and (h_i) such that $h_i \to 0$ as $i \to \infty$ and

$$\delta\big(f(x_n + h_i), f(x_n)\big) > \varepsilon \quad \text{whenever} \quad i < n\,.$$

For $n \geq 0$, consider the open sets:

$$A_n = \big\{h \in G : \ \delta\big(f(u_n + h), f(u_n)\big) > \varepsilon\big\}\,,$$

$$B_n = \big\{h \in G : \ \delta\big(f(v_n + h), f(v_n)\big) > \varepsilon\big\}\,.$$

For any $h \in G$ and each $n \geq 0$, it follows from the triangle inequality that either $\delta\big(f(u_n), f(u_n + h)\big) > \varepsilon$ or $\delta\big(f[v_n + (-v_n + u_n + h)], f(v_n)\big) > \varepsilon$. Thus, if we put $k_n = -u_n + v_n$, then

$$A_n \cup \big(k_n + B_n\big) \ = G \ \text{ for all } \ n \geq 0.$$

According to Lemma 1 (part (a)), for every neighbourhood V of 0 in G and for any infinite set $\Lambda \subseteq \mathbb{N}$, there exists an infinite set $\Lambda' \subseteq \Lambda$ such that either $V \cap \bigcap_{n\in\Lambda'} A_n \neq \emptyset$ or $V \cap \bigcap_{n\in\Lambda'} B_n \neq \emptyset$. Since G is metrizable, we can construct by induction a null sequence $(h_i) \subseteq G$ and a decreasing sequence (Λ_i) of infinite subsets of $\mathbb{N}$ such that, for each $i \geq 0$, either $h_i \in \bigcap_{n\in\Lambda_i} A_n$ or $h_i \in \bigcap_{n\in\Lambda_i} B_n$. Extracting if necessary a subsequence of (Λ_i), we may assume that either for all i and for all $n \in \Lambda_i$,

$$\delta\big(f(u_n + h_i), f(u_n)\big) > \varepsilon\,,$$

or for all i and for all $n \in \Lambda_i$,

$$\delta\big(f(v_n + h_i), f(v_n)\big) > \varepsilon\,.$$

Put $x_n = u_{\min \Lambda_n}$ in the first case, and $x_n = v_{\min \Lambda_n}$ in the second case. The sequences (x_n), (h_i) have the required property.

Remark. The same proof shows that if $f : G \to Y$ is continuous on G and SSC on some set $A \subseteq G$, then f is uniformly continuous on A.

Definition 2. We shall say that G has the *unconditional property (K)* if each null sequence $(k_n) \subseteq G$ has a subsequence (k'_n) such that $\sum k'_n$ is unconditionally convergent.

As may be guessed, this terminology refers to another one: by dropping the unconditionality condition, one gets the so-called *property (K)*, which seems to have been introduced by C. Kliś ([**Kli**]). In some cases, this property can be used as a substitute for completeness (see [**S**] for precise results in that direction).

Examples.

(1) Any completely metrizable group has the unconditional property *(K)*. Indeed, if G is completely metrizable, then it is complete for any translation-invariant metric d compatible with its topology ([**Kle**]), hence any series $\sum k'_n$ such that $d(0, k'_n) < 2^{-n}$ for all n is unconditionally convergent.

(2) It follows that G has the unconditional property *(K)* whenever the closed subgroup generated by every null sequence is completely metrizable. This happens for example for $G = \mathcal{D}(\Omega)$, the space of test functions on an open set $\Omega \subseteq \mathbb{R}^n$.

(3) If X is a Banach space, then $G = (X, w)$ has the unconditional property *(K)* if and only if X has the Schur property. One implication is trivial, and the other one follows from the classical Orlicz-Pettis Theorem on the equivalence of weak subseries convergence and norm subseries convergence in Banach spaces (see [**Di**]).

Theorem 1″. *If G has the unconditional property (K), then G is a nice group.*

Proof.

Assume that G has the unconditional property *(K)*. Keeping the same notation as in the proof of Theorem 1′, let $f : G \to (Y, \delta)$ be a function which is not uniformly sequentially continuous, with "witnesses" $(u_n), (v_n)$ and 2ε. We have to show that f is not SSC, and we may assume that it is sequentially continuous (otherwise, there is nothing to prove).

As above, put $A_n = \big\{z \in G : \delta\big(f(u_n + z), f(u_n)\big) > \varepsilon\big\}$, $B_n = \big\{z \in G : \delta\big(f(v_n + z), f(v_n)\big) > \varepsilon\big\}$, and $k_n = -u_n + v_n$. Then $k_n \to 0$ and $G = A_n \cup (B_n - k_n)$ for all n.

By the unconditional property *(K)*, we may assume that the series $\sum k_n$ is unconditionally convergent. Put $\Delta = \{0; 1\}^{\mathbb{N}}$, and, for $i \geq 0$, define $\varphi_i : \Delta \to G$ by $\varphi_i(\alpha) = \sum_{n \geq i} \alpha_n k_n$. By the unconditional convergence of $\sum k_n$, the sequence (φ_i) converges to 0 uniformly on Δ. Keeping in mind the proof of Theorem 1′, it is therefore enough to show that for each $i \in \mathbb{N}$ and each infinite set $\Lambda \subseteq \mathbb{N}$, there exists an infinite subset $\Lambda' \subseteq \Lambda$ such that either $\bigcap_{n \in \Lambda'} \varphi_i^{-1}(A_n) \neq \emptyset$ or $\bigcap_{n \in \Lambda'} \varphi_i^{-1}(B_n) \neq \emptyset$. This follows from Lemma 1 (part (b)).

Remark. It is proved in [**BKL**] that a *metrizable* group with property *(K)* is necessarily a Baire space; and if, in addition, the group has the Baire property in its completion, then it is actually complete. Thus, Theorem 1″ is of interest in the nonmetrizable case only.

To conclude this section, we show that nice groups can be quite ugly.

Proposition 3. *Nonmeasurable subgroups of $\mathbb{R}$ are nice; subgroups of $\mathbb{R}$ without the Baire property are nice.*

Proof.

If G is a subgroup of $\mathbb{R}$ without the Baire property, then G is nonmeager, hence nonmeager in any nonempty open set by translation. It follows easily that G is a Baire space, so we may apply Theorem 1′.

Now, assume that G is a nonmeasurable subgroup of $\mathbb{R}$. By Proposition 1, SSC functions on G extend continuously to $\mathbb{R}$. Therefore, we have to show that if $f : \mathbb{R} \to \mathbb{R}$ is continuous but not uniformly continuous, then $f_{|G}$ is not SSC on G.

By the proof of Theorem $1'$, it is enough to check that if (k_n) is a null sequence of real numbers and if (A_n), (B_n) are two sequences of open subsets of $\mathbb{R}$ such that $\mathbb{R} = A_n \cup (k_n + B_n)$ for all n, then, for each nonempty open set $V \subseteq \mathbb{R}$, there exists an infinite set $\Lambda \subseteq \mathbb{N}$ such that either $V \cap G \cap \bigcap_{n\in\Lambda} A_n \neq \emptyset$ or $V \cap G \cap \bigcap_{n\in\Lambda} B_n \neq \emptyset$. Since G is not Lebesgue null in any nonempty open set, this follows from Lemma 1 (part (c)).

Remark. We are unable to decide whether every compact group is nice.

II - Nasty groups

In this section, we exhibit a large and rather natural class of nasty groups.

We recall that a *character* of a topological abelian group H is a continuous homomorphism from H into the circle group $\mathbb{T}$.

If H is a locally compact abelian group, a compact set $L \subseteq H$ is said to be a *Dirichlet set* if there exists a sequence of characters (γ_k) tending to ∞ such that $\gamma_k(x) \to 1$ uniformly on L. The classical Dirichlet theorem asserts that finite subsets of $\mathbb{T}$ are Dirichlet sets, and the same is true in any second-countable nondiscrete LCA group. It is also well known that there exist uncountable Dirichlet sets; actually, in the sense of Baire category, "most" compact subsets of any second-countable nondiscrete LCA group are Dirichlet sets (see **[KeLo]**).

Notice that if (γ_k) is a sequence of characters of H such that, for some null sequence $(x_k) \subseteq H$, $\gamma_k(x_k)$ does not tend to 1 as $k \to \infty$, then γ_k has a subsequence converging to ∞; moreover, the converse is true if H is metrizable. Looking back at the definition of a Dirichlet set, this legitimates the following terminology.

Definition 3. We say that G is *Dirichlet-like* if there exists a sequence of characters (γ_k) and a null sequence $(x_k) \subseteq G$ such that $\gamma_k(x) \to 1$ pointwise on G, but $\gamma_k(x_k)$ does not tend to 1 as $k \to \infty$.

Lemma 2. *The following groups are Dirichlet-like.*

(1) *Countable dense subgroups of $\mathbb{R}$; more generally, dense subgroups of $\mathbb{R}$ generated by an increasing sequence of Dirichlet sets.*

(2) *Infinite dimensional normed spaces with a countable algebraic basis.*

(3) *Groups of the form (X, w) and (X^*, w^*), where X is a separable Banach space failing the Schur property.*

Proof.

(1) Let G be a dense subgroups of $\mathbb{R}$ generated by an increasing sequence of Dirichlet sets. By a standard diagonal argument, one can find a sequence $(\lambda_k) \subseteq \mathbb{R}$ tending to $+\infty$ such that $e^{i\lambda_k x} \to 1$ pointwise on G; and if G is dense in $\mathbb{R}$, then one can find a null sequence $(x_k) \subseteq G$ such that $e^{i\lambda_k x_k} \to -1$ as $k \to \infty$.

(2) Let $(X, \|\ \ \|)$ be a infinite-dimensional normed space with a coutable algebraic basis (e_k). By the Hahn-Banach theorem, we can choose a sequence $(x_k^*) \subseteq X^*$ such that

$||x_k^*|| \to \infty$ and $\langle x_k^*, e_j\rangle = 0$ whenever $j < k$. Since (e_k) is an algebraic basis for X, the sequence (x_k^*) is w^*-null; and since $||x_k^*|| \to \infty$, one can find a null sequence $(x_k) \subseteq X$ such that $\langle x_k^*, x_k\rangle = \pi$ for all k. The sequence $(\gamma_k) = (e^{ix_k^*})$ shows that X is Dirichlet-like.

(3) Let X be a separable Banach space failing the Schur property, and fix a weakly null sequence $(u_k) \subseteq X$ such that $||u_k|| > 3$ for all k, together with a norm-dense sequence $(d_j) \subseteq X$.

Since (u_k) is weakly null, one can construct by induction a subsequence (x_k) of (u_k) such that $\text{dist}\,\big(x_k, \text{span}\,\{d_j : j < k\}\big) > 1$ for all k; and by the Hahn-Banach theorem, one can find a bounded sequence $(x_k^*) \subseteq X^*$ such that $\langle x_k^*, x_k\rangle = \pi$ for all k, and $\langle x_k^*, d_j\rangle = 0$ whenever $j < k$. Since (d_j) is norm-dense in X, the sequence (x_k^*) is w^*-null, so the sequences $(e^{ix_k^*})$ and (e^{ix_k}) show that (X, w) and (X^*, w^*) are Dirichlet-like.

Theorem 2. *If G is Dirichlet-like and is not totally bounded, then G is a nasty group.*

Corollary 1. *Dense subgroups of $\mathbb{R}$ generated by an increasing sequence of Dirichlet sets are nasty; in particular, countable dense subgroups of $\mathbb{R}$ are nasty.*

Corollary 2. *Infinite dimensional normed space with a countable algebraic basis are nasty.*

The proof of Theorem 2 requires some additional definitions and lemmas.

Definition 4. Let (Y, δ) be a metric space, and let (f_k) be a sequence of functions from G into Y. We say that the sequence (f_k) is *equi-SSC* if, for any null sequence $(h_i) \subseteq G$,

$$\sup_k \Big[\inf_{i<n} \delta\big(f_k(x + h_i), f_k(x)\big)\Big] \to 0 \text{ as } n \to \infty,$$

uniformly on G.

The following remark will be essential for our purposes.

Basic example. *If (g_k) is a sequence of SSC functions from G into $\mathbb{C}$ such that, for each $h \in G$, we have $\lim_{k\to\infty}\big(g_k(x + h) - g_k(x)\big) = 0$ uniformly on G, then the sequence (g_k) is equi-SSC. This happens, for example, if $g_k(x) = \gamma_k(x) - 1$, where (γ_k) is a sequence of characters of G converging pointwise to* 1.

Proof.

Let (g_k) be as above, and fix a null sequence $(h_i) \subseteq G$ and a positive number ε.

Choose a positive integer K_0 such that $\sup_{x\in G} |g_k(x + h_0) - g_k(x)| < \varepsilon$ if $k > K_0$. Since $g_0, \ldots, g_{K_0}$ are SSC, there exists an integer N such that for each $k \le K_0$ and all $x \in G$, we have $\inf_{i<N} |g_k(x + h_i) - g_k(x)| < \varepsilon$. Then $\inf_{i<N} |g_k(x + h_i) - g_k(x)| < \varepsilon$ for each $x \in G$ and for all k.

The next lemma shows how to produce new equi-SSC sequences from old ones.

Lemma 3. *Let $(Y, |\ \ |)$ be a normed space, let $(g_k) \subseteq Y^G$ be a uniformly bounded, equi-SSC sequence of functions, and let $b : G \to \mathbb{R}$ be bounded and uniformly sequentially* *continuous. Let also $(u_k), (v_k) \subseteq G$, and define $f_k : G \to Y$ by $f_k(x) = b(u_k+x)g_k(v_k+x)$. Then (f_k) is equi-SSC.*

Proof.

One can write $|f_k(x+h)-f_k(x)| \leq C\big(|b(u_k+x+h)-b(u_k+x)|+|g_k(v_k+x+h)-g_k(v_k+x)|\big)$, for some constant C.

Definition 5. Let (F_k) be a sequence of subsets of G. We say that (F_k) is *separated* if there exists a neighbourhood of 0, $V \subseteq G$, such that $(F_k + V) \cap F_l = \emptyset$ if $k \neq l$. More generally, we say that (F_k) is *quasi-separated* if there exists a set $V \subseteq G$ satisfying the following properties:

(i) V "absorbs" every null sequence, which means that for every null sequence (h_i), there exists an integer i_0 such that $h_i \in V$ for all $i \geq i_0$;

(ii) $(F_k + V) \cap F_l = \emptyset$ if $k \neq l$.

Clearly, the two notions coincide if G is metrizable. On the other hand, if $G = (l_1, w)$ (the space l_1 with the weak topology) and if (e_k) is the canonical basis of l_1, then the sequence $\big(B(e_k, 1/3)\big)$ is quasi-separated because it is separated for the norm topology and l_1 has the Schur property, but it is not separated.

Of course, if (F_k) is quasi-separated, then the F_k's are pairwise disjoint.

The following lemma is our main tool for constructing SSC functions.

Lemma 4. *Let $(Y, |\ |)$ be a normed space, and let (f_k) be an equi-SSC sequence of functions from G into Y. Assume that the f_k's have quasi-separated supports, and put $f = \sum_{k\geq 0} f_k$. Then f is SSC.*

Proof.

Put $F_k = \operatorname{supp} f_k$, $k \geq 0$, and choose V according to Definition 5. Clearly, we may assume that V is symmetric. For any $x \in G$, there is at most one integer k such that $x \in F_k + V$. Let us denote this integer by $k(x)$ if it exists, and put $k(x) = 0$ if there is no such k.

For each null sequence $(h_i) \subseteq G$, there is an integer i_0 such that $h_i \in V$ if $i \geq i_0$. Then, for any $x \in G$, $f_k(x) = 0 = f_k(x + h_i)$ if $i \geq i_0$ and $k \neq k(x)$. Thus, for $i \geq i_0$, one can write $|f(x + h_i) - f(x)| = |f_{k(x)}(x + h_i) - f_{k(x)}(x)|$ for all $x \in G$. The lemma follows immediately.

Proposition 4. *Assume that G is not totally bounded. For any normed space $(Y, |\ |)$, the following assertions are equivalent.*

(1) G is nasty for Y.

(2) There exist an equi-SSC sequence $(g_k) \subseteq Y^G$ and a null sequence $(h_k) \subseteq G$ such that $g_k(0) = 0$ for all k but $g_k(h_k)$ does not tend to 0 as $k \to \infty$.

Proof.

Assume that (2) holds for some sequences (g_k), (h_k). Replacing g_k by $g_k/(1 + |g_k|)$, we may assume that the sequence (g_k) is uniformly bounded.

Since G is not totally bounded, there exist a sequence $(x_k) \subseteq G$ and a neighbourhood of 0, $U \subseteq G$, such that $-x_l + x_k \notin U$ if $k \neq l$. Choose a symmetric neighbourhood of 0, $V \subseteq G$, such that $V + V + V \subseteq U$, and let d be a uniformly continuous pseudometric on G such that $\{x \in G;\ d(0, x) \leq 1\} \subseteq V$. Finally, let $b_0 : \mathbb{R} \to \mathbb{R}$ be a uniformly continuous function such that $b_0(0) = 1$ and $\operatorname{supp} b_0 \subseteq [0, 1]$. Define $b : G \to \mathbb{R}$ by $b(x) = b_0\big(d(0, x)\big)$. The function b is bounded, uniformly continuous, and $b(0) = 1$. Moreover, $\operatorname{supp} b \subseteq V$, so that $(x_k + \operatorname{supp} b + V) \cap (x_l + \operatorname{supp} b) = \emptyset$ if $k \neq l$.

For each $k \geq 0$, define $f_k : G \to Y$ by

$$f_k(x) = b(-x_k + x)\, g_k(-x_k + x)\,.$$

By Lemma 3, the sequence (f_k) is equi-SSC, and since $\operatorname{supp} f_k \subseteq x_k + \operatorname{supp} b$ for all k, the f_k's have separated supports. Hence, by Lemma 4, the function $f = \sum f_k$ is SSC. On the other hand, this function is not uniformly sequentially continuous because $f(x_k + h_k) - f(x_k) = b(h_k) g_k(h_k)$ if k is large enough, whence $f(x_k + h_k) - f(x_k)$ does not tend to 0. Thus, G is nasty for Y.

Conversely, assume that G is nasty for Y, and let $f : G \to Y$ be an SSC function which is not uniformly sequentially continuous. Choose $(x_k) \subseteq G$ and a null sequence (h_k) such that $f(x_k + h_k) - f(x_k)$ does not tend to 0 as $k \to \infty$. The sequence $(g_k) \subseteq Y^G$ defined by $g_k(x) = f(x_k + x) - f(x_k)$ is equi-SSC, and $(g_k), (h_k)$ satisfy (2).

Corollary. *If G is metrizable, or if G is not totally bounded, then G is a nasty group if and only if it is nasty for some metric space Y.*

Proof.

We know that metrizable, totally bounded groups are nice, hence we may assume that G is not totally bounded.

If G is nasty for some metric space, it is also nasty for some normed linear space, because every metric space is uniformly homeomorphic to a subset of a normed space and uniformly equivalent metrics give rise to the same SSC functions. For such a normed space $(Y, |\ \ |)$, property (2) is satisfied. Now, the sequence $(|g_k|)$ shows that property (2) is also true for $\mathbb{R}$; hence G is a nasty group.

Proof of Theorem 2.

Keeping in mind the basic example, this follows immediately from Proposition 4.

Remark. Let us denote by $\mathcal{D}$, $\mathbf{N}$ and $\mathcal{I}$ respectively the families of Dirichlet-like, nasty and (meager+Lebesgue-null) dense subgroups of $\mathbb{R}$. Then $\mathcal{D} \subseteq \mathbf{N} \subseteq \mathcal{I}$ by Theorem 2 and Proposition 3. Moreover, it is clear that a dense subgroup of a nasty group is itself nasty. Thus, nastyness is a "smallness" property which lies between $\mathcal{D}$ and $\mathcal{I}$. We don't know if both inclusions are proper.

III - Banach spaces in their weak topology

Putting together the results obtained so far, we get an amusing characterization of the Schur property.

Proposition 5. *The following properties of a separable Banach space X are equivalent:*

(1) *X fails the Schur property;*

(2) *$G = (X, w)$ is Dirichlet-like;*

(3) *$G = (X, w)$ is a nasty group.*

Proof.

(1) implies (2) by Lemma 2, (2) implies (3) by Theorem 2, and (3) implies (1) by Theorem 1 because strong (resp. uniform) sequential continuity for the weak and the norm topology of X are equivalent if X has the Schur property (alternatively, one can apply Theorem $1''$).

Remark. In view of the Josefson-Nissenzweig theorem (see [**Di**]), Proposition 5 may be true even if X is nonseparable. In any case, the implication (*Schur property*) $\Rightarrow$ (*(X, w) is nice*) is always true.

A typical example of a Banach space with the Schur property is l_1 - this is Schur's Theorem, see [**Di**]. Moreover, it is well known that any infinite dimensional Banach space with the Schur property contains an isomorphic copy of l_1. Thus, at first sight, the following result may look a bit surprising.

Proposition 6. *If X is a Banach space which does not contain l_1, then every SSC function on $G = (X, w)$ is uniformly sequentially continuous on bounded sets.*

Proof.

By Rosenthal's l_1 Theorem (see [**Di**]), every bounded sequence in X admits a weak-Cauchy subsequence, hence the result follows from Proposition 1.

In view of the two above results, it is natural to wonder if there exist a Banach space X and an SSC function on (X, w) which is not uniformly sequentially continuous on bounded sets. Such an example can indeed be constructed on $X = c_0 \times l_1$, endowed with the norm $||(x, y)|| = \text{Max}\,(||x||_{c_0}, ||y||_{l_1})$.

Proposition 7. *There exists a function $f : c_0 \times l_1 \to \mathbb{R}$ which is SSC on $(c_0 \times l_1, w)$, but not uniformly sequentially continuous on the unit ball.*

Proof.

Let $b : l_1 \to \mathbb{R}$ be defined by $b(y) = 1 - 3\min(1/3, ||y||_1)$. By Schur's Theorem, b is uniformly sequentially continuous on (l_1, w). Moreover, $\operatorname{supp} b$ is the ball $B(0, 1/3)$. Hence, by Schur's Theorem again, if we denote by (e_k) the canonical basis of l_1, then the sequence $(e_k + \operatorname{supp} b) = \big(B(e_k, 1/3)\big)$ is quasi-separated. Now, define $f : c_0 \times l_1 \to \mathbb{R}$ by

$$f(x, y) = \sum_{k \geq 0} \tilde{x}_k\, b(y - e_k),$$

where $\tilde{x}_k = \min(1, |x_k|)$. The function f has the required properties: it is SSC on $c_0 \times l_1$ by Lemmas 3, 4, and it is not uniformly sequentially continuous on the unit ball because $f(h_k, e_k) - f(0, e_k) \equiv 1$, where (h_k) is the canonical basis of c_0.

IV - Sums and products; oscillating functions

In this last section, we consider real-valued functions.

Proposition 8. *The sum of two SSC functions is SSC; the product of two bounded SSC functions is SSC.*

Since addition is uniformly continuous and multiplication is uniformly continuous on bounded sets, it is enough to prove

Lemma 5. *If $f_0 : G \to (Y_0, \delta_0)$ and $f_1 : G \to (Y_1, \delta_1)$ are both SSC, then the pair $(f_0, f_1) : G \to Y_0 \times Y_1$ is also SSC.*

Proof.

Of course, $Y_0 \times Y_1$ is endowed with one of the usual product metrics.

Fix $f_j : G \to Y_j$ $(j = 0, 1)$ and assume that (f_0, f_1) is not SSC. This means that there exist a positive number ε and two sequences (h_i), $(x_n) \subseteq G$ such that $h_i \to 0$ and $\operatorname{Max}\left\{\delta_0\big(f_0(x_n + h_i), f_0(x_n)\big), \delta_1\big(f_1(x_n + h_i), f_1(x_n)\big)\right\} \geq \varepsilon$ for $n > i$.

For each infinite set $I \subseteq \mathbb{N}$, let us denote by $I^{(2)}$ the set of all ordered pairs of elements of I, that is, $I^{(2)} = \{(i, n) \in I^2;\ i < n\}$. Now, put

$$A_0 = \{(i, n) \in \mathbb{N}^{(2)};\ \delta_0\big(f_0(x_n + h_i), f_0(x_n)\big) \geq \varepsilon\},$$
$$A_1 = \{(i, n) \in \mathbb{N}^{(2)};\ \delta_1\big(f_1(x_n + h_i), f_1(x_n)\big) \geq \varepsilon\}.$$

By assumption, one has $A_0 \cup A_1 = \mathbb{N}^{(2)}$. Hence, by Ramsey's Theorem for pairs of integers, there is an infinite set $I \subseteq \mathbb{N}$ such that either $I^{(2)} \subseteq A_0$ or $I^{(2)} \subseteq A_1$. Thus, we may assume for instance that $\delta_0\big(f_0(x_n + h_i), f_0(x_n)\big) \geq \varepsilon$ whenever $n > i$. This means that f_0 is not SSC.

Remark. The proof of Proposition 1 given in **[DM]** also used Ramsey's Theorem - for triples of integers.

Before stating our last result, let us look back at the function $f : \mathbb{D} \to \mathbb{R}$ defined in the introduction. Clearly, this function is "oscillating at infinity", in the following sense.

Definition 6. If G is a dense subgroup of $\mathbb{R}$, we say that a function $f : G \to \mathbb{R}$ is *oscillating at infinity* if there exists $a > 0$ such that, for all positive numbers K, ε and for every integer N, there exist $x_1, \ldots, x_{2N} \in G$ such that $K < x_1 < x_2 < \ldots < x_{2N}$ and

(i) $x_{i+1} - x_i < \varepsilon$ for all $i < 2N$;

(ii) $f(x_{2k}) - f(x_{2l-1}) \geq a$ for all $k, l \leq N$.

Our last result asserts that this property is shared by all SSC functions which are not uniformly continuous.

Proposition 9. *If G is a dense subgroup of $\mathbb{R}$, then SSC functions on G are either uniformly continuous, or oscillating at infinity.*

Proof.

Let $f : G \to \mathbb{R}$ be SSC, and assume that f is not uniformly continuous. Fix $K, \varepsilon > 0$ and a positive integer N.

By assumption, there exist $a > 0$ and two sequences $(u_p), (v_p) \subseteq G$ such that $u_p < v_p$, $v_p - u_p \to 0$ and $|f(v_p) - f(u_p)| > 3a$. Since f is uniformly continuous on bounded sets, we may assume that $u_p > K$ for all p; and by extracting subsequences, we may also assume for instance that $f(v_p) - f(u_p) > 3a$, $p \geq 0$.

Let $(h_i) \subseteq G$ be a null sequence such that $0 < h_i < \varepsilon$ for all i.

Since f is strongly sequentially continuous, we can find an integer m such that, for all p, $\inf_{i<m} \left|f(u_p + h_i) - f(u_p)\right| < a/N$. Therefore, there exists i_0 such that, for infinitely many p's, $\left|f(u_p + h_{i_0}) - f(u_p)\right| < a/N$. In particular, there exists p_0 such that $u_{p_0} < v_{p_0} < u_{p_0} + h_{i_0}$ and $\left|f(u_{p_0} + h_{i_0}) - f(u_{p_0})\right| < a/N$. Repeating this argument, we can produce two increasing sequences of integers (p_n) and (i_n) such that $u_{p_n} < v_{p_n} < u_{p_n} + h_{i_n}$ and $\left|f(u_{p_n} + h_{i_n}) - f(u_{p_n})\right| < a/N$ for all n. Thus, we may assume that there exists a null sequence $(k_n) \subseteq G$ such that $u_n < v_n < u_n + k_n$ and $\left|f(u_n + k_n) - f(u_n)\right| < a/N$, $n \geq 0$. Similarly, we may assume that there exists another null sequence $(\ell_n) \subseteq G$ such that $v_n < u_n + k_n < v_n + \ell_n$ and $\left|f(v_n + \ell_n) - f(v_n)\right| < a/N$ for all n. At this point, each quadruple $(u_n, v_n, u_n + k_n, v_n + l_n)$ satisfies conditions (i),(ii), with $a(3 - 2/N)$ instead of a. Proposition 9 follows by iterating this procedure N times.

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