DOES A TYPICAL ℓ_p-SPACE CONTRACTION HAVE A NON-TRIVIAL INVARIANT SUBSPACE?

by

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Abstract. — Given a Polish topology τ on $\mathcal{B}_1(X)$, the set of all contraction operators on $X = \ell_p$, $1 \leqslant p < \infty$ or $X = c_0$, we prove several results related to the following question: does a typical $T \in \mathcal{B}_1(X)$ in the Baire Category sense has a non-trivial invariant subspace? In other words, is there a dense G_δ set $\mathcal{G} \subseteq (\mathcal{B}_1(X), \tau)$ such that every $T \in \mathcal{G}$ has a non-trivial invariant subspace? We mostly focus on the Strong Operator Topology and the Strong* Operator Topology.

1. Introduction

Unless otherwise specified, all the Banach spaces – and hence all the Hilbert spaces – considered in this paper are complex, infinite-dimensional, and *separable*. If X is a Banach space, we denote by $\mathcal{B}(X)$ the space of all bounded linear operators on X endowed with its usual norm. For any $M > 0$,

$$\mathcal{B}_M(X) := \{T \in \mathcal{B}(X) \ ; \ \|T\| \leqslant M\}$$

is the closed ball of radius M in $\mathcal{B}(X)$. In particular, $\mathcal{B}_1(X)$ is the set of all *contractions* of the Banach space X.

In this paper, we will be interested in *typical* properties of Banach space contractions. The word "typical" is to be understood in the Baire category sense: given a Baire topological space $\mathcal{Z}$ and any property (P) of elements of $\mathcal{Z}$, we say that *a typical* $z \in \mathcal{Z}$ *satisfies* (P) if the set $\{z \in \mathcal{Z} \ ; \ z \text{ satisfies (P)}\}$ is comeager in $\mathcal{Z}$, *i.e.* this set contains a dense G_δ subset of $\mathcal{Z}$. So, given a Banach space X, we need topologies on $\mathcal{B}_1(X)$ which turn $\mathcal{B}_1(X)$ into a Baire space. The operator norm topology does so, but it appears to be too strong to get interesting results by Baire category arguments; in particular, the fact that $(\mathcal{B}_1(X), \|\cdot\|)$ is "usually" non-separable is a real disadvantage. However, there are quite a few weaker topologies on $\mathcal{B}(X)$ whose restriction to each ball $\mathcal{B}_M(X)$ is Polish, *i.e.* completely metrizable and separable. We will mostly focus on two of them: the *Strong*

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Operator Topology ($\mathtt{SOT}$) and the *Strong* Operator Topology* ($\mathtt{SOT}^*$). Recall that $\mathtt{SOT}$ is just the topology of pointwise convergence, and that $\mathtt{SOT}^*$ is the topology of pointwise convergence for operators and their adjoints: a net (T_i) in $\mathcal{B}(X)$ converges to $T \in \mathcal{B}(X)$ with respect to $\mathtt{SOT}$ if and only if $T_i x \to Tx$ in norm for every $x \in X$, and $T_i \to T$ with respect to $\mathtt{SOT}^*$ if and only if $T_i x \to Tx$ for every $x \in X$ and $T_i^* x^* \to T^* x^*$ for every $x^* \in X^*$. The *Weak Operator Topology* ($\mathtt{WOT}$) will also make an occasional appearance ($T_i \to T$ with respect to $\mathtt{WOT}$ if and only if $T_i x \to Tx$ weakly for every $x \in X$). Even though these topologies behave badly when considered on the whole space $\mathcal{B}(X)$, each closed ball $\mathcal{B}_M(X)$ is indeed Polish when endowed with $\mathtt{SOT}$, and the same holds true for $\mathtt{SOT}^*$ if one assumes additionally that X^* is separable. (For $\mathtt{WOT}$, it is safer to assume X is reflexive; and then $\mathcal{B}_M(X)$ is even compact and metrizable.)

The study of typical properties of contractions was initiated, in a Hilbertian setting, by Eisner [12] and Eisner-Mátrai [13]. Given a Hilbert space H (complex, infinite-dimensional and separable), Eisner studied in [12] properties of typical operators $T \in \mathcal{B}_1(H)$ for the Weak Operator Topology and proved that a typical $T \in (\mathcal{B}_1(H), \mathtt{WOT})$ is unitary. The situation turns out to be completely different if one considers the Strong Operator Topology: indeed, it was shown in [13] that a typical $T \in (\mathcal{B}_1(H), \mathtt{SOT})$ is unitarily equivalent to the backward shift of infinite multiplicity acting on $\ell_2\big(\mathbb{Z}_+, \ell_2\big)$. So the behaviour of typical contractions on the Hilbert space is essentially fully understood in the $\mathtt{SOT}$ case, and rather well so in the $\mathtt{WOT}$ case. The general picture in the $\mathtt{SOT}^*$ case is more complicated, and this was studied in some detail in [18].

Let X be a Banach space, and let τ be a topology on $\mathcal{B}(X)$ turning the ball $\mathcal{B}_1(X)$ into a Polish space. If $\mathcal{G}$ is a subset of $\mathcal{B}(X)$ such that $\mathcal{G}_1 := \mathcal{G} \cap \mathcal{B}_1(X)$ is a G_δ subset of $\mathcal{B}_1(X)$ for the topology τ, then $(\mathcal{G}_1, \tau)$ is also a Polish space in its own right. Our initial goal in this paper was to determine, for certain suitable subsets $\mathcal{G} \subseteq \mathcal{B}(X)$, whether a typical contraction T from $\mathcal{G}$ has a non-trivial invariant subspace, *i.e.* a closed linear subspace $E \subseteq X$ with $E \neq \{0\}$ and $E \neq X$ such that $T(E) \subseteq E$.

This is of course motivated by the famous *Invariant Subspace Problem*, which asks, for a given Banach space X, whether *every* operator $T \in \mathcal{B}(X)$ has a non-trivial invariant subspace. The Invariant Subspace Problem was solved in the negative by Enflo in the 70's [14] for some peculiar Banach space; and then by Read [32], who subsequently exhibited in several further papers examples of operators without invariant subspaces on some classical Banach spaces like ℓ_1 and c_0. See [33], [35], and also [17] for a unified approach to these constructions. On the other hand, there are also lots of positive results. To name a few: the classical Lomonosov Theorem from [28] (any operator whose commutant contains a non-scalar operator commuting with a non-zero compact operator has a non-trivial invariant subspace); the Brown-Chevreau-Pearcy Theorem from [5] (any Hilbert space contraction whose spectrum contains the whole unit circle has a non-trivial invariant subspace); and more recently, the construction by Argyros and Haydon [1] of Banach spaces X for which the Invariant Subspace Problem has a *positive* answer. We refer to the books [31] and [8], as well as to the survey [7], for a comprehensive overview of the known methods yielding the existence of non-trivial invariant subspaces for various classes of operators on Banach spaces.

A problem closely related to the Invariant Subspace Problem is the Invariant *Subset* Problem, which asks (for a given Banach space X) whether every operator on X admits a non-trivial invariant closed set; equivalently, if there exists at least one $x \neq 0$ in X whose orbit under the action of T is not dense in X. Again, this question was solved in the negative by Read in [34], who constructed an operator on the space ℓ_1 with no non-trivial invariant closed set. Further examples of such operators on a class of Banach spaces including ℓ_1 and c_0 were exhibited in [17]. One may also add that the Invariant Subset Problem has been a strong motivation for the study of *hypercyclic* operators. Recall that an operator $T \in \mathcal{L}(X)$ is said to be hypercyclic if there exists some vector $x \in X$ whose orbit under the action of T is dense in X; in which case x is said to be hypercyclic for T. With this terminology, the Invariant Subset Problem for X asks whether there exists an operator $T \in \mathcal{L}(X)$ such that every $x \neq 0$ is hypercyclic for T. We refer to the books [4] and [19] for more information on hypercyclicity and other dynamical properties of linear operators.

Despite considerable efforts, the Invariant Subspace Problem and the Invariant Subset Problem remain stubbornly open in the reflexive setting, and in particular for the Hilbert space. Therefore, it seems quite natural to address the *a priori* more tractable "generic" version of the problem; and this is meaningful even on spaces like ℓ_1 or c_0, since the mere existence of counterexamples says nothing about the typical character of the property of having a non-trivial invariant subspace.

We will mostly focus on operators on $X = \ell_p$, $1 \leqslant p < \infty$ or $X = c_0$. This already offers a wide range of possibilities: the spaces ℓ_1 and c_0 are known to support operators without non-trivial invariant subspaces (and even without non-trivial invariant closed sets), whereas the spaces ℓ_p for $1 < p < \infty$ are all reflexive – so it is not known if they support operators without non-trivial invariant subspaces – but have rather different geometric properties in the three cases $p = 2$, $1 < p < 2$ and $p > 2$. The main part of our work will consist in the study of the spaces $(\mathcal{B}_1(X), \tau)$ for one of the topologies $\tau = \texttt{SOT}$ or $\tau = \texttt{SOT}^*$ (when $\tau = \texttt{SOT}^*$, we have to exclude $X = \ell_1$ since it has a non-separable dual).

It follows from the topological 0 - 1 law that on any of the spaces $X = \ell_p$ or c_0, either a typical contraction has a non-trivial invariant subspace, or a typical contraction does not have a non-trivial invariant subspace (see Proposition 3.2). In view of the complexity of the known counterexamples to the Invariant Subspace Problem on ℓ_1 and c_0, and of the very serious difficulty encountered when trying to construct such a counterexample on a reflexive space, it seems reasonable to expect that the correct alternative is the first one, *i.e.* a typical contraction on X does have a non-trivial invariant subspace; but our efforts to prove that have remained unsuccesful, except in the "trivial" case $X = \ell_2$ (see below) and for $X = \ell_1$. All is not lost, however, since we do obtain a few significant results related to our initial goal. In the next section, we describe in some detail the contents of the paper.

1a. Notations. — The following notations will be used throughout the paper.

- We denote by $\mathbb{D}$ the open unit disk in $\mathbb{C}$, and by $\mathbb{T}$ the unit circle

- If $(x_i)_{i\in I}$ is any family of vectors in a Banach space X, we denote by $[\,x_i\,;\ i \in I\,]$ its closed linear span in X.

- When $X = \ell_p$, $1 \leqslant p < \infty$ or $X = c_0$, we denote by $(e_j)_{j\geqslant 0}$ the canonical basis of X, and by $(e_j^*)_{j\geqslant 0} \subseteq X^*$ the associated sequence of coordinate functionals. For any set I of nonnegative integers, we denote by $P_I : X \to [e_i\,;\, i \in I]$ the canonical projection of X onto $[e_i\,;\, i \in I]$. When $I = [0, N]$ for some $N \geqslant 0$, we may also write P_N instead of $P_{[0,N]}$. For every $N \geqslant 0$, we set $E_N := [e_0, \dots, e_N]$ and $F_N := [e_j;\ j > N]$.

- If T is a bounded operator on a (complex) Banach space X, we denote by $\sigma(T)$ its spectrum, by $\sigma_e(T)$ its essential spectrum, by $\sigma_p(T)$ its point spectrum – *i.e.* the eigenvalues of T – and by $\sigma_{ap}(T)$ its approximate point spectrum, *i.e.* the set of all $\lambda \in \mathbb{C}$ such that $T - \lambda$ is not bounded below.

2. Main results

In Section 3, we review some properties of typical contractions of ℓ_2 for the topology $\mathtt{SOT}$, and we prove some general properties of $\mathtt{SOT}$-typical contractions of the spaces ℓ_p and c_0. We mention in particular the following result (labeled as Proposition 3.9).

Proposition 2.1. — *Let $X = \ell_p$, $1 \leqslant p < \infty$, or $X = c_0$. A typical $T \in (\mathcal{B}_1(X), \mathtt{SOT})$ has the following spectral properties: $T - \lambda$ has dense range for every $\lambda \in \mathbb{C}$, and $\sigma(T) = \sigma_{ap}(T) = \overline{\mathbb{D}}$.*

When $X = \ell_2$, the Eisner-Mátrai result from [13] mentioned above implies that a typical operator T in $(\mathcal{B}_1(\ell_2), \mathtt{SOT})$ satisfies much stronger properties: T^* is a non-surjective isometry, $T - \lambda$ is surjective for every $\lambda \in \mathbb{D}$, and every $\lambda \in \mathbb{D}$ is an eigenvalue of T with infinite multiplicity. It immediately follows that an $\mathtt{SOT}$-typical contraction on ℓ_2 has an enormous amount of invariant subspaces.

A natural step towards the understanding of the behaviour of $\mathtt{SOT}$-typical contractions on other ℓ_p-spaces is to determine whether they enjoy similar properties. Somewhat surprisingly, it turns out that $\mathtt{SOT}$-typical contractions on ℓ_1 behave very much like $\mathtt{SOT}$-typical contractions on ℓ_2 in this respect. Indeed, we prove in Section 4 the following result (labeled as Theorem 4.1):

Theorem 2.2. — *Let $X = \ell_1$. A typical $T \in (\mathcal{B}_1(X), \mathtt{SOT})$ satisfies the following properties: T^* is a non-surjective isometry, $T - \lambda$ is surjective for every $\lambda \in \mathbb{D}$, and every $\lambda \in \mathbb{D}$ is an eigenvalue of T with infinite multiplicity. In particular, a typical $T \in (\mathcal{B}_1(X), \mathtt{SOT})$ has non-trivial invariant subspaces.*

The situation is radically different when $X = \ell_p$ for $1 < p < \infty$ and $p \neq 2$, or when $X = c_0$. It is not hard to show that in this case, the set of all co-isometries is nowhere dense in $(\mathcal{B}_1(X), \mathtt{SOT})$, so that a statement analogous to the first part of Theorem 2.2 cannot be true (see Proposition 5.1). Moreover, we prove in Section 5 that, at least when $X = \ell_p$ for $p > 2$ or $X = c_0$, the typical behaviour regarding the eigenvalues is in fact to have no eigenvalue at all. More precisely, we obtain the following results (see Theorem 5.2 and Theorem 5.3).

Theorem 2.3. — *If $X = c_0$, then a typical $T \in (\mathcal{B}_1(X), \mathtt{SOT})$ has no eigenvalue. If $X = \ell_p$ with $p > 2$ then, for any $M > 1$, a typical $T \in (\mathcal{B}_1(X), \mathtt{SOT})$ is such that $(MT)^*$ is hypercyclic.*

The conclusion in the ℓ_p-case is indeed stronger than asserting that T has no eigenvalue, since it is well known that the adjoint of a hypercyclic operator cannot have eigenvalues.

Our proofs in the c_0-case and in the ℓ_p-case are quite different. In the c_0-case, we make use of the so-called *Banach-Mazur* game. In the ℓ_p-case, we observe that the result is already known *for the topology* $\mathtt{SOT}^*$ by [18], and then we prove the following rather unexpected fact, which might be useful in other situations: if $p > 2$, then any $\mathtt{SOT}^*$-comeager subset of $\mathcal{B}_1(\ell_p)$ is also $\mathtt{SOT}$-comeager (see Theorem 5.12). Our proof of this latter result heavily relies on properties of the ℓ_p-norm which are specific to the case $p > 2$, and so it does not extend to the case $1 < p < 2$. In fact, we have not been able to determine whether an $\mathtt{SOT}$ - typical contraction on $X = \ell_p$, $1 < p < 2$ has eigenvalues. We do not know either if hypercyclicity of $(MT)^*$ is typical in the c_0 - case.

Theorem 2.3 might suggest that in fact, an $\mathtt{SOT}$ - typical contraction on ℓ_p, $p > 2$ does *not* have a non-trivial invariant subspace. If it were true, this would solve the Invariant Subspace Problem for ℓ_p, presumably with rather "soft" arguments; so we would not bet anything on it. To support this feeling, a result of Müller [30] can be brought to use to prove that for $X = \ell_p$ with $1 < p < \infty$, a typical $T \in (\mathcal{B}_1(X), \mathtt{SOT})$ has a non-trivial invariant *closed convex cone* (this is Proposition 5.26). When $X = c_0$, we show that a typical $T \in (\mathcal{B}_1(X), \mathtt{SOT})$ has a non-zero *non-supercyclic* vector – *i.e.* a vector $x \neq 0$ whose scaled orbit $\{\lambda T^n x;\ n \geqslant 0\,,\ \lambda \in \mathbb{C}\}$ is not dense in X – and hence a non-trivial invariant star-shaped closed set (this is Proposition 5.27).

The following table summarizes most of our knowledge about $\mathtt{SOT}$ - typical contractions on ℓ_p or c_0.

	$p = 1$	$1 < p < 2$	$p = 2$	$2 < p < \infty$	c_0
$\sigma(T)$	$\overline{\mathbb{D}}$ (Prop 3.9)	$\overline{\mathbb{D}}$ (Prop 3.9)	$\overline{\mathbb{D}}$ (Eisner-Mátrai)	$\overline{\mathbb{D}}$ (Prop 3.9)	$\overline{\mathbb{D}}$ (Prop 3.9)
T^* is an isometry	Yes (Thm 4.1)	No (Prop 5.1)	Yes (Eisner-Mátrai)	No (Prop 5.1)	No (Prop 5.1)
T has a non-trivial invariant closed cone	Yes (Thm 4.1)	Yes (Prop 5.26)	Yes (Eisner-Mátrai)	Yes (Prop 5.26)	?
T has a non-zero non-supercyclic vector	Yes (Thm 4.1)	Yes (Prop 5.26)	Yes (Eisner-Mátrai)	Yes (Prop 5.26)	Yes (Prop 5.27)
$\sigma_p(T)$	$\mathbb{D}$ (Thm 4.1)	?	$\mathbb{D}$ (Eisner-Mátrai)	$\varnothing$ (Thm 5.3)	$\varnothing$ (Thm 5.2)
T has a non-trivial invariant subspace	Yes (Thm 4.1)	?	Yes (Eisner-Mátrai)	?	?

TABLE 1. Properties of $\mathtt{SOT}$ - typical $T \in \mathcal{B}_1(X)$

In Section 6, we change the setting and consider a subclass $\mathcal{T}_1(X)$ of contractions on $X = \ell_p$, $1 \leqslant p < \infty$ or $X = c_0$ defined as follows: if $(e_j)_{j \geqslant 0}$ denotes the canonical basis of X, then $\mathcal{T}_1(X)$ consists of all operators $T \in \mathcal{B}_1(X)$ which are "triangular plus 1" with respect to (e_j), with positive entries on the first subdiagonal:

$$\mathcal{T}_1(X) = \big\{T \in \mathcal{B}_1(X)\ ;\ Te_j \in [\,e_0, \dots, e_{j+1}\,] \text{ and } \langle Te_j, e_{j+1}\rangle > 0 \text{ for every } j \geqslant 0\big\}.$$

The class $\mathcal{T}_1(X)$ is rather natural for at least two reasons. Firstly, in the Hilbertian setting, any cyclic operator $T \in \mathcal{B}_1\big(\ell_2\big)$ is unitarily equivalent to some operator in $\mathcal{T}_1\big(\ell_2\big)$. Secondly, all the operators without non-trivial invariant subspaces constructed by Read in [32], [33], [35] belong to the classes $\mathcal{T}_1(\ell_1)$ or $\mathcal{T}_1\big(c_0\big)$. The extensions considered in [17] are also of this form, as well as the operators with few non-trivial invariant subspaces constructed in [16]. Thus, "Read's type operators" on $X = \ell_p$ or c_0 belong to $\mathcal{T}_1(X)$, and hence it is quite natural to investigate whether an $\mathtt{SOT}$ - typical or $\mathtt{SOT}^*$- typical operator from $\mathcal{T}_1(X)$ has a non-trivial invariant subspace. Again, our results here are rather partial. We prove the following "local" version of the Eisner-Mátrai result from [13] (see Corollary 6.6).

Theorem 2.4. — *Let $X = \ell_2$. A typical $T \in (\mathcal{T}_1(X), \mathtt{SOT})$ is unitarily equivalent to the backward shift of infinite multiplicity acting on $\ell_2\big(\mathbb{Z}_+, \ell_2\big)$. In particular, it has eigenvalues, and hence non-trivial invariant subspaces.*

In view of the results of Section 4, it is not too surprising that we are also able to settle the case $X = \ell_1$ (see Proposition 6.7):

Theorem 2.5. — *Let $X = \ell_1$. A typical $T \in (\mathcal{T}_1(X), \mathtt{SOT})$ has all the properties listed in* Theorem 2.2. *In particular, it has a non-trivial invariant subspace.*

We move over in Section 7 to the study of typical contractions on $X = \ell_p$, $1 < p < \infty$ with respect to the topology $\mathtt{SOT}^*$. The spectral properties of $\mathtt{SOT}^*$-typical contractions are rather different from those of $\mathtt{SOT}$-typical contractions, at least for $p = 2$, since for example a typical $T \in (\mathcal{B}(\ell_2), \mathtt{SOT}^*)$ has no eigenvalue (see [13] or [18]). Regarding invariant subspaces, the situation is exactly the same as in the $\mathtt{SOT}$-case: we can prove that an $\mathtt{SOT}^*$-typical contraction on $X = \ell_p$ has a non-trivial invariant subspace only in the Hilbertian case $p = 2$ (recall that $p = 1$ is not considered in the $\mathtt{SOT}^*$ setting); and to do that, we have to use a quite non-trivial result, namely the Brown-Chevreau-Pearcy Theorem from [5]. We are also able to prove (using the result of Müller [30]) that for any $1 < p < \infty$, a typical $T \in (\mathcal{B}_1(\ell_p), \mathtt{SOT}^*)$ has a non-trivial invariant closed cone. These facts are summarized in the following table.

	$1 < p < 2$	$p = 2$	$2 < p < \infty$
$\sigma(T)$	$\overline{\mathbb{D}}$ (Prop 7.2)	$\overline{\mathbb{D}}$ (Prop 7.2)	$\overline{\mathbb{D}}$ (Prop 7.2)
T^* is an isometry	No (Prop 7.1)	No (Prop 7.1)	No (Prop 7.1)
T has a non-trivial invariant closed cone	Yes (Cor 7.4)	Yes (Cor 7.4)	Yes (Cor 7.4)
$\sigma_p(T)$	$\varnothing$ (Prop 7.1)	$\varnothing$ (Prop 7.1)	$\varnothing$ (Prop 7.1)
T has a non-trivial invariant subspace	?	Yes (Cor 7.3)	?

TABLE 2. Properties of $\mathtt{SOT}^*$-typical $T \in \mathcal{B}_1(X)$

Since $\mathtt{SOT}^*$-typical contractions of a Hilbert space are by far not so well understood as $\mathtt{SOT}$-typical contractions, it is interesting to investigate whether they satisfy some of the standard criteria implying the existence of a non-trivial invariant subspace. What about the Lomonosov Theorem, for instance? In this direction, we obtain the following result (this is Theorem 7.5):

Theorem 2.6. — *Let $X = \ell_2$. A typical $T \in (\mathcal{B}_1(X), \mathtt{SOT}^*)$ does not commute with any non-zero compact operator.*

It should be apparent that this paper rises more questions than it answers. We collect some of them in Section 8.

3. Some general properties of $\mathtt{SOT}$-typical contractions

For the convenience of the reader, we begin this section by giving a quick proof of the fact that when X is a (separable) Banach space, the closed balls $(\mathcal{B}_M(X), \mathtt{SOT})$ are indeed Polish spaces. The proof for $\mathtt{SOT}^*$ when X^* is separable would be essentially the same.

Lemma 3.1. — *For any (separable) Banach space X and any $M > 0$, the ball $\mathcal{B}_M(X)$ is a Polish space when endowed with the topology* $\mathtt{SOT}$.

Proof. — Let $\mathbf{Q}$ be a countable dense subfield of $\mathbb{C}$, and let $Z \subseteq X$ be countable dense subset of X which is also a vector space over $\mathbf{Q}$. Consider the set

$$\mathcal{L}_M := \big\{L : Z \to X \; ; \; L \text{ is } \mathbf{Q}\text{-linear and } \forall z \in Z \; : \; \|L(z)\| \leqslant M\|z\|\big\}.$$

This is a closed subset of X^Z with respect to the product topology, hence a Polish space. Moreover, it is easily checked that the map $T \mapsto T_{|Z}$ is a homeomorphism from $(\mathcal{B}_M(X), \texttt{SOT})$ onto $\mathcal{L}_M$. So $(\mathcal{B}_M(X), \texttt{SOT})$ is Polish. $\square$

Next, we prove a general result showing that all the properties we will be considering in this paper are either typical or "atypical". If X is a (separable) Banach space, let us denote by $\mathrm{Iso}(X)$ the group of all surjective linear isometries $J : X \to X$. We say that a set $\mathcal{A} \subseteq \mathcal{B}_1(X)$ is $\mathrm{Iso}(X)$ - *invariant* if $J\mathcal{A}J^{-1} = \mathcal{A}$ for every $J \in \mathrm{Iso}(X)$.

Proposition 3.2. — *Let $X = \ell_p$, $1 \leqslant p < \infty$ or $X = c_0$. If $\mathcal{A} \subseteq (\mathcal{B}_1(X), \texttt{SOT})$ has the Baire property and is* $\mathrm{Iso}(X)$ *- invariant, then $\mathcal{A}$ is either meager or comeager in $\mathcal{B}_1(X)$.*

Proof of Proposition 3.2. — Recall first that $\mathrm{Iso}(X)$ becomes a Polish group when endowed with $\texttt{SOT}$. This is well known and true for any (separable) Banach space X; but we give a quick sketch of proof for convenience of the reader. First, $\mathrm{Iso}(X)$ is a G_δ subset of $(\mathcal{B}_1(X), \texttt{SOT})$, because an operator $J \in \mathcal{B}_1(X)$ belongs to $\mathrm{Iso}(X)$ if and only if it is an isometry with dense range (the first condition obviously defines a closed set, and the second one is easily seen to define a G_δ set). So $(\mathrm{Iso}(X), \texttt{SOT})$ is a Polish space. Next, multiplication is continuous on $\mathrm{Iso}(X) \times \mathrm{Iso}(X)$ since it is continuous on $\mathcal{B}_1(X) \times \mathcal{B}_1(X)$. Finally, if (J_n) is a sequence in $\mathrm{Iso}(X)$ such that $J_n \to J \in \mathrm{Iso}(X)$, then $\|J_n^{-1}x - J^{-1}x\| = \|x - J_nJ^{-1}x\| \to 0$ for every $x \in X$, so $J_n^{-1} \to J^{-1}$; hence the map $J \mapsto J^{-1}$ is continuous on $\mathrm{Iso}(X)$.

The Polish group $\mathrm{Iso}(X)$ acts continuously by conjugacy on $\mathcal{B}_1(X)$. By the *topological* 0 - 1 *law* (see [24, Theorem 8.46]), it is enough to show that the action is topologically transitive, *i.e.* that for any pair $(\mathcal{U}, \mathcal{V})$ of non-empty open subsets of $(\mathcal{B}_1(X), \texttt{SOT})$, one can find $J \in \mathrm{Iso}(X)$ such that $(J\mathcal{U}J^{-1}) \cap \mathcal{V} \neq \varnothing$.

Choose $A \in \mathcal{U}$ and $B \in \mathcal{V}$. For any integer $N \geqslant 0$, set $A_N := P_NAP_N$ and $B_N := P_NBP_N$, considered as operators on $E_N = [e_0, \dots, e_N]$. Denote by $\widetilde{A}_N$ and $\widetilde{B}_N$ the copies of A_N and B_N living on $\widetilde{E}_N := [e_{N+1}, \dots, e_{2N+1}]$. If N is large enough, then the operator $T_N := A_N \oplus \widetilde{B}_N$ – considered as an operator on X – belongs to $\mathcal{U}$, and $S_N := B_N \oplus \widetilde{A}_N$ belongs to $\mathcal{V}$. Let us fix such an integer N, and let $J \in \mathrm{Iso}(X)$ be the isometry exchanging e_j and e_{N+1+j} for $j = 0, \dots, N$ and such that $Je_k = e_k$ for all $k > 2N+1$. Then $JT_NJ^{-1} = S_N$; so we have indeed $(J\mathcal{U}J^{-1}) \cap \mathcal{V} \neq \varnothing$. $\square$

Corollary 3.3. — *If $X = \ell_p$ or c_0 then, either a typical $T \in \mathcal{B}_1(X)$ has a non-trivial invariant subspace, or a typical $T \in \mathcal{B}_1(X)$ does not have a non-trivial invariant subspace.*

Proof. — Let $\mathcal{A}$ be the set of all $T \in \mathcal{B}_1(X)$ having a non-trivial invariant subspace. It is clear that $\mathcal{A}$ is $\mathrm{Iso}(X)$ - invariant; so we just have to show that $\mathcal{A}$ has the Baire property. Now, if $T \in \mathcal{B}_1(X)$ then

$$T \in \mathcal{A} \iff \exists (x, x^*) \in X \times B_{X^*} \; : \; \big(x \neq 0 \, , \; x^* \neq 0 \quad \text{and} \quad \forall n \in \mathbb{Z}_+ \; : \; \langle x^*, T^n x\rangle = 0\big).$$

Since the relation under brackets defines a Borel (in fact, F_σ) subset of the Polish space $\mathcal{B}_1(X) \times X \times (B_{X^*}, w^*)$, this shows that $\mathcal{A}$ is an *analytic* subset of $\mathcal{B}_1(X)$. In particular, $\mathcal{A}$ has the Baire property. (See [24] for background on analytic sets.) $\square$

As mentioned in the introduction, we know what is actually true only for $X = \ell_2$ and $X = \ell_1$. Likewise, one can say that either a typical $T \in \mathcal{B}_1(X)$ has eigenvalues, or a typical $T \in \mathcal{B}_1(X)$ does not have eigenvalues; but we do not know what is true for $X = \ell_p$, $1 < p < 2$.

When X is a Hilbert space (so that we call it H), the $\texttt{SOT}$ - typical properties of contractions are essentially fully understood, thanks to the following result of Eisner and Mátrai [13]. Let us denote by $\ell_2(\mathbb{Z}_+, \ell_2)$ the infinite direct sum of countably many copies of ℓ_2, and let B_∞ be the canonical backward shift acting on $\ell_2(\mathbb{Z}_+, \ell_2)$, *i.e.* the operator defined by $B_\infty(x_0, x_1, x_2, \dots) := (x_1, x_2, x_3, \dots)$ for every $(x_0, x_1, x_2, \dots) \in \ell_2(\mathbb{Z}_+, \ell_2)$.

Theorem 3.4. — *A typical $T \in (\mathcal{B}_1(H), \texttt{SOT})$ is unitarily equivalent to B_∞.*

This theorem is proved in [13] by showing that a typical $T \in (\mathcal{B}_1(H), \texttt{SOT})$ has the following properties: T^* is an isometry, $\dim\ker(T) = \infty$, and $\|T^n x\| \to 0$ as $n \to \infty$ for every $x \in H$. The result then follows from the Wold decomposition theorem.

Corollary 3.5. — *A typical $T \in (\mathcal{B}_1(H), \texttt{SOT})$ has the following properties.*

(a) *T^* is a non-surjective isometry.*
(b) *$T - \lambda$ is surjective for every $\lambda \in \mathbb{D}$.*
(c) *Every $\lambda \in \mathbb{D}$ is an eigenvalue of T with infinite multiplicity.*
(d) $\sigma(T) = \sigma_e(T) = \sigma_{ap}(T) = \overline{\mathbb{D}}$.
(e) $\|T^n x\| \to 0$ *for every* $x \in H$.

In particular, it follows immediately from (c) that an $\texttt{SOT}$ - typical contraction on H has a wealth of invariant subspaces. So we may state

Corollary 3.6. — *A typical $T \in (\mathcal{B}_1(H), \texttt{SOT})$ has a non-trivial invariant subspace.*

One may also note that property (a) by itself implies as well that T has a non-trivial invariant subspace. Indeed, by a classical result of Godement [15], any Banach space isometry has a non-trivial invariant subspace. Hence, if X is a reflexive Banach space, then any operator $T \in \mathcal{B}(X)$ such that T^* is an isometry has a non-trivial invariant subspace.

In the next three propositions, we show that something remains of Corollary 3.5 in a non-Hilbertian setting.

Proposition 3.7. — *Let X be a Banach space. A typical $T \in (\mathcal{B}_1(X), \texttt{SOT})$ is such that $\|T^n x\| \to 0$ as $n \to \infty$ for all $x \in X$.*

Proof. — This result is in fact proved in [13, Lemma 5.9], but we give the details for convenience of the reader. The key point is that if $T \in \mathcal{B}_1(X)$ is a contraction and $x \in X$, then the sequence $\|T^n x\|$ is non-increasing, and hence $T^n x \to 0$ if and only if $\inf_{n\in\mathbb{N}} \|T^n x\| = 0$. Let us denote by $\mathcal{C}_0$ the set of all $T \in \mathcal{B}_1(X)$ such that $\forall x \in X \;:\; \|T^n x\| \to 0$. Let also Z be a countable dense subset of X. By what we have just said, the following equivalence holds true for any $T \in \mathcal{B}_1(X)$:

$$T \in \mathcal{C}_0 \iff \forall z \in Z\ \forall K \in \mathbb{N}\ \exists n \;:\; \|T^n z\| < 1/K.$$

Since the map $T \mapsto T^n$ is $\texttt{SOT}$-continuous on $\mathcal{B}_1(X)$ for each $n \in \mathbb{N}$, this shows that $\mathcal{C}_0$ is a G_δ subset of $(\mathcal{B}_1(X), \texttt{SOT})$. Moreover, $\mathcal{C}_0$ is dense in $\mathcal{B}_1(X)$ because it contains every operator T with $\|T\| < 1$. ∎

Recall that a Banach space X is said to have the *Metric Approximation Property* (for short, the MAP) if, for any compact set $K \subseteq X$ and every $\varepsilon > 0$, one can find a finite rank operator $R \in \mathcal{B}(X)$ with $\|R\| \leqslant 1$ such that $\|Rx - x\| < \varepsilon$ for all $x \in K$. Equivalently, X has the MAP if and only if there exists a net $(R_i) \subseteq \mathcal{B}_1(X)$ consisting of finite rank operators such that $R_i \xrightarrow{\texttt{SOT}} Id$ (and since X is assumed to be separable, one can in fact take a *sequence* (R_i) with this property). See e.g. [27] for more on this notion.

Proposition 3.8. — *Let X be a Banach space which has the* MAP*. Then, the set of all $T \in \mathcal{B}_1(X)$ such that $0 \in \sigma_{ap}(T)$ is a dense G_δ subset of $(\mathcal{B}_1(X), \texttt{SOT})$. In particular, a typical $T \in (\mathcal{B}_1(X), \texttt{SOT})$ is non-invertible.*

Proof. — Set $\mathcal{G} := \{T \in \mathcal{B}_1(X);\ 0 \in \sigma_{ap}(T)\}$. Then,

$$T \in \mathcal{G} \iff \forall K \in \mathbb{N}\ \exists x \in S_X\ :\ \|Tx\| < 1/K,$$

so $\mathcal{G}$ is a G_δ subset of $(\mathcal{B}_1(X), \texttt{SOT})$. Moreover, $\mathcal{G}$ contains all finite rank operators in $\mathcal{B}_1(X)$, and the latter are $\texttt{SOT}$ - dense in $\mathcal{B}_1(X)$ because X has the MAP. □

In the particular case of ℓ_p and c_0, we have the following proposition, which provides some spectral properties of $\texttt{SOT}$ - typical contractions on these spaces:

Proposition 3.9. — *Assume that $X = \ell_p$, $1 \leqslant p < \infty$ or that $X = c_0$. A typical $T \in (\mathcal{B}_1(X), \texttt{SOT})$ has the following properties: $T - \lambda$ has dense range for every $\lambda \in \mathbb{C}$, and $\sigma(T) = \sigma_{ap}(T) = \overline{\mathbb{D}}$.*

Proof. — Let us set

$$\mathcal{G}_1 := \big\{T \in \mathcal{B}_1(X);\ T - \lambda \text{ has dense range for every } \lambda \in \mathbb{C}\big\},$$

and

$$\mathcal{G}_2 := \big\{T \in \mathcal{B}_1(X);\ \sigma(T) = \sigma_{ap}(T) = \overline{\mathbb{D}}\big\}.$$

By the Baire Category Theorem, it is enough to show that $\mathcal{G}_1$ and $\mathcal{G}_2$ are both dense G_δ subsets of $(\mathcal{B}_1(X), \texttt{SOT})$.

Let Z be a countable dense subset of X. If $T \in \mathcal{B}_1(X)$, then

$$T \notin \mathcal{G}_1 \iff \exists \lambda \in \mathbb{C}\ \exists K \in \mathbb{N}\ \exists z \in Z\ :\ \forall z' \in Z\ :\ \|(T - \lambda)z' - z\| \geqslant 1/K.$$

This shows that $\mathcal{B}_1(X) \backslash \mathcal{G}_1$ is the projection of an F_σ subset of $\mathbb{C} \times \mathcal{B}_1(X)$, where $\mathcal{B}_1(X)$ is endowed with the topology $\texttt{SOT}$. Since $\mathbb{C}$ is σ-compact, it follows that $\mathcal{B}_1(X) \backslash \mathcal{G}_1$ is F_σ in $(\mathcal{B}_1(X), \texttt{SOT})$, *i.e.* $\mathcal{G}_1$ is G_δ.

Similarly, if $T \in \mathcal{B}_1(X)$ then (since $\sigma_{ap}(T) \subseteq \sigma(T) \subseteq \overline{\mathbb{D}}$), we have that $T \notin \mathcal{G}_2$ if and only if $T - \lambda$ is bounded below for some $\lambda \in \overline{\mathbb{D}}$, *i.e.*

$$T \notin \mathcal{G}_2 \iff \exists \lambda \in \overline{\mathbb{D}}\ \exists K \in \mathbb{N}\ \forall z \in Z\ :\ \|(T - \lambda)z\| \geqslant 1/K\,\|z\|;$$

and since $\overline{\mathbb{D}}$ is compact, it follows that $\mathcal{G}_2$ is a G_δ subset of $(\mathcal{B}_1(X), \texttt{SOT})$.

A straightforward adaptation of the proof of [18, Proposition 2.3] or [4, Proposition 2.23] shows that for any $M > 1$, the set of all hypercyclic operators in $\mathcal{B}_M(X)$ is a dense G_δ subset of $(\mathcal{B}_M(X), \texttt{SOT})$. In particular, an $\texttt{SOT}$ - typical $R \in \mathcal{B}_M(X)$ is such that $aR + b$ has dense range for any $a, b \in \mathbb{C}$ with $(a, b) \neq (0, 0)$. Since the map $T \mapsto MT$ is a homeomorphism from $(\mathcal{B}_1(X), \texttt{SOT})$ onto $(\mathcal{B}_M(X), \texttt{SOT})$, it follows that $\mathcal{G}_1$ is dense in $(\mathcal{B}_1(X), \texttt{SOT})$.

Now, let us show that $\mathcal{G}_2$ is dense in $\mathcal{B}_1(X)$. Recall that $(e_j)_{j \geqslant 0}$ is the canonical basis of X, that for any $N \geqslant 0$, $P_N : X \to [e_0, \dots, e_N]$ is the canonical projection map onto $E_N = [e_0, \dots, e_N]$, and that $F_N = [e_j;\ j > N]$. Choose an operator $B_N \in \mathcal{B}_1(F_N)$ such that $B_N - \lambda$ is not bounded below for any $\lambda \in \overline{\mathbb{D}}$ (for example, the backward shift built on $(e_j)_{j \geqslant N+1}$ will do). If $A \in \mathcal{B}_1(X)$ is arbitrary, then $T_N := P_N A_{|E_N} \oplus B_N$ satisfies $\|T_N\| \leqslant 1$ and is such that $T - \lambda$ is not bounded below for any $\lambda \in \overline{\mathbb{D}}$; that is, $T_N \in \mathcal{G}_2$. Since $T_N \xrightarrow{\texttt{SOT}} A$ as $N \to \infty$, this shows that $\mathcal{G}_2$ is indeed dense in $(\mathcal{B}_1(X), \texttt{SOT})$. $\square$

Proposition 3.9 says in particular that a typical $T \in \mathcal{B}_1(\ell_p)$ has the largest possible spectrum. However, it is worth mentioning that there are also lots of operators $T \in \mathcal{B}_1(\ell_p)$ whose spectrum is rather small:

Proposition 3.10. — *Let $X = \ell_p$, $1 \leqslant p < \infty$. The set of operators $T \in \mathcal{B}_1(X)$ such that $\sigma(T) \subseteq \mathbb{T}$ is dense in $(\mathcal{B}_1(X), \texttt{SOT})$.*

Perhaps surprisingly, the proof of this result is not straightforward. For technical reasons, it will be convenient to identify the space X with $Y := \ell_p(\mathbb{Z})$ endowed with its canonical basis $(f_k)_{k \in \mathbb{Z}}$.

For each $N \geqslant 0$, let $Y_N := [\, f_k \,;\, |k| \leqslant N \,]$. To any operator $A \in \mathcal{B}(Y_N)$ and any bounded sequence of positive numbers $\omega = (\omega_k)_{k \in \mathbb{Z}}$ – such a sequence ω will be called a *weight sequence* – we associate the operator $S_{A,\omega}$ on Y defined by

$$S_{A,\omega} f_k := \begin{cases} A f_k + \omega_{k-(2N+1)} f_{k-(2N+1)} & \text{if } |k| \leqslant N, \\ \omega_{k-(2N+1)} f_{k-(2N+1)} & \text{if } |k| > N. \end{cases}$$

Some of the properties of these operators, related to linear dynamics, can be found in [18, Proposition 2.14 and Remark 2.15].

In the next two lemmas, the integer N and the operator $A \in \mathcal{B}(Y_N)$ are fixed. The first lemma gives an estimate on the norm of $S_{A,\omega}$.

Lemma 3.11. — *For any $\varepsilon > 0$, there exists $\delta > 0$ depending only on ε and $\|A\|$ such that, for any weight sequence ω satisfying $\sup_{-(3N+1) \leqslant k \leqslant N} \omega_k < \delta$, one has*

$$\|S_{A,\omega}\| \leqslant \max \Big((\|A\|^p + \varepsilon^p)^{1/p}, \sup_{k \notin [-(3N+1), N]} \omega_k \Big).$$

Proof of Lemma 3.11. — For any set of integers I we denote by P_I the canonical projection of $Y = \ell_p(\mathbb{Z})$ onto $[\, f_k \,;\, k \in I \,]$. For every $y = \sum_{k \in \mathbb{Z}} y_k f_k$, we have

$$\|S_{A,\omega}\, y\|^p = \Big\| A\, P_{[-N,N]} y + \sum_{k=N+1}^{3N+1} \omega_{k-(2N+1)} y_k f_{k-(2N+1)} \Big\|^p + \sum_{k \notin [N+1, 3N+1]} \omega_{k-(2N+1)}^p |y_k|^p$$

Let $c_1, c_2 \in \mathbb{R}_+$ be such that $(a+b)^p \leqslant a^p + c_1 b^p + c_2 a^{p-1} b$ for every $a, b \geqslant 0$. Using Hölder's inequality, we se that for any $u, v \in \ell_p$, we have

$$\|u + v\|^p \leqslant \|u\|^p + c_1 \|v\|^p + c_2 \|u\|^{p-1} \|v\|.$$

So we get

$$\begin{aligned}\|\, S_{A,\omega}\, y \,\|^p \leqslant \|\, A \,\|^p \,\|\, P_{[-N,N]}y \,\|^p + c_1 \sup_{N+1\leqslant k\leqslant 3N+1} \omega^p_{k-(2N+1)} \,\|\, P_{[N+1,3N+1]}y \,\|^p \\ + c_2 \,\|\, A \,\|^{p-1} \|\, P_{[-N,N]}\, y \,\|^{p-1} \sup_{N+1\leqslant k\leqslant 3N+1} \omega_{k-(2N+1)} \,\|\, P_{[N+1,3N+1]}\, y \,\| \\ + \sup_{k\in[-N,N]} \omega^p_{k-(2N+1)} \,\|\, P_{[-N,N]}\, y \,\|^p \\ + \sup_{k\notin[-N,3N+1]} \omega^p_{k-(2N+1)} \,\|\, (I - P_{[-N,3N+1]})\, y \,\|^p.\end{aligned}$$

Since $\sup_{-N\leqslant k\leqslant 3N+1} \omega_{k-(2N+1)} = \sup_{-(3N+1)\leqslant k\leqslant N} \omega_k < \delta$ and since

$$\|\, P_{[-N,N]}\, y \,\|^{p-1} \,\|\, P_{[N+1,3N+1]}\, y \,\| \leqslant \|\, P_{[-N,3N+1]}\, y \,\|^p,$$

this yields that

$$\begin{aligned}\|\, S_{A,\omega}\, y \,\|^p &\leqslant \Big(\|\, A \,\|^p + \delta^p\Big) \,\|\, P_{[-N,N]}\, y \,\|^p + \big(c_1\delta^p + c_2 \,\|\, A \,\|^{p-1}\delta\big) \,\|\, P_{[-N,3N+1]}\, y \,\|^p \\ &\quad + \sup_{k\notin[-(3N+1),N]} \omega^p_k \,\|\, (I - P_{[-N,3N+1]})\, y \,\|^p \\ &\leqslant \Big(\|\, A \,\|^p + (c_1+1)\delta^p + c_2 \,\|\, A \,\|^{p-1}\delta\Big) \,\|\, P_{[-N,3N+1]}y \,\|^p \\ &\quad + \sup_{k\notin[-(3N+1),N]} \omega^p_k \,\|\, (I - P_{[-N,3N+1]})\, y \,\|^p.\end{aligned}$$

Choosing δ such that $(c_1+1)\delta^p + c_2 \,\|\, A \,\|^{p-1}\delta < \varepsilon^p$, we obtain that

$$\|\, S_{A,\omega} \,\| \leqslant \max\Bigg(\big(\,\|\, A \,\|^p + \varepsilon^p\big)^{1/p}, \sup_{k\notin[-(3N+1),N]} \omega_k\Bigg),$$

as claimed. □

The second lemma gives a description of the point spectrum of these operators $S_{A,\omega}$. For any weight sequence ω and any λ in $\mathbb{C}$, let us define two subsets $\Lambda^-_{\omega,\lambda}$ and $\Lambda^+_{\omega,\lambda}$ of $[-N, N]$ as follows:

$$\Lambda^-_{\omega,\lambda} := \Bigg\{k \in [-N,N]\ ;\ \sum_{i\geqslant 1} \Big|\, \frac{\omega_{k-(2N+1)}\cdots\omega_{k-i(2N+1)}}{\lambda^i} \,\Big|^p < \infty\Bigg\} \quad \text{with} \quad \Lambda^-_{\omega,0} = \varnothing,$$

$$\Lambda^+_{\omega,\lambda} := \Bigg\{k \in [-N,N]\ ;\ \sum_{i\geqslant 0} \Big|\, \frac{\lambda^i}{\omega_{k+i(2N+1)}\cdots\omega_k} \,\Big|^p < \infty\Bigg\}.$$

Lemma 3.12. — *Let $\lambda \in \mathbb{C}$. Then λ is an eigenvalue of $S_{A,\omega}$ if and only if there exists a non-zero vector $u \in Y_N$ such that* $\operatorname{supp}(u) \subseteq \Lambda^-_{\omega,\lambda}$ *and* $\operatorname{supp}\big((A-\lambda)u\big) \subseteq \Lambda^+_{\omega,\lambda}$. *In particular, if $\Lambda^-_{\omega,\lambda} = \varnothing$ then λ is not an eigenvalue of $S_{A,\omega}$.*

Here, we denote by $\operatorname{supp}(y)$ the *support* of a vector $y = \sum_{k\in\mathbb{Z}} y_k f_k$, *i.e.* the set of all $k \in \mathbb{Z}$ such that $y_k \neq 0$.

Proof of Lemma 3.12. — The number λ is an eigenvalue of $S_{A,\omega}$ if and only if there exists a non-zero vector $y \in Y$ such that

$$(A-\lambda)\,P_{[-N,N]}\,y + \sum_{|k|\leqslant N} \omega_k y_{k+(2N+1)} f_k = 0 \tag{1}$$

$$\omega_k y_{k+(2N+1)} = \lambda\, y_k \qquad \text{for every } k \geqslant N+1 \tag{2}$$

$$\omega_{-k} y_{-k+(2N+1)} = \lambda\, y_{-k} \qquad \text{for every } k \geqslant N+1 \tag{3}$$

It follows from these expressions that $\lambda = 0$ is never an eigenvalue of $S_{A,\omega}$. We suppose for the rest of the proof that $\lambda \neq 0$. We deduce from equations (1), (2), and (3) that

$$y_{k+(2N+1)} = -\frac{1}{\omega_k}\,\langle f_k^*,\,(A-\lambda)\,P_{[-N,N]}y\rangle \qquad \text{for every } |k| \leqslant N,$$

(given $P_{[-N,N]}y$, these equations determine the values of y_j for $j \in [N+1, 3N+1]$)

$$y_{l+i(2N+1)} = \frac{\lambda^i}{\omega_{l+(i-1)(2N+1)}\cdots\omega_l}\, y_l \qquad \text{for every } l \geqslant N+1 \text{ and every } i \geqslant 1,$$

(so the values of y_l for $l \in [N+1, 3N+1]$ determine the values of y_j for $j > 3N+1$)

and

$$y_{l-i(2N+1)} = \frac{\omega_{l-(2N+1)}\cdots\omega_{l-i(2N+1)}}{\lambda^i}\, y_l \qquad \text{for every } l \leqslant N \text{ and every } i \geqslant 1$$

(so the values of y_l for $l \in [-N, N]$ determine the values of y_j for $j < -N$).

Hence, if u is a vector of Y_N, there exists a vector $y \in Y$ with $P_{[-N,N]}\,y = u$ and $y \in \ker(S_{A,\omega} - \lambda)$ if and only if the series

$$\sum_{i\geqslant 1} \left|\frac{\omega_{k-(2N+1)}\cdots\omega_{k-i(2N+1)}}{\lambda^i}\right|^p \cdot |u_k|^p \quad \text{and} \quad \sum_{i\geqslant 0} \left|\frac{\lambda^i}{\omega_{k+i(2N+1)}\cdots\omega_k}\right|^p \cdot \left|\langle f_k^*, (A-\lambda)u\rangle\right|^p$$

are convergent for every $k \in [-N, N]$. This is equivalent to the conditions that, for every $k \in [-N, N]$, either $u_k = 0$ or $k \in \Lambda^-_{\omega,\lambda}$, and either $\langle f_k^*, (A-\lambda)u\rangle = 0$ or $k \in \Lambda^+_{\omega,\lambda}$. Lemma 3.12 follows. □

With the two lemmas above at our disposal, we can now proceed to the

Proof of Proposition 3.10.. — Let $\mathcal{U}$ be a non-empty open set in $(\mathcal{B}_1(Y), \texttt{SOT})$. We want to show that $\mathcal{U}$ contains an operator T such that $\sigma(T) \subseteq \mathbb{T}$.

Choose an integer N, an operator $A \in \mathcal{B}(Y_N)$ with $\|A\| < 1$, and $\varepsilon > 0$, such that any operator $T \in \mathcal{B}_1(Y)$ satisfying $\|(T-A)f_k\| < \varepsilon$ for all $k \in [-N, N]$ belongs to the open set $\mathcal{U}$. Then, choose a weight sequence $\omega = (\omega_k)_{k\in\mathbb{Z}}$ such that ω_k is extremely small if $-(3N+1) \leqslant k \leqslant N$, and otherwise $\omega_k = 1$. By Lemma 3.11 and since $\|A\| < 1$, the operator $T := S_{A,\omega}$ satisfies $\|T\| \leqslant 1$, provided that the weights ω_k are sufficiently small for $-(3N+1) \leqslant k \leqslant N$. Moreover, Tf_k is very close to Af_k for $k \in [-N, N]$. Hence $T \in \mathcal{U}$.

It remains to show that $\sigma(T) \subseteq \mathbb{T}$. First, observe that since all but finitely many weights ω_k are equal to 1, the set $\Lambda^-_{\omega,\lambda}$ is obviously empty for any $\lambda \in \mathbb{D}$. By Lemma 3.12, it follows that T has no eigenvalue in $\mathbb{D}$. Let us show that, in fact, $\sigma(T) \cap \mathbb{D} = \varnothing$. Let $\lambda \in \mathbb{D}$ be arbitrary. Since all but finitely many weights ω_k are equal to 1, we see that the operator T is a finite rank perturbation of S^{2N+1}, where S is the unweighted bilateral shift on

$Y = \ell_p(\mathbb{Z})$. Hence, $T - \lambda$ is a finite rank perturbation of $S^{2N+1} - \lambda$. Now, S^{2N+1} is a surjective isometry, so $S^{2N+1} - \lambda$ is invertible because $|\lambda| < 1$. Therefore, $T - \lambda$ is a Fredholm operator with index 0. But $T - \lambda$ is also one-to-one since T has no eigenvalue in $\mathbb{D}$. So $T - \lambda$ is invertible, *i.e.* $\lambda \notin \sigma(T)$. This terminates the proof of Proposition 3.10. □

4. SOT - typical contractions on ℓ_1

We consider in this section the case of the space $X = \ell_1$, which turns out to behave much like ℓ_2 as far as the properties considered in Corollary 3.5 are concerned:

Theorem 4.1. — *Let $X = \ell_1$. A typical $T \in (\mathcal{B}_1(X), \mathtt{SOT})$ has the following properties.*

(a) *T^* is a non-surjective isometry.*
(b) *$T - \lambda$ is surjective for every $\lambda \in \mathbb{D}$.*
(c) *Every $\lambda \in \mathbb{D}$ is an eigenvalue of T with infinite multiplicity.*
(d) *$\sigma(T) = \sigma_{ap}(T) = \overline{\mathbb{D}}$.*
(e) *$\|T^n x\| \to 0$ for every $x \in X$.*

Proof. — We already know that (e) holds true on any Banach space X; and (d) follows from (c) because $\sigma_{ap}(T)$ is a closed set containing $\sigma_p(T)$. So we just need to prove (a), (b) and (c).

(a) Since $X = \ell_1$ has the MAP, we know by Proposition 3.8 that a typical $T \in (\mathcal{B}_1(X), \mathtt{SOT})$ is such that T^* is non-invertible; so it is enough to show that a typical $T \in (\mathcal{B}_1(X), \mathtt{SOT})$ is such that T^* is an isometry. Let us denote by $\mathcal{I}_*$ the set of all $T \in \mathcal{B}_1(X)$ such that T^* is an isometry; we are going to show that $\mathcal{I}_*$ is a dense G_δ subset of $(\mathcal{B}_1(X), \mathtt{SOT})$.

The proof that $\mathcal{I}_*$ is G_δ works on any separable Banach space X. This relies on the following fact.

Fact 4.2. — The map $(T, x^*) \mapsto \|T^* x^*\|$ is lower-semicontinuous on the set $(\mathcal{B}_1(X), \mathtt{SOT}) \times (B_{X^*}, w^*)$.

Proof of Fact 4.2. — This is clear since $\|T^* x^*\| = \sup_{x \in B_X} |\langle x^*, Tx\rangle|$ and each map

$$(T, x^*) \mapsto \langle x^*, Tx\rangle, \quad x \in X$$

is continuous on $(\mathcal{B}_1(X), \mathtt{SOT}) \times (B_{X^*}, w^*)$. □

If $T \in \mathcal{B}_1(X)$, then

$$T \notin \mathcal{I}_* \iff \exists x^* \in B_{X^*}\ \exists \alpha \in \mathbb{Q}\ :\ \|T^* x^*\| \leqslant \alpha < \|x^*\|.$$

The condition $\|T^* x^*\| \leqslant \alpha$ defines a closed subset of $(\mathcal{B}_1(X), \mathtt{SOT}) \times (B_{X^*}, w^*)$ by Fact 4.2; and the condition $\|x^*\| > \alpha$ defines an open subset of (B_{X^*}, w^*), and hence an F_σ set in (B_{X^*}, w^*) because (B_{X^*}, w^*) is metrizable. So we see that $\mathcal{B}_1(X) \backslash \mathcal{I}_*$ is the projection along B_{X^*} of an F_σ subset of $(\mathcal{B}_1(X), \mathtt{SOT}) \times (B_{X^*}, w^*)$. Since (B_{X^*}, w^*) is compact, this shows that $\mathcal{I}_*$ is G_δ.

Now, let us show that $\mathcal{I}_*$ is SOT - dense in $\mathcal{B}_1(X)$ when $X = \ell_1$. Given an arbitrary $A \in \mathcal{B}_1(X)$, set $T_N := P_N A P_N + B_N(I - P_N)$, where $B_N : F_N \to X$ is the operator defined by $B_N e_{N+1+k} = e_k$ for every $k \geqslant 0$. Then $\|T_N e_j\| \leqslant 1$ for every $j \in \mathbb{Z}_+$, and hence $\|T_N\| \leqslant 1$ since we are working on $X = \ell_1$. Moreover, if $x^* \in X^*$ is arbitrary, then $\|T_N^* x^*\| \geqslant |\langle x^*, T_N e_{N+1+k}\rangle| = |\langle x^*, e_k\rangle|$ for every $k \geqslant 0$, and hence $\|T^* x^*\| \geqslant \|x^*\|$ since

$X^* = \ell_\infty$. So we have $T_N \in \mathcal{I}_*$ for all $N \geqslant 0$; and since $T_N \xrightarrow{\texttt{SOT}} A$ as $N \to \infty$, we conclude that $\mathcal{I}_*$ is indeed dense in $(\mathcal{B}_1(X), \texttt{SOT})$.

To prove (b) and (c), observe that if $T \in \mathcal{B}_1(X)$ is such that T^* is an isometry, then $T^* - \lambda$ is bounded below for every $\lambda \in \mathbb{D}$. By (a) and since the adjoint of an operator R is bounded below if and only if R is surjective, it follows that a typical $T \in (\mathcal{B}_1(X), \texttt{SOT})$ is such that $T - \lambda$ is surjective for every $\lambda \in \mathbb{D}$; which is (b). Since a typical $T \in (\mathcal{B}_1(X), \texttt{SOT})$ is also such that $T - \lambda$ is non-invertible for every $\lambda \in \mathbb{D}$ (by Proposition 3.9), it follows that a typical $T \in \mathcal{B}_1(X)$ is such that $T - \lambda$ is not one-to-one for every $\lambda \in \mathbb{D}$. However, in order to obtain (c), we still have to prove that for a typical $T \in (\mathcal{B}_1(X), \texttt{SOT})$, the multiplicity of every $\lambda \in \mathbb{D}$ as an eigenvalue of T is infinite. The proof of this statement relies on the following lemma:

Lemma 4.3. — *Let $X = \ell_1$. Let $\overline{\alpha} = (\alpha_i)_{i \geqslant 0}$ be a sequence of positive numbers with $\alpha_0 = \alpha_1 = 1$ and $\sum_{i \geqslant 0} \alpha_i < \infty$. Let also $\overline{N} = (N_i)_{i \geqslant 0}$ be a strictly increasing sequence of nonnegative integers with $N_0 = 0$. For every $q \geqslant 0$, let $\mathcal{D}_q(\overline{\alpha}, \overline{N})$ be the subset of $\mathcal{B}_1(X)$ defined as follows:*

$$T \in \mathcal{D}_q(\overline{\alpha}, \overline{N}) \iff \forall\, 0 \leqslant i_0 < \cdots < i_l \leqslant q \quad \textit{with} \quad \Big\| T\Big(\sum_{j=0}^{l} \alpha_j e_{N_{i_j}} \Big) \Big\| \leqslant \alpha_{l+1}$$
$$\exists\, i > q \ : \ \Big\| T\Big(\sum_{j=0}^{l} \alpha_j e_{N_{i_j}} + \alpha_{l+1} e_{N_i} \Big) \Big\| < \alpha_{l+2}.$$

Then $\mathcal{D}_q(\overline{\alpha}, \overline{N})$ is a dense open subset of $(\mathcal{B}_1(X), \texttt{SOT})$.

Proof of Lemma 4.3. — The set $\mathcal{D}_q(\overline{\alpha}, \overline{N})$ is clearly $\texttt{SOT}$-open in $\mathcal{B}_1(X)$. Let $T_0 \in \mathcal{B}_1(X)$, and let $\mathcal{U}$ be an $\texttt{SOT}$-open neighborhood of T_0 in $\mathcal{B}_1(X)$. We choose $r \geqslant q$ such that any operator $T \in \mathcal{B}_1(X)$ satisfying $Te_n = T_0 e_n$ for every $0 \leqslant n \leqslant N_r$ belongs to $\mathcal{U}$. Next, we enumerate the (finite) set Σ_q consisting of all sequences $(i_0, \dots, i_l)$ with

$$i_0 < \cdots < i_l \leqslant q \quad \text{and} \quad \Big\| T_0\Big(\sum_{j=0}^{l} \alpha_j e_{N_{i_j}} \Big) \Big\| \leqslant \alpha_{l+1}$$

as $\Sigma_q = \{\sigma_1, \dots, \sigma_s\}$, and write for each $k = 1, \dots, s$ the sequence σ_k as $(i_{k,0}, \dots, i_{k,l_k})$ for a certain integer $l_k \geqslant 0$. We have thus

$$\Big\| T_0\Big(\sum_{j=0}^{l_k} \alpha_j e_{N_{i_{k,j}}} \Big) \Big\| \leqslant \alpha_{l_k+1} \quad \text{for every} \quad k = 1, \dots, s.$$

Let now $i_1, \dots, i_s$ be integers with $r < i_1 < \cdots < i_s$, and define an operator T on X by setting

$$Te_n := \begin{cases} T_0 e_n & \text{for every } 0 \leqslant n \leqslant N_r, \\ -\dfrac{1}{\alpha_{l_k+1}}\, T_0\Big(\displaystyle\sum_{j=0}^{l_k} \alpha_j e_{N_{i_{k,j}}} \Big) & \text{if } n = N_{i_k} \text{ for some } 1 \leqslant k \leqslant s, \\ 0 & \text{in all other cases.} \end{cases}$$

Since $\|Te_n\| \leqslant 1$ for every $n \geqslant 0$ and $X = \ell_1$, we see that $\|T\| \leqslant 1$. Also,

$$\text{for every } k = 1, \dots, s, \qquad T\Big(\sum_{j=0}^{l_k} \alpha_j e_{N_{i_{k,j}}}\Big) = T_0\Big(\sum_{j=0}^{l_k} \alpha_j e_{N_{i_{k,j}}}\Big)$$

since $i_{k,j} \leqslant q \leqslant r$ and T and T_0 coincide on $[\, e_n \ ; \ 0 \leqslant n \leqslant N_r \,]$. Hence

$$T\Big(\sum_{j=0}^{l_k} \alpha_j e_{N_{i_{k,j}}} + \alpha_{l_k+1} e_{N_{i_k}}\Big) = 0 \qquad \text{for every } k = 1, \dots, s.$$

This implies that the operator T belongs to $\mathcal{D}_q(\overline{\alpha}, \overline{N})$. We have thus proved that $\mathcal{D}_q(\overline{\alpha}, \overline{N})$ is SOT - dense in $\mathcal{B}_1(X)$. □

The reason why these sets $\mathcal{D}_q(\overline{\alpha}, \overline{N})$ are introduced is given in the next lemma:

Lemma 4.4. — *Let $\overline{\alpha}$ and $\overline{N}$ satisfy the assumptions of* Lemma 4.3. *If $T \in \mathcal{B}_1(X)$ belongs to the set $\bigcap_{q\geqslant 0} \mathcal{D}_q(\overline{\alpha}, \overline{N})$, there exists a non-zero vector $x \in [\, e_{N_i} \ ; \ i \geqslant 0 \,]$ such that $Tx = 0$.*

Proof of Lemma 4.4. — Since $\alpha_0 = \alpha_1 = 1$ and $N_0 = 0$, we have $\|T(\alpha_0 e_{N_0})\| \leqslant \alpha_1$. Since $T \in \mathcal{D}_q(\overline{\alpha}, \overline{N})$, setting $i_0 = 0$, we obtain that there exists $i_1 > 0$ such that

$$\|T(\alpha_0 e_{N_{i_0}} + \alpha_1 e_{N_{i_1}})\| < \alpha_2.$$

An induction argument along the same line then shows the existence of a strictly increasing sequence of nonnegative integers $(i_j)_{j\geqslant 0}$ such that

$$\Big\|T\Big(\sum_{j=0}^{l} \alpha_j e_{N_{i_j}}\Big)\Big\| < \alpha_{l+1} \quad \text{for every} \quad l \geqslant 0.$$

The vector $x = \sum_{j\geqslant 0} \alpha_j e_{N_{i_j}}$ belongs to $[\, e_{N_i} \ ; \ i \geqslant 0 \,]$, is non-zero since $\alpha_j > 0$ for every $j \geqslant 0$, and satisfies $Tx = 0$. □

Thanks to Lemma 4.3 and 4.4, it is now not difficult to prove that an SOT - typical operator $T \in \mathcal{B}_1(X)$ satisfies $\dim \ker T = \infty$. Indeed, let $(\overline{N}_n)_{n\geqslant 1}$ be an infinite sequence consisting of strictly increasing sequences $\overline{N}_n = (N_{i,n})_{i\geqslant 0}$ of nonnegative integers with $N_{0,n} = 0$, with the property that the sets $\{N_{i,n} \ ; \ i \geqslant 1\}$, $n \geqslant 1$, are pairwise disjoint. Let also $\overline{\alpha} = (\alpha_i)_{i\geqslant 0}$ be any sequence of positive numbers with $\alpha_0 = \alpha_1 = 1$ and $\sum_{i\geqslant 0} \alpha_i < \infty$. By Lemma 4.3, the set

$$\mathcal{G} := \bigcap_{n\geqslant 1} \bigcap_{q\geqslant 0} \mathcal{D}_q(\overline{\alpha}, \overline{N}_n)$$

is a dense G_δ subset of $(\mathcal{B}_1(X), \texttt{SOT})$. The set $\mathcal{U}_0 \subseteq \mathcal{B}_1(X)$ consisting of operators $T \in \mathcal{B}_1(X)$ such that $Te_0 \neq 0$ being open and dense in $(\mathcal{B}_1(X), \texttt{SOT})$, $\mathcal{G} \cap \mathcal{U}_0$ is also a dense G_δ subset of $(\mathcal{B}_1(X), \texttt{SOT})$. By Lemma 4.4, if T belongs to $\mathcal{G} \cap \mathcal{U}_0$, there exists, for each $n \geqslant 1$, a non-zero vector $x_n \in [\, e_{N_{i,n}} \ ; \ i \geqslant 0 \,]$ such that $Tx_n = 0$. Since $Te_0 \neq 0$, x_n is not colinear to e_0, and the assumption that the sets $\{N_{i,n}, \ i \geqslant 1\}$, $n \geqslant 1$, are pairwise disjoint implies that the vectors x_n, $n \geqslant 1$, are linearly independent. So $\dim \ker T = \infty$.

At this point, we know that a typical $T \in \mathcal{B}_1(X)$ is such that $T - \lambda$ is surjective for every $\lambda \in \mathbb{D}$ and $\dim \ker(T) = \infty$. These two properties imply that $\dim \ker(T - \lambda) = \infty$ for every $\lambda \in \mathbb{D}$. Indeed, $T - \lambda$ is semi-Fredholm for every $\lambda \in \mathbb{D}$ since it is surjective. By the continuity of the Fredholm index, $\operatorname{ind}(T - \lambda)$ does not depend on $\lambda \in \mathbb{D}$. Now, we have $\operatorname{ind}(T) = \infty$ since T is surjective and $\dim \ker(T) = \infty$. So $\operatorname{ind}(T - \lambda) = \infty$ for every $\lambda \in \mathbb{D}$; and since $T - \lambda$ is surjective, it follows that $\dim \ker(T - \lambda) = \infty$. (See e.g. [23, p. 229-244] for a treatment of the Fredholm index for operators on Banach spaces.) This terminates the proof of assertion (c), and hence the proof of Theorem 4.1. $\square$

Remark 4.5. — Here is an alternative proof of part (c) in Theorem 4.1. Using [13, Lemma 5.13] (which holds true on ℓ_p for any $1 \leqslant p < \infty$), it is not too hard to show that for any $1 \leqslant p < \infty$, a typical $T \in \mathcal{B}_1(\ell_p)$ is not a Fredholm operator. Now, by (a) in Theorem 4.1, a typical $T \in \mathcal{B}_1(\ell_1)$ is surjective. Hence, a typical $T \in \mathcal{B}_1(\ell_1)$ satisfies $\dim\ker(T) = \infty$. Knowing that, one concludes the proof as above using the Fredholm index.

As an immediate consequence of property (c) in Theorem 4.1, we obtain:

Corollary 4.6. — *Let $X = \ell_1$. A typical $T \in (\mathcal{B}_1(X), \mathtt{SOT})$ has a non-trivial invariant subspace.*

5. SOT-typical contractions on ℓ_p and c_0

We have proved in the previous sections that a typical contraction $T \in (\mathcal{B}_1(X), \mathtt{SOT})$ for $X = \ell_2$ or $X = \ell_1$ has the property that T^* is an isometry on X^*. We begin this section by observing that if $X = \ell_p$ with $1 < p < \infty$ and $p \neq 2$, or if $X = c_0$, this is no longer true. Thus one cannot follow this route to show that a typical $T \in (\mathcal{B}_1(X), \mathtt{SOT})$ has a non-trivial invariant subspace.

Proposition 5.1. — *Let $X = c_0$ or ℓ_p, with $1 < p < \infty$ and $p \neq 2$. The set of all $T \in \mathcal{B}_1(X)$ such that T^* is an isometry is nowhere dense in $(\mathcal{B}_1(X), \mathtt{SOT})$. In particular, a typical $T \in (\mathcal{B}_1(X), \mathtt{SOT})$ is such that T^* is not an isometry.*

Proof. — The dual space of X is equal to ℓ_q for some $1 \leqslant q < \infty$ with $q \neq 2$. Now, the isometries of ℓ_q (or more generally the isometries of L_q spaces) are completely described by a classical result of Lamperti [25]. All we need to know is that if S is an isometry of ℓ_q, then S maps functions with disjoint supports to functions with disjoint supports: if $f, g \in \ell_q$ and $f \cdot g = 0$, then $(Sf) \cdot (Sg) = 0$ (see e.g. [6, Corollary 8.4] for a proof of this fact).

So if $T \in \mathcal{B}_1(X)$ is such that T^* is an isometry, then $(T^*e_0^*) \cdot (T^*e_1^*) = 0$. In other words, the set

$$\{T \in \mathcal{B}_1(X);\ T^* \text{ is an isometry}\}$$

is contained in

$$\mathcal{F} := \{T \in \mathcal{B}_1(X);\ \forall j \in \mathbb{Z}_+ \ :\ \langle e_0^*, Te_j\rangle = 0 \text{ or } \langle e_1^*, Te_j\rangle = 0\}.$$

The set $\mathcal{F}$ is obviously SOT-closed in $\mathcal{B}_1(X)$, and it is easily seen that its complement is SOT-dense in $\mathcal{B}_1(X)$. In fact, it is clear that for any fixed $j \geqslant 0$, the open set

$$\mathcal{O}_j := \{T \in \mathcal{B}_1(X);\ \langle e_0^*, Te_j\rangle \neq 0 \text{ and } \langle e_1^*, Te_j\rangle \neq 0\}$$

is already SOT-dense in $\mathcal{B}_1(X)$. This concludes the proof. $\square$

In view of the results from Sections 3 and 4 above, it is also a natural question to ask whether, for $X = \ell_p$ with $1 < p < \infty$ and $p \neq 2$ or $X = c_0$, a typical $T \in (\mathcal{B}_1(X), \mathtt{SOT})$ admits eigenvalues or not. This question turns out to be surprisingly difficult, and we are only able to answer it for $X = c_0$ and $X = \ell_p$ with $p > 2$.

Theorem 5.2. — *Let $X = c_0$. A typical $T \in (\mathcal{B}_1(X), \mathtt{SOT})$ has no eigenvalue.*

In the ℓ_p - case, we obtain the following stronger result.

Theorem 5.3. — *Let $X = \ell_p$ with $p > 2$. For any $M > 1$, a typical $T \in (\mathcal{B}_1(X), \texttt{SOT})$ is such that $(MT)^*$ is hypercyclic. In particular, a typical $T \in (\mathcal{B}_1(X), \texttt{SOT})$ has no eigenvalue.*

Recall that the adjoint of a hypercyclic operator cannot have any eigenvalue. Indeed, suppose that S is a hypercyclic operator acting on a Banach space Y, and suppose that $S^*y^* = \lambda y^*$ for some $y^* \in Y^*$ and $\lambda \in \mathbb{C}$. Then $\langle y^*, S^n y\rangle = \lambda^n \langle y^*, y\rangle$ for every $y \in Y$ and every $n \geqslant 0$. It follows that the set $\{\langle y^*, S^n y\rangle \,;\, n \geqslant 0\}$ is never dense in $\mathbb{C}$, and thus the vector y cannot be hypercyclic for the operator S.

The reason why we have to assume, in the statement of Theorem 5.3, that $p > 2$, lies in the well-known fact that the ℓ_p - norms have very different convexity and smoothness properties when $p > 2$ and when $1 < p < 2$. Many classical inequalities on ℓ_p - norms, like the Clarkson inequalities for instance, reverse when one moves over from the case $p > 2$ to the case $1 < p < 2$. Also, if $p > 2$, the p-th power $\|\cdot\|_p^p$ of the ℓ_p - norm is $\mathcal{C}^2$ - smooth, while it is only $\mathcal{C}^1$ - smooth when $1 < p < 2$ (see [11, Chapter 5] for more on smoothness of L^p - norms). More prosaically, we will also use our assumption that $p > 2$ through the following classical inequality (see e.g. [22, Lemma 2.1]):

Lemma 5.4. — *Let u and v be complex numbers with $v \neq 0$. If $2 < p < \infty$, we have*

$$|u+v|^p + |u-v|^p > 2\,|u|^p + p\,|u|^{p-2}\,|v|^2.$$

(If $0 < p < 2$ and $u \neq 0$, the reverse inequality holds.)

The proofs of Theorems 5.2 and 5.3 use rather different ideas. However, it is possible to give a direct proof of a weaker yet apparently non-trivial statement that follows the same pattern in the c_0 - case and in the ℓ_p - case. (The second part of the statement follows from Proposition 3.8.)

Corollary 5.5. — *Let $X = \ell_p$ with $p > 2$ or $X = c_0$. A typical $T \in (\mathcal{B}_1(X), \texttt{SOT})$ is one-to-one; and hence a typical $T \in \mathcal{B}_1(X)$ is not surjective.*

Accordingly we will first give this direct proof of Corollary 5.5, and then we will prove Theorem 5.2 and Theorem 5.3.

5a. Direct proof of Corollary 5.5. — To show that a typical $T \in \mathcal{B}_1(X)$ is one-to-one when $X = \ell_p$ for some $p > 2$ or $X = c_0$, we are going to write down explicitely a G_δ subset $\mathcal{G}$ of $(\mathcal{B}_1(X), \texttt{SOT})$ such that every $T \in \mathcal{G}$ is one-to-one and $\mathcal{G}$ is dense in $\mathcal{B}_1(X)$.

The definition of $\mathcal{G}$ is rather technical looking, but it does not require $p > 2$ in the ℓ_p case; so assume temporarily that $X = \ell_p$, $1 \leqslant p < \infty$ or $X = c_0$. For every integer $k \geqslant 1$, let $\mathcal{A}_k \subseteq \mathcal{B}_1(X)$ be defined in the following way:

$$T \in \mathcal{A}_k \iff \exists \varepsilon > 0,\ \exists\, 0 < \delta_1 \leqslant \delta_2,\ \exists\, N > k,\ \exists\, r_1, r_2 \in \mathbb{Z}_+ \quad \text{such that}$$

(i) $\big\|e_{r_1}^* T P_{(N,\infty)}\big\| < \varepsilon \cdot 10^{-(k+1)}$ and $\big\|e_{r_2}^* T P_{(N,\infty)}\big\| < \varepsilon \cdot 10^{-(k+1)}$;

(ii) $\big|\langle e_{r_1}^*, Tx\rangle\big| > |\varepsilon x_k| - |\delta_1 x_N| - \varepsilon \cdot 10^{-(k+1)}$ and $\big|\langle e_{r_2}^*, Tx\rangle\big| > |\delta_2 x_N| - \varepsilon \cdot 10^{-(k+1)}$ for every $x \in B_{E_N}$.

Now set

$$\mathcal{G} := \bigcap_{k \geqslant 1} \mathring{\mathcal{A}}_k,$$

where $\mathring{\mathcal{A}}_k$ is the $\texttt{SOT}$ - interior of $\mathcal{A}_k$ relative to $\mathcal{B}_1(X)$.

Lemma 5.6. — *Any operator $T \in \bigcap_{k\geqslant 1} \mathcal{A}_k$ is one-to-one.*

Proof. — Indeed, let $T \in \bigcap_{k\geqslant 1} \mathcal{A}_k$, and suppose that $x = \sum_{j\geqslant 0} x_j e_j \in X$ satisfies $\|x\| = 1$ and $Tx = 0$. Since $\|x\| = 1$, there exists $k \geqslant 0$ such that $|x_k| > 5 \cdot 10^{-(k+1)}$.

Since $T \in \mathcal{A}_k$, we may choose ε, δ_1, δ_2, N, r_1, and r_2 such that conditions (i) and (ii) above are satisfied. On the one hand, we have

$$\begin{aligned} 0 = |\langle e^*_{r_1}, Tx\rangle| &\geqslant |\langle e^*_{r_1}, TP_{[0,N]}x\rangle| - \|e^*_{r_1} TP_{(N,\infty)}\| \\ &> \varepsilon\, |x_k| - \delta_1 |x_N| - 2\varepsilon \cdot 10^{-(k+1)} > 3\varepsilon \cdot 10^{-(k+1)} - \delta_1 |x_N|. \end{aligned}$$

On the other hand,

$$\begin{aligned} 0 = |\langle e^*_{r_2}, Tx\rangle| &\geqslant |\langle e^*_{r_2}, TP_{[0,N]}x\rangle| - \|e^*_{r_2} TP_{(N,\infty)}\| \\ &> \delta_2 |x_N| - 2\varepsilon \cdot 10^{-(k+1)}. \end{aligned}$$

Hence $3\varepsilon \cdot 10^{-(k+1)} < \delta_1 |x_N| \leqslant \delta_2 |x_N| < 2\varepsilon \cdot 10^{-(k+1)}$, which is a contradiction. □

Lemma 5.7. — *Assume that $X = \ell_p$ for some $p > 2$ or $X = c_0$. Then, for every integer $k \geqslant 1$, the* `SOT` *-interior of $\mathcal{A}_k$ relative to $\mathcal{B}_1(X)$ is dense in $(\mathcal{B}_1(X), \texttt{SOT})$.*

Proof. — Fix $k \geqslant 1$. Let $\mathcal{U}$ be a non-empty open subset of $(\mathcal{B}_1(X), \texttt{SOT})$, and pick a finite rank operator $B \in \mathcal{U}$. There exist $\varepsilon \in (0, 1/4)$ and an integer $N > k + 1$ such that $\mathcal{U}$ contains all operators $T \in \mathcal{B}_1(X)$ satisfying $\|(T - B)e_j\| < 4\varepsilon$ for every $j = 0, \dots, N - 1$. We also fix an integer $M > N$ such that $\operatorname{Ran}(B) \subseteq E_M$.

Our aim is to exhibit an operator $A \in \mathcal{B}_1(X)$ belonging to the `SOT`-interior $\mathring{\mathcal{A}}_k$ of $\mathcal{A}_k$ in $\mathcal{B}_1(X)$, and satisfying $\|(A - B)e_j\| < 4\varepsilon$ for every $j = 0, \dots, N - 1$. We will have to treat separately in our discussion the cases $X = \ell_p$ and $X = c_0$.

Case 1. Suppose that $X = \ell_p$ for some $p > 2$, and let $\delta > 0$. We define an operator A in the following way:

$$Ae_j := \begin{cases} (1 - 2\varepsilon)\, Be_j & \text{if } 0 \leqslant j < N \text{ and } j \neq k, \\ (1 - 2\varepsilon)\, Be_k + \varepsilon\, e_{M+1} & \text{if } j = k, \\ \delta(e_{M+1} + e_{M+2}) & \text{if } j = N, \\ 0 & \text{if } j > N. \end{cases}$$

By the Intermediate Value Theorem, we can now choose $\delta > 0$ in such a way that we have $\|A\| = 1$. Obviously, $\|(A - B)e_j\| < 4\varepsilon$ for every $0 \leqslant j < N$. What remains to be shown is that if $T \in \mathcal{B}_1(X)$ is sufficiently close to A for the topology `SOT`, then $T \in \mathcal{A}_k$.

For every $x = \sum_{j\geqslant 0} x_j e_j \in X$, we have

$$\langle e^*_{M+1}, Ax\rangle = \varepsilon x_k + \delta x_N \quad \text{and} \quad \langle e^*_{M+2}, Ax\rangle = \delta x_N.$$

Moreover, $AP_{(N,\infty)} = 0$, and so A belongs to $\mathcal{A}_k$, the data witnessing this being ε, $\delta_1 = \delta_2 = \delta$, N, $r_1 = M + 1$, and $r_2 = M + 2$. If T is sufficiently close to A with respect to `SOT`, it will satisfy property (ii) of the definition of $\mathcal{A}_k$ for the same choices of ε, δ_1, δ_2, N, r_1, and r_2 (since the ball B_{E_N} is compact, the condition given by property (ii) is `SOT`-open). Let us now show that if T is sufficiently close to A for `SOT`, it also satisfies property (i).

Let $x \in E_N$ be a norming vector for A (*i.e.* $\|Ax\| = \|x\| = 1$). Since $\|A_{|E_{N-1}}\| \leqslant 1 - \varepsilon < 1$, x does not belong to E_{N-1}, so that we have $x_N \neq 0$. Hence $\langle e^*_{r_2}, Ax\rangle \neq 0$. Moreover, we also have $\langle e^*_{r_1}, Ax\rangle \neq 0$. Indeed, suppose that $\langle e^*_{r_1}, Ax\rangle = 0$, and consider the vector $x' = x - 2x_N e_N$. It satisfies $\|x'\| = \|x\| = 1$, and

$$Ax' = Ax - 2x_N\delta(e_{M+1} + e_{M+2}) = P_{[0,M]}Ax - 2x_N\delta e_{M+1} - x_N\delta e_{M+2}.$$

So

$$\|Ax'\|^p = \|P_{[0,M]}Ax\|^p + |2\delta x_N|^p + |\delta x_N|^p,$$

while

$$\|Ax\|^p = \|P_{[0,M]}Ax\|^p + |\delta x_N|^p = 1.$$

Hence $\|Ax'\| > 1$, which is a contradiction. So $\langle e^*_{r_1}, Ax\rangle \neq 0$. Let

$$c := \min\big(|\langle e^*_{r_1}, Ax\rangle|, |\langle e^*_{r_2}, Ax\rangle|\big),$$

and let $\eta \in (0,1)$. Any operator $T \in \mathcal{B}_1(X)$ which is sufficiently close to A for the topology $\texttt{SOT}$ satisfies the following properties:

$$|\langle e^*_{r_i}, Tx\rangle| \geqslant c/2,\ i = 1,2, \quad \text{and} \quad \|Tx\|^p \geqslant 1 - \eta.$$

The inequality from Lemma 5.4 now comes into play: for every $y \in F_N = [\, e_j\ ;\ j > N\,]$, for any $\lambda > 0$, and for every $j \geqslant 0$, we have

$$|\langle e^*_j, Tx + \lambda Ty\rangle|^p + |\langle e^*_j, Tx - \lambda Ty\rangle|^p \geqslant 2|\langle e^*_j, Tx\rangle|^p + p\lambda^2\, |\langle e^*_j, Tx\rangle|^{p-2}\, |\langle e^*_j, Ty\rangle|^2.$$

Summing over j, we obtain that

$$\|Tx + \lambda Ty\|^p + \|Tx - \lambda Ty\|^p \geqslant 2\|Tx\|^p + p\lambda^2 \sum_{j\geqslant 0} |\langle e^*_j, Tx\rangle|^{p-2}\, |\langle e^*_j, Ty\rangle|^2.$$

Hence we have for $i = 1, 2$,

$$\|Tx + \lambda Ty\|^p + \|Tx - \lambda Ty\|^p \geqslant 2\|Tx\|^p + p\lambda^2\, |\langle e^*_{r_i}, Tx\rangle|^{p-2}\, |\langle e^*_{r_i}, Ty\rangle|^2.$$

So if T is $\texttt{SOT}$ - close to A,

$$\|Tx + \lambda Ty\|^p + \|Tx - \lambda Ty\|^p \geqslant 2(1-\eta) + p\,\lambda^2\,(c/2)^{p-2}\, |\langle e^*_{r_i}, Ty\rangle|^2$$

for every $y \in F_N$ and every $\lambda > 0$.

On the other hand, since $\|T\| \leqslant 1$, and since the vectors x and y have disjoint supports, we also have

$$\begin{aligned}\|Tx + \lambda Ty\|^p + \|Tx - \lambda Ty\|^p &\leqslant \|x + \lambda y\|^p + \|x - \lambda y\|^p \\ &= 2(\|x\|^p + \lambda^p\|y\|^p) \\ &= 2 + 2\lambda^p\|y\|^p.\end{aligned}$$

Hence

$$|\langle e^*_{r_i}, Ty\rangle|^2 \leqslant \frac{2}{p\,(c/2)^{p-2}}\Big(\frac{\eta}{\lambda^2} + \lambda^{p-2}\Big)$$

for every $\lambda > 0$, and every $y \in F_N$ such that $\|y\| \leqslant 1$. If we choose first $\lambda > 0$ sufficiently small, and then $\eta \in (0,1)$ sufficiently small, we obtain that for all $T \in (\mathcal{B}_1(X), \texttt{SOT})$ sufficiently close to A,

$$\|e^*_{r_i} T P_{(N,\infty)}\| < \varepsilon \cdot 10^{-k}, \quad i = 1, 2$$

which is exactly property (i) from the definition of $\mathcal{A}_k$. This terminates the proof of Lemma 5.7 in the ℓ_p - case.

Case 2. Suppose that $X = c_0$. In this case, we define the operator A in the following way:

$$Ae_j := \begin{cases} Be_j & \text{if } 0 \leqslant j < N \text{ and } j \neq k, \\ Be_k + \varepsilon\, e_{M+1} & \text{if } j = k, \\ (1-\varepsilon)\, e_{M+1} + e_{M+2} & \text{if } j = N, \\ 0 & \text{if } j > N. \end{cases}$$

Then $\|A\| = 1$, and clearly $\|(A-B)e_j\| < 4\varepsilon$ for every $j = 0, \dots, N-1$. Moreover, for every $x = \sum_{j \geqslant 0} x_j e_j \in X$,

$$\langle e^*_{M+1}, Ax\rangle = \varepsilon x_k + (1-\varepsilon) x_N \quad \text{and} \quad \langle e^*_{M+2}, Ax\rangle = x_N.$$

Also $AP_{(N,\infty)} = 0$, and so A belongs to $\mathcal{A}_k$ for the choices of the associated data as ε, $\delta_1 = 1-\varepsilon$, $\delta_2 = 1$, N, $r_1 = M+1$, and $r_2 = M+2$. If T is sufficiently close to A with respect to $\mathtt{SOT}$, (ii) is clearly satisfied with the same choices of ε, δ_1, δ_2, N, r_1, and r_2, and what remains to be shown is that (i) is also satisfied.

The argument for this is much more direct than in the previous case, because if we set $u_1 := e_k + e_N$ and $u_2 := e_N$, we have $\langle e^*_{r_i}, Au_i\rangle = 1 = \|u_i\|$, $i = 1, 2$.

Let $\eta \in (0,1)$. If T is sufficiently close to A for $\mathtt{SOT}$ then $|\langle e^*_{r_i}, Tu_i\rangle| > 1 - \eta$, and hence $\|e^*_{r_i} TP_{[0,N]}\| > 1 - \eta$, $i = 1, 2$. The definition of the c_0 - norm implies that

$$\|e^*_{r_i} T\| = \|e^*_{r_i} TP_{[0,N]}\| + \|e^*_{r_i} TP_{(N,\infty)}\|,$$

and thus $\|e^*_{r_i} TP_{(N,\infty)}\| \leqslant \eta$, $i = 1, 2$. Choosing $\eta = \varepsilon \cdot 10^{-k}$, we obtain that any $T \in (\mathcal{B}_1(X), \mathtt{SOT})$ sufficiently close to A satisfies (i) with the choices of ε, δ_1, δ_2, N, r_1, and r_2 as above, and so $T \in \mathcal{A}_k$. ☐

Proof of Corollary 5.5. — Assuming that $X = \ell_p$, $p > 2$ or $X = c_0$, it follows from the above two lemmas that the set $\mathcal{G} = \bigcap_{k \geqslant 1} \mathring{\mathcal{A}}_k$ is a dense G_δ subset of $(\mathcal{B}_1(X), \mathtt{SOT})$ consisting of one-to-one operators. ☐

5b. Proof of Theorem 5.2. — We will make use of the so-called *Banach-Mazur game*, which is often a very effective tool for showing that a given set is comeager.

Let $\mathcal{E}$ be a Polish space, and let $\mathcal{A} \subseteq \mathcal{E}$. The Banach-Mazur game $\mathbf{G}(\mathcal{A})$ is an infinite game with two players, denoted by I and II. The two players play alternatively open sets $\mathcal{U}_0 \supseteq \mathcal{U}_1 \supseteq \mathcal{U}_2 \supseteq \cdots$; so, player I plays the open sets of even indices $\mathcal{U}_{2k}$ and player II plays the open sets of odd indices $\mathcal{U}_{2k+1}$. Then, player II wins the run if $\bigcap_{n \geqslant 0} \mathcal{U}_n \subseteq \mathcal{A}$. The key result concerning this game is the following (see e.g. [24, p. 51] for a proof): the set $\mathcal{A}$ is comeager in $\mathcal{E}$ if and only if player II has a winning strategy in $\mathbf{G}(\mathcal{A})$. Moreover, nothing changes if the open sets $\mathcal{U}_0, \mathcal{U}_1, \mathcal{U}_2, \dots$ are required to be picked from a given basis for the topology of $\mathcal{E}$.

In our setting, the space $\mathcal{E}$ is $(\mathcal{B}_1(c_0), \mathtt{SOT})$, and $\mathcal{A}$ is the set of all operators $T \in \mathcal{B}_1(c_0)$ having no eigenvalue. The basic open sets will be of the form

$$\mathcal{U}(N, A, \varepsilon) := \big\{T \in \mathcal{B}_1(c_0);\ \|Te_j - Ae_j\| < \varepsilon \text{ for } j = 0, \dots, N\big\},$$

where $N \in \mathbb{N}$, $A \in \mathcal{B}_1(E_N)$ and $0 < \varepsilon \leqslant 1$. So player I will play open sets of the form $\mathcal{U}_{2k} = \mathcal{U}(N_{2k}, A_{2k}, \varepsilon_{2k})$, and player II will play open sets of the form $\mathcal{U}_{2k+1} = \mathcal{U}(N_{2k+1}, A_{2k+1}, \varepsilon_{2k+1})$.

Let $(\alpha_k)_{k\geqslant 0}$ be a sequences of positive real numbers, with $\alpha_k \leqslant 1$. Let also $0 < c \leqslant 1/2$ and $C > 1$. We are going to describe a strategy for player II depending on (α_k), c and C, and then to show that this strategy is winning if (α_k), c and C are suitably chosen.

Assume that player I has just played a basic open set $\mathcal{U}_{2k} = \mathcal{U}(A_{2k}, N_{2k}, \varepsilon_{2k})$. Let us set

$$\tau_k := \alpha_k \varepsilon_{2k},$$

and choose a τ_k-net $\Lambda_k = \{\lambda_1, \dots, \lambda_{L_k}\}$ for the unit circle $\mathbb{T}$, with $\lambda_1 := 1$. Let us also choose an integer $R_k \geqslant 2$ such that

$$\left(\frac{1-\tau_k/2}{1-\tau_k}\right)^{R_k-2} > \frac{C}{\varepsilon_{2k}}.$$

Now, set

$$N_{2k+1} := N_{2k} + (L_k+1)R_k - 1,$$

and define an operator $A_{2k+1} \in \mathcal{B}(E_{N_{2k+1}})$ as follows :

$$A_{2k+1}e_j := A_{2k}e_j \qquad \text{if } j \leqslant N_{2k} \text{ and } j \neq k;$$

$$A_{2k+1}e_k := A_{2k}e_k + \sum_{i=1}^{L_k} \lambda_i \frac{\varepsilon_{2k}}{2}\, e_{N_{2k}+iR_k};$$

$$A_{2k+1}e_{N_{2k}+iR_k} := \lambda_i \left(1 - \frac{\varepsilon_{2k}}{2}\right) e_{N_{2k}+iR_k} \qquad \text{for every } 1 < i \leqslant L_k;$$

$$A_{2k+1}e_{N_{2k}+R_k} := \lambda_1 \left(1 - \frac{\varepsilon_{2k}}{2}\right) e_{N_{2k}+R_k} + e_{N_{2k}+R_k+1};$$

$$A_{2k+1}e_{N_{2k}+R_k+s} := e_{N_{2k}+R_k+s+1} \qquad \text{for every } 1 \leqslant s < R_k - 1;$$

$$A_{2k+1}e_j := 0 \qquad \text{otherwise.}$$

Observe that in fact $A_{2k+1} \in \mathcal{B}_1(E_{N_{2k+1}})$, *i.e.* $\|A_{2k+1}\| \leqslant 1$. Indeed, if $x = \sum_{j\geqslant 0} x_j e_j \in E_{N_{2k+1}}$, then

$$\begin{aligned}\|A_{2k+1}x\| = \max\Bigg\{ &\max_{0\leqslant j\leqslant N_{2k}} |\langle e_j^*, A_{2k}x\rangle|, \\ &\max_{1\leqslant i\leqslant L_k} \left|\frac{\varepsilon_{2k}}{2}\, x_k + \left(1-\frac{\varepsilon_{2k}}{2}\right) x_{N_{2k}+iR_k}\right|, \max_{1\leqslant s<R_k-1} |x_{N_{2k}+R_k+s}| \Bigg\} \\ \leqslant \|x\|.&\end{aligned}$$

Finally, set

$$\varepsilon_{2k+1} := c\,\tau_k\,\varepsilon_{2k}.$$

Then, the strategy of player II is to play the open set $\mathcal{U}(N_{2k+1}, A_{2k+1}, \varepsilon_{2k+1})$. Note that this is indeed allowed: since we are assuming that $c \leqslant 1/2$ and since $\tau_k \leqslant 1$, we have $\|A_{2k+1}e_j - A_{2k}e_j\| \leqslant \varepsilon_{2k}/2$ for $j = 0, \dots, N_{2k}$, and hence $\mathcal{U}(N_{2k+1}, A_{2k+1}, \varepsilon_{2k+1}) \subseteq \mathcal{U}(N_{2k}, A_{2k}, \varepsilon_{2k})$.

Before proving that this strategy is winning for II if (α_k), c and C are suitably chosen, we point out the following fact.

Fact 5.8. — Let $(\mathcal{U}_n)_{n\geqslant 0}$ be a run in the Banach-Mazur game where player II has followed the strategy described above. If $T\in\bigcap_{n\geqslant 0}\mathcal{U}_n$ then, for any $k\geqslant 0$, every $1\leqslant i\leqslant L_k$ and every $1\leqslant s<R_k-1$, we have

$$\|e^*_{N_{2k}+iR_k}TP_{\mathbb{Z}_+\setminus\{k,N_{2k}+iR_k\}}\|\leqslant 2\varepsilon_{2k+1}\quad\text{and}\quad\|e^*_{N_{2k}+R_k+s}TP_{\mathbb{Z}_+\setminus\{N_{2k}+R_k+s-1\}}\|\leqslant\varepsilon_{2k+1}.$$

Proof of Fact 5.8. — If $x\in c_0$ and $\|x\|\leqslant 1$, then $\|e_k+e_{N_{2k}+iR_k}+\omega P_{\mathbb{Z}_+\setminus\{k,N_{2k}+iR_k\}}x\|=1$ for every $\omega\in\mathbb{T}$, by definition of the c_0 - norm; so $\|T(e_k+e_{N_{2k}+iR_k}+\omega P_{\mathbb{Z}_+\setminus\{k,N_{2k}+iR_k\}}x)\|\leqslant 1$ for every $\omega\in\mathbb{T}$ and hence

$$\big|\langle e^*_{N_{2k}+iR_k},T(e_k+e_{N_{2k}+iR_k})\rangle\big|+\big|\langle e^*_{N_{2k}+iR_k},TP_{\mathbb{Z}_+\setminus\{k,N_{2k}+iR_k\}}x)\rangle\big|\leqslant 1.$$

On the other hand,

$$\begin{aligned}\big|\langle e^*_{N_{2k}+iR_k},T(e_k+e_{N_{2k}+iR_k})\rangle\big|&\geqslant\big|\langle e^*_{N_{2k}+iR_k}A_{2k+1}(e_k+e_{N_{2k}+iR_k})\rangle\big|\\&\qquad-\big\|(T-A_{2k+1})(e_k+e_{N_{2k}+iR_k})\big\|\\&=1-\big\|(T-A_{2k+1})(e_k+e_{N_{2k}+iR_k})\big\|\\&\geqslant 1-2\varepsilon_{2k+1}.\end{aligned}$$

Thus, we see that $|\langle e^*_{N_{2k}+iR_k},TP_{\mathbb{Z}_+\setminus\{k,N_{2k}+iR_k\}}x)\rangle|\leqslant 2\varepsilon_{2k+1}$ for every $x\in B_{c_0}$, *i.e.* that

$$\|e^*_{N_{2k}+iR_k}TP_{\mathbb{Z}_+\setminus\{k,N_{2k}+iR_k\}}\|\leqslant 2\varepsilon_{2k+1}.$$

Likewise, if $x\in B_{c_0}$ then, on the one hand,

$$\big|\langle e^*_{N_{2k}+R_k+s},Te_{N_{2k}+R_k+s-1}\rangle\big|+\big|\langle e^*_{N_{2k}+R_k+s},TP_{\mathbb{Z}_+\setminus\{N_{2k}+R_k+s-1\}}x\rangle\big|\leqslant 1$$

and, on the other hand,

$$\begin{aligned}\big|\langle e^*_{N_{2k}+R_k+s},Te_{N_{2k}+R_k+s-1}\rangle\big|&\geqslant\big|\langle e^*_{N_{2k}+R_k+s},A_{2k+1}e_{N_{2k}+R_k+s-1}\rangle\big|\\&\qquad-\|(T-A_{2k+1})e_{N_{2k}+R_k+s-1}\|\\&\geqslant 1-\varepsilon_{2k+1};\end{aligned}$$

and this shows that $\|e^*_{N_{2k}+R_k+s}TP_{\mathbb{Z}_+\setminus\{N_{2k}+R_k+s-1\}}\|\leqslant\varepsilon_{2k+1}$. □

We can now prove

Lemma 5.9. — *Assume that $c<1/24$, and choose $\eta\geqslant 16c$ such that $8c+\eta<1$. If $C>4/\eta$ and $\prod_{k=0}^{\infty}\frac{1-\alpha_k}{1+4\alpha_k}\geqslant 8c+\eta$, then the strategy described above is winning for player* II.

Proof. — Let $(\mathcal{U}_n)_{n\geqslant 0}$ be a run in the game where player II has followed the strategy above, and let $T\in\bigcap_{n\geqslant 0}\mathcal{U}_n$. We have to show that T has no eigenvalue. Towards a contradiction, assume that $Tx=\lambda x$ for some $x\in c_0$ such that $\|x\|=1$ and some $\lambda\in\mathbb{C}$. Note that $|\lambda|\leqslant 1$ since $\|T\|\leqslant 1$. In what follows, we write $x=\sum_{j\geqslant 0}x_je_j$.

Claim 5.10. — If $k\geqslant 0$ is such that $|x_k|\geqslant 8c+\eta$, then $1-\tau_k\leqslant|\lambda|$.

Proof of Claim 5.10. — Assume that $|\lambda| < 1 - \tau_k$ for some $k \geqslant 0$ such that $|x_k| \geqslant 8c + \eta$. Looking at the $(N_{2k} + R_k)$-th coordinate of $Tx - \lambda x$, we may write

$$\begin{aligned}
0 &= \left|\langle e^*_{N_{2k}+R_k}, Tx\rangle - \lambda\, x_{N_{2k}+R_k}\right| \\
&\geqslant \left|\langle e^*_{N_{2k}+R_k}, TP_{\{k,N_{2k}+R_k\}}x\rangle - \lambda\, x_{N_{2k}+R_k}\right| - \left|\langle e^*_{N_{2k}+R_k}, TP_{\mathbb{Z}_+\setminus\{k,N_{2k}+R_k\}}x\rangle\right| \\
&\geqslant \left|\langle e^*_{N_{2k}+R_k}, A_{2k+1}(x_k e_k + x_{N_{2k}+R_k} e_{N_{2k}+R_k})\rangle - \lambda\, x_{N_{2k}+R_k}\right| - \|(T - A_{2k+1})x_k e_k\| \\
&\qquad - \|(T - A_{2k+1})x_{N_{2k}+R_k} e_{N_{2k}+R_k}\| - \left|\langle e^*_{N_{2k}+R_k}, TP_{\mathbb{Z}_+\setminus\{k,N_{2k}+R_k\}}x\rangle\right| \\
&\geqslant \left|\frac{\varepsilon_{2k}}{2}\, x_k + \left(1 - \frac{\varepsilon_{2k}}{2} - \lambda\right)\, x_{N_{2k}+R_k}\right| - 4\varepsilon_{2k+1},
\end{aligned}$$

where we have used Fact 5.8. So $\left|\frac{\varepsilon_{2k}}{2}\, x_k + \left(1 - \frac{\varepsilon_{2k}}{2} - \lambda\right)\, x_{N_{2k}+R_k}\right| \leqslant 4\varepsilon_{2k+1}$. Since $|x_k| \geqslant 8c + \eta$, $\varepsilon_{2k+1} = c\tau_k\varepsilon_{2k} \leqslant c\,\varepsilon_{2k}$ and $\left|1 - \frac{\varepsilon_{2k}}{2} - \lambda\right| \leqslant 2$, it follows that

$$|x_{N_{2k}+R_k}| \geqslant \frac{(8c + \eta)\varepsilon_{2k}/2 - 4c\,\varepsilon_{2k}}{2} \geqslant \frac{\eta}{4}\,\varepsilon_{2k}.$$

Now, for any $1 \leqslant s < R_k - 1$, we have

$$\begin{aligned}
0 &= \left|\langle e^*_{N_{2k}+R_k+s}, Tx\rangle - \lambda\, x_{N_{2k}+R_k+s}\right| \\
&\geqslant \left|\langle e^*_{N_{2k}+R_k+s}, Tx_{N_{2k}+R_k+s-1} e_{N_{2k}+R_k+s-1}\rangle - \lambda\, x_{N_{2k}+R_k+s}\right| \\
&\qquad - \left\|e^*_{N_{2k}+R_k+s} TP_{\mathbb{Z}_+\setminus\{N_{2k}+R_k+s-1\}}\right\| \\
&\geqslant \left|\langle e^*_{N_{2k}+R_k+s}, A_{2k+1} x_{N_{2k}+R_k+s-1} e_{N_{2k}+R_k+s-1}\rangle - \lambda\, x_{N_{2k}+R_k+s}\right| \\
&\qquad - \|(T - A_{2k+1})e_{N_{2k}+R_k+s-1}\| - \left\|e^*_{N_{2k}+R_k+s} TP_{\mathbb{Z}_+\setminus\{N_{2k}+R_k+s-1\}}\right\| \\
&\geqslant |x_{N_{2k}+R_k+s-1} - \lambda\, x_{N_{2k}+R_k+s}| - 2\varepsilon_{2k+1} \\
&\geqslant |x_{N_{2k}+R_k+s-1}| - |\lambda|\ |x_{N_{2k}+R_k+s}| - 2\varepsilon_{2k+1}.
\end{aligned}$$

In other words:

$$|x_{N_{2k}+R_k+s}| \geqslant \frac{|x_{N_{2k}+R_k+s-1}| - 2\varepsilon_{2k+1}}{|\lambda|}.$$

Since $\varepsilon_{2k+1} = c\,\tau_k\varepsilon_{2k} \leqslant \frac{\eta}{16}\,\tau_k\varepsilon_{2k}$ and $|\lambda| < 1 - \tau_k$, it follows that

$$|x_{N_{2k}+R_k+s}| \geqslant \frac{|x_{N_{2k}+R_k+s-1}| - (\eta/8)\,\tau_k\varepsilon_{2k}}{1 - \tau_k} \qquad \text{for every } 1 \leqslant s < R_k - 1.$$

Recall that $|x_{N_{2k}+R_k}| \geqslant \frac{\eta\,\varepsilon_{2k}}{4}$. With $s := 1$, we get

$$|x_{N_{2k}+R_k+1}| \geqslant |x_{N_{2k}+R_k}|\,\frac{1 - \tau_k/2}{1 - \tau_k} \geqslant \frac{\eta\varepsilon_{2k}}{4}.$$

Iterating this $R_k - 2$ times, we obtain (keeping in mind that $C > 4/\eta$)

$$|x_{N_{2k}+R_k+R_k-2}| \geqslant \frac{\eta\,\varepsilon_{2k}}{4}\left(\frac{1 - \tau_k/2}{1 - \tau_k}\right)^{R_k-2} \geqslant \frac{\eta\,\varepsilon_{2k}}{4} \times \frac{C}{\varepsilon_{2k}} > 1,$$

which is a contradiction since $\|x\| = 1$. □

Claim 5.11. — If $k \geqslant 0$ is such that $|x_k| \geqslant 8c + \eta$, then one can find $k' > k$ such that

$$|x_{k'}| \geqslant \frac{1 - \alpha_k}{1 + 4\alpha_k}\,|x_k|.$$

Proof of Claim 5.11. — By Claim 5.10, we know that $|\lambda| \geqslant 1 - \tau_k$. Since the set $\Lambda_k = \{\lambda_1, \dots, \lambda_{L_k}\}$ is a τ_k - net for $\mathbb{T}$, it follows that one can find $1 \leqslant i \leqslant L_k$ such that $|\lambda - \lambda_i| < 2\tau_k$. Then, we may write

$$\begin{aligned}
0 &= \big|\langle e^*_{N_{2k}+iR_k}, Tx\rangle - \lambda\, x_{N_{2k}+iR_k}\big| \\
&\geqslant \big|\langle e^*_{N_{2k}+iR_k}, T(x_k e_k + x_{N_{2k}+iR_k} e_{N_{2k}+iR_k})\rangle - \lambda\, x_{N_{2k}+iR_k}\big| - \big\|e^*_{N_{2k}+iR_k} T P_{\mathbb{Z}_+\setminus\{k, N_{2k}+iR_k\}}\big\| \\
&\geqslant \big|\langle e^*_{N_{2k}+iR_k}, A_{2k+1}(x_k e_k + x_{N_{2k}+iR_k} e_{N_{2k}+iR_k})\rangle - \lambda\, x_{N_{2k}+iR_k}\big| - \|(T - A_{2k+1}) x_k e_k\| \\
&\qquad - \|(T - A_{2k+1}) x_{N_{2k}+iR_k} e_{N_{2k}+iR_k}\| - \big\|e^*_{N_{2k}+iR_k} T P_{\mathbb{Z}_+\setminus\{k, N_{2k}+iR_k\}}\big\| \\
&\geqslant \left|\lambda_i \frac{\varepsilon_{2k}}{2}\, x_k + \left(\lambda_i\left(1 - \frac{\varepsilon_{2k}}{2}\right) - \lambda\right) x_{N_{2k}+iR_k}\right| - 4\varepsilon_{2k+1}.
\end{aligned}$$

So we have $\big|\lambda_i \frac{\varepsilon_{2k}}{2}\, x_k + \big(\lambda_i\big(1 - \frac{\varepsilon_{2k}}{2}\big) - \lambda\big) x_{N_{2k}+iR_k}\big| \leqslant 4\varepsilon_{2k+1} = 4c\tau_k\varepsilon_{2k}$. Since $|x_k| \geqslant 8c + \eta$ and $\big|\lambda_i\big(1 - \frac{\varepsilon_{2k}}{2}\big) - \lambda\big| \leqslant \frac{\varepsilon_{2k}}{2} + 2\tau_k$, it follows that

$$|x_{N_{2k}+iR_k}| \geqslant \frac{\varepsilon_{2k}/2 - 4c\tau_k\varepsilon_{2k}/(8c+\eta)}{\varepsilon_{2k}/2 + 2\tau_k}\, |x_k| = \frac{1 - 8c\tau_k/(8c+\eta)}{1 + 4\tau_k/\varepsilon_{2k}}\, |x_k|.$$

Since $\tau_k = \alpha_k \varepsilon_{2k}$ and $\varepsilon_{2k} \leqslant 1$, this implies that

$$|x_{N_{2k}+iR_k}| \geqslant \frac{1 - \alpha_k}{1 + 4\alpha_k}\, |x_k|.$$

So we may set $k' := N_{2k} + iR_k$. □

It is now easy to conclude the proof of Lemma 5.9. Since $\|x\| = 1$, we can choose $k_0 \geqslant 0$ such that $|x_{k_0}| = 1$. Then, since we are assuming that $\prod_{k=0}^{\infty} \frac{1-\alpha_k}{1+4\alpha_k} \geqslant 8c + \eta$, we may use Claim 5.32 to find an increasing sequence of integers $(k_l)_{l \geqslant 0}$ such that $|x_{k_l}| \geqslant 8c + \eta$ for all $l \geqslant 0$. Since $x \in c_0$, this is the required contradiction. □

By Lemma 5.9, the proof of Theorem 5.2 is now complete.

5c. Outline of the proof of Theorem 5.3. — The proof of Theorem 5.3 will now occupy us for a while. The main step will be to prove the following statement, which interrelates in a rather curious way the topologies SOT and SOT* on $\mathcal{B}_1(\ell_p)$ when $p > 2$.

Theorem 5.12. — *Let $X = \ell_p$, $p > 2$. If $\mathcal{O} \subseteq \mathcal{B}_1(X)$ is* SOT* *- open, then the relative* SOT *- interior of $\mathcal{O}$ in $\mathcal{B}_1(X)$ is* SOT* *- dense in $\mathcal{O}$, and hence* SOT *- dense as well.*

From this, we immediately deduce

Corollary 5.13. — *Let $X = \ell_p$, $p > 2$. If $\mathcal{G} \subseteq \mathcal{B}_1(X)$ is* SOT* *- G_δ and* SOT *- dense in $\mathcal{B}_1(X)$, then $\mathcal{G}$ is* SOT *- comeager in $\mathcal{B}_1(X)$. Therefore, any* SOT* *- comeager subset of $\mathcal{B}_1(X)$ is* SOT *- comeager as well.*

Proof. — Choose a sequence $(\mathcal{O}_k)_{k\in\mathbb{N}}$ of SOT*- open subsets of $\mathcal{B}_1(X)$ such that $\bigcap_{k\in\mathbb{N}} \mathcal{O}_k = \mathcal{G}$. Then each set $\mathcal{O}_k$ is SOT - dense in $\mathcal{B}_1(X)$. Denoting by $\overset{\circ}{\mathcal{O}}_k$ the relative SOT - interior of $\mathcal{O}_k$ in $\mathcal{B}_1(X)$, it follows from Theorem 5.12 that each $\overset{\circ}{\mathcal{O}}_k$ is (open and) dense in $(\mathcal{B}_1(X), \texttt{SOT})$. Hence $\mathcal{G}$ is SOT - comeager since it contains $\bigcap_{k\in\mathbb{N}} \overset{\circ}{\mathcal{O}}_k$. □

The other ingredient is the following result. This was proved for $p = 2$ in [18], and essentially the same proof would give the result for an arbitrary p.

Proposition 5.14. — *Let $X = \ell_p$, $1 < p < \infty$. For any real number $M > 1$, the set $\{T \in \mathcal{B}_1(X);\ (MT)^*$ is hypercyclic and $\|T\| = 1\}$ is a dense G_δ subset of $\big(\mathcal{B}_1(X), \texttt{SOT}^*\big)$.*

Proof. — The proof of [18, Proposition 2.3], which is given for $p = 2$ but works in fact for any $1 < p < \infty$, shows that the set of hypercyclic operators is G_δ and dense in $\big(\mathcal{B}_M(X), \texttt{SOT}^*\big)$. Since the map $T \mapsto (MT)^*$ is a homeomorphism from $\big(\mathcal{B}_1(X), \texttt{SOT}^*\big)$ onto $\big(\mathcal{B}_M(X^*), \texttt{SOT}^*\big)$, it follows that the set $\{T \in \mathcal{B}_1(X);\ (MT)^*$ is hypercyclic $\}$ is G_δ and dense in $\big(\mathcal{B}_1(X), \texttt{SOT}^*\big)$. Moreover, the set $\{T \in \mathcal{B}_1(X);\ \|T\| = 1\}$ is also G_δ and dense in $\big(\mathcal{B}_1(X), \texttt{SOT}^*\big)$; in fact, $\texttt{SOT}$ - G_δ and $\texttt{SOT}^*$- dense. ☐

Proof of Theorem 5.3. — The result follows immediatey from Corollary 5.13 and Proposition 5.14. ☐

5d. Outline of the proof of Theorem 5.12. — Let us recall our notation: the canonical basis of $X = \ell_p$ is denoted by $(e_j)_{j\geqslant 0}$, its dual basis by $(e_j^*)_{j\geqslant 0}$, and for each $N \geqslant 0$,

$$E_N = [\, e_0, \dots, e_N\,] \qquad \text{and} \qquad F_N = [\, e_j\ ;\ j > N\,].$$

The canonical projection of X onto E_N is denoted by P_N. For any operator $T \in \mathcal{B}(X)$, we identify the operator $P_N T P_N \in \mathcal{B}(X)$ with $P_N T_{|E_N} \in \mathcal{B}(E_N)$. So we consider $P_N T P_N$ both as an operator on X and as an operator on E_N. Likewise, we may consider any operator on $A \in \mathcal{B}(E_N)$ as an operator on X by identifying it with $P_N A P_N$.

In what follows, we will use the following rather *ad hoc* terminology.

Recall that a vector x is said to be *norming* for an operator A if $\|x\| = 1$ and $\|Ax\| = \|A\|$. Given $N \geqslant 0$, we will say that an operator $A \in \mathcal{B}(E_N)$ is *absolutely exposing* if the set of all norming vectors for A is as small as possible, i.e consists only of unimodular multiples of a single vector $x_0 \in S_{E_N}$. We denote by $\mathcal{E}_1(E_N)$ the set of absolutely exposing operators $A \in \mathcal{B}_1(E_N)$. Moreover, we say that an absolutely exposing operator $A \in \mathcal{B}(E_N)$ is *evenly distributed* if for any norming vector $x \in S_{E_N}$ for A, we have $\langle e_j^*, x\rangle \neq 0$ and $\langle e_j^*, Ax\rangle \neq 0$ for every $j \in [0, N]$.

The key steps in the proof of Theorem 5.12 are the next two propositions, whose proofs are postponed to a later section.

Proposition 5.15. — *Let $X = \ell_p$, $1 < p < \infty$. Let also N be a non-negative integer, and let $A \in \mathcal{B}_1(E_N)$. For any $\varepsilon > 0$, there exists an operator $B \in \mathcal{B}_1(E_{2N+1})$ such that*

(i) $\|B\| = 1$;
(ii) *B is absolutely exposing;*
(iii) *B is evenly distributed;*
(iv) $\|BP_N - A\| < \varepsilon$.

Moreover, if $\|A\|$ is sufficiently close to 1, then one may require that in fact

(iv') $\|B - A\| < \varepsilon$.

This first proposition is valid on any ℓ_p - space with $p > 1$. Our assumption that $p > 2$ will be needed only in the next statement:

Proposition 5.16. — *Let $X = \ell_p$, $p > 2$. Fix an integer $M \geqslant 0$, and suppose that $B \in \mathcal{B}_1(E_M)$ is such that $\|B\| = 1$, B is absolutely exposing, and B is evenly distributed. Then, for any $\varepsilon > 0$, there exists $\delta > 0$ such that the following holds true:*

$$\text{if } T \in \mathcal{B}_1(X) \text{ satisfies } \|P_M T P_M - B\| < \delta, \text{ then } \|P_M T(I - P_M)\| < \varepsilon.$$

Taking the above two propositions for granted, we can now give the

Proof of Theorem 5.12. — For any $\mathcal{A}\subseteq\mathcal{B}_1(X)$, we denote by $\mathring{\mathcal{A}}$ the relative SOT-interior of $\mathcal{A}$ in $\mathcal{B}_1(X)$. It is in fact enough to show that if $\mathcal{O}\subseteq\mathcal{B}_1(X)$ is SOT*-open and $\mathcal{O}\neq\varnothing$, then $\mathring{\mathcal{O}}\neq\varnothing$.

Since the unit sphere of $\mathcal{B}(X)$ is SOT*-dense in $\mathcal{B}_1(X)$, one can find $T_0\in\mathcal{O}$ with $\|T\|=1$. Choose $K\in\mathbb{N}$ and ε such that for any $T\in\mathcal{B}_1(X)$, the following implication holds true:

$$\big(\,\|(T-T_0)P_K\|<\varepsilon\quad\text{and}\quad\|(T^*-T_0^*)P_K^*\|<\varepsilon\,\big)\implies T\in\mathcal{O}.$$

Next, choose $L\geqslant K$ such that, setting $A_N:=P_NT_0P_N$, we have

$$\forall N\geqslant L\;:\;\|(A_N-T_0)P_K\|<\varepsilon/3\quad\text{and}\quad\|(A_N^*-T_0^*)P_K^*\|<\varepsilon/3.$$

Now, let $\alpha>0$ be very small, and fix $N\geqslant L$ such that $\|A_N\|>1-\alpha$.

If α is small enough then, by Proposition 5.15, one can find $M>N$ and $B\in\mathcal{B}_1(E_M)$ such that $\|B\|=1$, B is absolutely exposing and evenly distributed, and $\|B-A_N\|<\varepsilon/3$.

By Proposition 5.16, one can find $\gamma>0$ such that for any $T\in\mathcal{B}_1(X)$, the following implication holds true:

$$\|P_MTP_M-B\|<\gamma\implies\|P_MT(I-P_M)\|<\varepsilon/6.$$

We assume without loss of generality that $\gamma<\varepsilon/6$. Then $\|P_MT-B\|<\gamma+\varepsilon/6<\varepsilon/3$ if $\|P_MTP_M-B\|<\gamma$; and since $P_MTP_M-B=P_M(TP_M-B)$, we see that

$$\|TP_M-B\|<\gamma\implies\|P_MT-B\|<\varepsilon/3.$$

Now, set

$$\mathcal{U}:=\big\{T\in\mathcal{B}_1(X);\;\|TP_M-B\|<\gamma\big\}.$$

The set $\mathcal{U}$ is SOT-open, and $\mathcal{U}\neq\varnothing$ since $B\in\mathcal{U}$. Let us show that $\mathcal{U}\subseteq\mathcal{O}$. It is enough to check that if $T\in\mathcal{U}$, then $\|(T-T_0)P_K\|<\varepsilon$ and $\|(T^*-T_0^*)P_K^*\|<\varepsilon$.

On the one hand, we have

$$\|(T-T_0)P_K\|\leqslant\|(T-A_N)P_K\|+\|(A_N-T_0)P_K\|<\|(T-A_N)P_K\|+\varepsilon/3.$$

Moreover, $\|(T-A_N)P_K\|\leqslant\|(T-B)P_K\|+\|(B-A_N)P_K\|<\|(T-B)P_K\|+\varepsilon/3$, and $\|(T-B)P_K\|=\|(TP_M-B)P_K\|<\gamma<\varepsilon/3$. Hence $\|(T-T_0)P_K\|<\varepsilon$.

On the the other hand, $\|(T^*-T_0^*)P_K^*\|<\|(T^*-A_N^*)P_K^*\|+\varepsilon/3<\|(T^*-B^*)P_K^*\|+2\varepsilon/3$. Moreover, since $\|TP_M-B\|<\gamma$, we have $\|P_MT-B\|<\varepsilon/3$ and hence $\|T^*P_M^*-B^*\|<\varepsilon/3$. So we get $\|(T^*-B^*)P_K^*\|=\|(T^*P_M^*-B^*)P_K^*\|<\varepsilon/3$, and hence $\|(T^*-T_0^*)P_K^*\|<\varepsilon$. □

5e. Proofs of Propositions 5.15 and 5.16. — Since it is the shortest, we start with the proof of Proposition 5.16.

Proof of Proposition 5.16. — Let $\delta>0$ (how small δ needs to be will be determined during the proof) and let $T\in\mathcal{B}_1(X)$ be such that $\|P_MTP_M-B\|<\delta$. Let $x_1\in S_{E_M}$ be a norming vector for B. For every non-zero vector $y\in F_M=[\,e_i\,;\,i>M]$ and every scalar $\mu>0$, we have by Lemma 5.4 that

$$\begin{aligned}|\langle e_j^*,Tx_1+\mu Ty\rangle|^p+|\langle e_j^*,Tx_1-\mu Ty\rangle|^p\geqslant 2\,|\langle e_j^*,Tx_1\rangle|^p&\\ +\,p\,\mu^2\,|\langle e_j^*,Tx_1\rangle|^{p-2}|\langle e_j^*,Ty\rangle|^2&\end{aligned}$$

for every $j \geqslant 0$, which can be rewritten as

$$p\mu^2\, |\langle e_j^*, Tx_1\rangle|^{p-2}\, |\langle e_j^*, Ty\rangle|^2 \leqslant |\,\langle e_j^*, Tx_1 + \mu Ty\rangle\,|^p + |\,\langle e_j^*, Tx_1 - \mu Ty\rangle\,|^p - 2\,|\,\langle e_j^*, Tx_1\rangle\,|^p.$$

Summing over $j \geqslant 0$ yields that

$$\begin{aligned} p\mu^2 \sum_{j\geqslant 0} |\langle e_j^*, Tx_1\rangle|^{p-2}\,|\,\langle e_j^*, Ty\rangle\,|^2 &\leqslant \|T(x_1+\mu y)\|^p + \|T(x_1-\mu y)\|^p - 2\,\|Tx_1\|^p \\ &\leqslant \|x_1+\mu y\|^p + \|x_1-\mu y\|^p - 2\|Tx_1\|^p. \end{aligned}$$

Since $\|Tx_1\| > 1-\delta$ and since the vectors x_1 and y have disjoint supports, it follows that

$$p\,\mu^2 \sum_{j\geqslant 0} |\langle e_j^*, Tx_1\rangle|^{p-2}\,|\langle e_j^*, Ty\rangle|^2 \leqslant 2\left(1+\mu^p\,\|y\|^p - (1-\delta)^p\right).$$

Let us temporarily write

$$\alpha := \sum_{j\geqslant 0} |\langle e_j^*, Tx_1\rangle|^{p-2}\,|\langle e_j^*, Ty\rangle|^2.$$

Then $\|y\|^p\mu^p - (p\alpha/2)\mu^2 + 1 - (1-\delta)^p \geqslant 0$ for every $\mu > 0$. Optimizing this inequality with respect to μ, *i.e.* taking $\mu := \left(\frac{\alpha}{\|y\|^p}\right)^{1/(p-2)}$, we obtain

$$\frac{\alpha^{\,p/(p-2)}}{\|y\|^{\,2p/(p-2)}}\left(1-\frac{p}{2}\right) + \left(1-(1-\delta)^p\right) \geqslant 0;$$

which can be rewritten as

$$\frac{p-2}{2}\left(\frac{\alpha}{\|y\|^2}\right)^{p/(p-2)} \leqslant 1-(1-\delta)^p \leqslant p\delta$$

i.e.

$$0 \leqslant \alpha \leqslant \|y\|^2\cdot K_p\cdot \delta^{(p-2)/p}, \quad \text{where} \quad K_p := \left(\frac{2p}{p-2}\right)^{(p-2)/p}. \tag{5.1}$$

Our assumption on the operator B implies that

$$\gamma := \min_{j\in[0,M]} |\langle e_j^*, Bx_1\rangle| > 0.$$

Let δ be so small that every operator $T \in \mathcal{B}_1(E_M)$ with $\|P_MTP_M - B\| < \delta$ satisfies $\min_{j\in[0,M]} |\langle e_j^*, Tx_1\rangle| > \gamma/2$. For every $j \in [0,M]$, we have by (5.1) that

$$|\langle e_j^*, Ty\rangle|^2 \leqslant \|y\|^2\cdot K_p\cdot\left(\frac{2}{\gamma}\right)^{p-2}\cdot\delta^{(p-2)/p},$$

i.e.

$$|\langle e_j^*, Ty\rangle|^p \leqslant \|y\|^p\cdot K_p^{p/2}\cdot\left(\frac{2}{\gamma}\right)^{\frac{p(p-2)}{2}}\cdot\delta^{(p-2)/2}.$$

Summing over $j \in [0,M]$, we obtain that

$$\|P_MTy\| \leqslant \|y\|\cdot K_p^{1/2}\cdot M^{1/p}\cdot\left(\frac{2}{\gamma}\right)^{\frac{p-2}{2}}\cdot\delta^{\frac{p-2}{2p}}.$$

If we choose $\delta > 0$ so small that $K_p^{1/2}\cdot M^{1/p}\cdot(2/\gamma)^{\frac{p-2}{2}}\cdot\delta^{\frac{p-2}{2p}} < \varepsilon$ (this condition depends on p, M, and B, but not on T), we obtain that $\|P_MT(I-P_M)\| < \varepsilon$, which is exactly the inequality we were looking for. □

It remains to prove Proposition 5.15, and this will be a bit longer. Recall that for each $N \geqslant 1$, we denote by $\mathcal{E}_1(E_N)$ the set of absolutely exposing operators of $\mathcal{B}_1(E_N)$. We begin by a series of rather elementary lemmas which shed some light on the behaviour of absolutely exposing contractions.

Lemma 5.17. — *The set $\mathcal{E}_1(E_N)$ is dense in $\mathcal{B}_1(E_N)$.*

Proof. — Let $A \in \mathcal{B}_1(E_N)$ with $A \neq 0$ and $\|A\| < 1$. Let $x_0 \in S_{E_N}$ be such that $\|Ax_0\| = \|A\|$, and let $x_0^* \in E_N$ be such that $\|x_0\| = \langle x_0^*, x_0\rangle = 1$. Let R_0 be the rank 1 operator on E_N defined by $R_0(x) := \langle x_0^*, x\rangle\, Ax_0$, $x \in E_N$; and for any $\delta > 0$, let $A_\delta := A + \delta R_0$. Then $A_\delta x_0 = (1+\delta)Ax_0$, and $\|A_\delta x_0\| = (1+\delta)\|Ax_0\| = (1+\delta)\|A\|$. Since $\|R_0\| = \|A\|$, $\|A_\delta\| \leqslant (1+\delta)\|A\|$, and hence $\|A_\delta\| = (1+\delta)\|A\|$. So $\|A_\delta x_0\| = \|A_\delta\|$. Moreover, if $x \in S_{E_N}$ is such that $\|A_\delta x\| = \|A_\delta\|$, then $\|Ax + \delta\,\langle x_0^*, x\rangle\, Ax_0\| = (1+\delta)\|A\|$. So $\langle x_0^*, x\rangle \neq 0$ and

$$\begin{aligned}(1+\delta)\|A\| = \|Ax + \delta\langle x_0^*, x\rangle\, Ax_0\| &\leqslant \|A\|\, \|x + \delta\,\langle x_0^*, x\rangle\, x_0\|\\ &\leqslant \|A\|\,\big(\|x\| + \delta\,|\langle x_0^*, x\rangle|\,\|x_0\|\big) \leqslant \|A\|\,(1+\delta).\end{aligned}$$

Since $\|A\| > 0$, it follows that $\|x + \delta\,\langle x_0^*, x\rangle\, x_0\| = \|x\| + \|\delta\,\langle x_0^*, x\rangle\, x_0\|$, and since $\langle x_0^*, x\rangle \neq 0$, this implies that x is colinear to x_0, by strict convexity of the ℓ_p - norm. So A_δ is absolutely exposing. Given $\varepsilon > 0$, one can choose $\delta > 0$ so small that $\|A_\delta\| < 1$ and $\|A - A_\delta\| < \varepsilon$, and this proves that $\mathcal{E}_1(E_N)$ is dense in $\mathcal{B}_1(E_N)$. ☐

Lemma 5.18. — *Let $A \in \mathcal{E}_1(E_N)$, and let $x_0 \in S_{E_N}$ be a norming vector for A. For every $\varepsilon > 0$, there exists $\delta > 0$ such that any vector $x \in S_{E_N}$ such that $\|Ax\| > (1-\delta)\|A\|$ satisfies* $\operatorname{dist}(x, \mathbb{T}x_0) < \varepsilon$.

Proof. — Fix $\varepsilon > 0$ and, towards contradiction, suppose that for every $n \geqslant 1$, there exists a vector $x_n \in S_{E_N}$ such that $\|Ax_n\| > (1 - 2^{-n})\|A\|$ and $\operatorname{dist}(x_n, \mathbb{T}x_0) \geqslant \varepsilon$. Without loss of generality, we can suppose that the sequence $(x_n)_{n\geqslant 1}$ converges to a vector $x_\infty \in S_{E_N}$. Then $\|Ax_\infty\| = \|A\|$ and thus $x_\infty \in \mathbb{T}x_0$, which is a contradiction. ☐

Lemma 5.19. — *Let $A \in \mathcal{E}_1(E_N)$, and let $x_0 \in S_{E_N}$ be a norming vector for A. For every $\varepsilon > 0$, there exists $\gamma > 0$ with the following property: for every operator $B \in \mathcal{B}_1(E_N)$ with $\|A - B\| < \gamma$ and any norming vector $x \in S_{E_N}$ for B, we have* $\operatorname{dist}(x, \mathbb{T}x_0) < \varepsilon$.

Proof. — Given $A \in \mathcal{B}_1(E_N)$ and $\varepsilon > 0$, let $\delta > 0$ be given by Lemma 5.18: if $x \in S_{E_N}$ is such that $\|Ax\| > (1-\delta)\|A\|$, then $\operatorname{dist}(x, \mathbb{T}x_0) < \varepsilon$. Let $\gamma > 0$ be so small that $\|A\| - 2\gamma \geqslant (1-\delta)\|A\|$. If $B \in \mathcal{B}_1(E_N)$ is such that $\|B - A\| < \gamma$, and if $x \in S_{E_N}$ is a norming vector for B, then $\big|\,\|B\| - \|Ax\|\,\big| < \gamma$, and since $\big|\,\|B\| - \|A\|\,\big| < \gamma$, one has $\big|\,\|A\| - \|Ax\|\,\big| < 2\gamma$. Hence $\|Ax\| > \|A\| - 2\gamma \geqslant (1-\delta)\|A\|$, and so $\operatorname{dist}(x, \mathbb{T}x_0) < \varepsilon$. ☐

Some of the arguments in the proofs to come will involve a duality mapping $\mathbf{J} : \ell_p \to \ell_{p'}$, where $1/p + 1/p' = 1$. Recall that we are assuming that $1 < p < \infty$. The map $\mathbf{J}$ is defined as follows: if $x \in X = \ell_p$, then $\mathbf{J}(x)$ is the unique element of $X^* = \ell_{p'}$ such that $\langle \mathbf{J}(x), x\rangle = \|\mathbf{J}(x)\|\,\|x\| = \|x\|^p$ (there are several natural ways of "normalizing" the duality mapping when defining it on the whole space rather than just on the unit sphere). Explicitely, we have:

$$\mathbf{J}(x) = \sum_{j\geqslant 1} \overline{\langle e_j^*, x\rangle} \cdot \big|\langle e_j^*, x\rangle\big|^{p-2}\, e_j^*, \quad x \in X.$$

Recall also that an absolutely exposing operator $B \in \mathcal{E}_1(E_N)$ is said to be evenly distributed if for every norming vector $x_1 \in S_{E_N}$ for B, we have $\langle e_j^*, x_1\rangle \neq 0$ and $\langle e_j^*, Bx_1\rangle \neq 0$ for all $j \in [0, N]$.

Proposition 5.20. — *Let $A \in \mathcal{E}_1(E_N)$ be such that $A \neq 0$ and $\|A\| < 1$. For every $\varepsilon > 0$, there exists an operator $B \in \mathcal{E}_1(E_N)$ such that $\|B - A\| < \varepsilon$ and B is evenly distributed.*

Proof. — The proof will proceed in two steps.

Step 1. *We prove that given $A \in \mathcal{E}_1(E_N)$ with $A \neq 0$ and $\|A\| < 1$, and given $\varepsilon > 0$, there exists $C \in \mathcal{E}_1(E_N)$ with $\|C - A\| < \varepsilon$ such that for any norming vector $x_1 \in S_{E_N}$ for C, and for every $j \in [0, N]$, we have $\langle e_j^*, x_1\rangle \neq 0$.*

Let $x_0 \in S_{E_N}$ be any norming vector for A, and let $J := \{j \in [0, N]\ ;\ \langle e_j^*, x_0\rangle \neq 0\}$. If $J = [0, N]$, there is nothing to prove, so suppose that $J \neq [0, N]$. Observe that since x_0 is supported on J and $\|Ax_0\| = \|A\|$, we have $\|\, A_{|E_J} \,\| = \|A\|$, where $E_J = [\, e_j\ ;\ j \in J]$. Consider now the sets

$$J' := \big\{\, j \in [0, N]\ ;\ \forall\, l \in [0, N],\ \exists\, s \in \mathbb{C}\backslash\{0\} \text{ such that } \|A + s\, e_l \otimes e_j^*\| \leqslant \|A\|\big\}$$

and

$$K := [0, N]\backslash(J \cup J').$$

We first observe that J' is actually included in J:

Lemma 5.21. — *For every $j \in J'$, we have $\langle e_j^*, x_0\rangle \neq 0$.*

Proof of Lemma 5.21. — Let $j \in J'$, and suppose that $\langle e_j^*, x_0\rangle = 0$. Since $x_0 \in S_{E_N}$ is a norming vector for A, the function $\phi : \mathbb{R} \to \mathbb{R}$ defined by

$$\phi(t) = \frac{\|A(x_0 + te_j)\|^p}{\|x_0 + te_j\|^p}$$

attains its maximum at $t = 0$. As $\langle e_j^*, x_0\rangle = 0$, $\|x_0 + te_j\|^p = 1 + t^p$, and so

$$\phi(t) = \frac{\|A(x_0 + te_j)\|^p}{1 + t^p}, \quad t \in \mathbb{R}.$$

Since $p > 1$, ϕ is differentiable on $\mathbb{R}$, and thus $\phi'(0) = 0$. Using the fact that for any complex numbers a and b, the function $\varphi_{a,b}$ defined by $\varphi_{a,b}(t) = |a + tb|^p$, $t \in \mathbb{R}$, is differentiable on $\mathbb{R}$ with $\varphi'_{a,b}(0) = p \cdot |a|^{p-2} \cdot \mathfrak{Re}(a\,\bar{b})$, we obtain that

$$\phi(0) = p \cdot \sum_{k=0}^{N} |\langle e_k^*, Ax_0\rangle|^{p-2} \cdot \mathfrak{Re}\big(\langle e_k^*, Ax_0\rangle\, \overline{\langle e_k^*, Ae_j\rangle}\big) = p \cdot \mathfrak{Re}\big(\langle \mathbf{J}(Ax_0), Ae_j\rangle\big) = 0.$$

Applying the same argument to the function $\psi : \mathbb{R} \to \mathbb{R}$ defined by

$$\psi(t) = \frac{\|A(x_0 + ite_j)\|^p}{1 + t^p}, \quad t \in \mathbb{R},$$

we obtain that $\psi'(0) = p \cdot \mathfrak{Im}\left(\langle \mathbf{J}(Ax_0), Ae_j\rangle\right) = 0$. Hence $\langle \mathbf{J}(Ax_0), Ae_j\rangle = 0$. In other words, the vector Ae_j belongs to the kernel of the functional $\mathbf{J}(Ax_0)_{|E_N}$, which we denote by H. Since $Ax_0 \neq 0$, H is a hyperplane of E_N.

Fix now $l \in [0, N]$. Since $j \in J'$, there exists $s \neq 0$ such that $\|A + s\, e_l \otimes e_j^*\| \leqslant \|A\|$. Setting $B = A + s\, e_l \otimes e_j^*$, we have $\|B\| \leqslant \|A\|$ and $Bx_0 = Ax_0 + s\,\langle e_j^*, x_0\rangle\, e_l = Ax_0$ since we have supposed that $\langle e_j^*, x_0\rangle = 0$. Hence $\|B\| = \|A\|$ and x_0 is a norming vector for the operator B. Reasoning as above, we obtain that the vector Be_j belongs to the kernel of the functional $\mathbf{J}(Bx_0)_{|E_N}$, which is H. So $Be_j = Ae_j + s\, e_l \in H$. Since $Ae_j \in H$ and $s \neq 0$, it follows that e_l belongs to H. This being true for every $l \in [0, N]$, we have $H = E_N$, which is a contradiction. So $\langle e_j^*, x_0\rangle \neq 0$, and Lemma 5.21 is proved. $\square$

We have thus shown that $J' \subseteq J$, and hence we can write the set $[0, N]$ as the disjoint union of the sets J and K.

Lemma 5.22. — *Suppose that $K \neq \varnothing$. Let $k \in K$ and $\gamma > 0$. There exists an operator $D \in \mathcal{E}_1(E_N)$ with $D \neq 0$, $\|D\| < 1$ and $\|D - A\| < \gamma$ such that, for any norming vector $x_2 \in S_{E_N}$ for D, we have $\langle e_j^*, x_2\rangle \neq 0$ for all $j \in J \cup \{k\}$.*

Proof of Lemma 5.22. — Since k belongs to K, it does not belong to J', and hence there exists $l \in [0, N]$ such that $\|A + s\, e_l \otimes e_k^*\| > \|A\|$ for every $s > 0$. For each $s > 0$, set $B_s := A + s\, e_l \otimes e_k^*$. Then $\|B_s - A\| = s$, and the operator induced by B_s on $E_{[0,N]\setminus\{k\}}$ coincides with A on $E_{[0,N]\setminus\{k\}}$. Since $\|B_s\| > \|A\|$, it follows that if $x \in S_{E_N}$ is a norming vector for B_s, then $\langle e_k^*, x\rangle \neq 0$.

On the other hand, since $\langle e_j^*, x_0\rangle \neq 0$ for every $j \in J$, Lemma 5.19 implies that if s is sufficiently small, any norming vector $x \in S_{E_N}$ for B_s is such that $\langle e_j^*, x\rangle \neq 0$ for every $j \in J$. Combining this with the observation above, we deduce that if $s \neq 0$ is sufficiently small, then any norming vector $x \in S_{E_N}$ for B_s is such that $\langle e_j^*, x\rangle \neq 0$ for every $j \in J \cup \{k\}$. By Lemmas 5.17 and 5.19, we can now choose $B \in \mathcal{E}_1(E_N)$ so close to B_s (with s small enough chosen first) that any norming vector $x_2 \in S_{E_N}$ for B satisfies $\langle e_j^*, x_2\rangle \neq 0$ for every $j \in J \cup \{k\}$. $\square$

We now have all the tools handy to finish the proof of Step 1 of Proposition 5.20. Suppose that K has cardinality $r \geqslant 1$, and write $K = \{k_1, \dots, k_r\}$. Choose $0 < \gamma < \varepsilon/r$, and apply Lemma 5.22 r times successively, to obtain operators $D_i \in \mathcal{E}_1(E_N)$, $i \in [1, r]$, with $D_i \neq 0$, $\|D_i\| = 1$, $\|D_{i+1} - D_i\| < \gamma$ for every $i \in [1, r-1]$, and such that for every norming vector $y_i \in S_{E_N}$ for the operator D_i, one has that $\langle e_j^*, y_i\rangle \neq 0$ for every $j \in J \cup \{k_1, \dots, k_i\}$. Then, the operator $C := D_r$ belongs to $\mathcal{E}_1(E_N)$, satisfies $\|C - A\| < \varepsilon$, and whenever $x \in S_{E_N}$ is a norming vector for C, we have $\langle e_j^*, x\rangle \neq 0$ for every $j \in J \cup K$. Since $J \cup K = [0, N]$ by Lemma 5.21, this proves the statement we were looking for.

Step 2. *We now prove the statement of Proposition 5.20, namely that given $A \in \mathcal{E}_1(E_N)$ with $A \neq 0$ and $\|A\| < 1$, and given $\varepsilon > 0$, there exists $B \in \mathcal{E}_1(E_N)$ with $\|A - B\| < \varepsilon$ such that for any norming vector $x_2 \in S_{E_N}$ for B, and for every $j \in [0, N]$, we have $\langle e_j^*, x_2\rangle \neq 0$ and $\langle e_j^*, Bx_2\rangle \neq 0$.*

Let $C \in \mathcal{E}_1(E_N)$ be given by Step 1, with $C \neq 0$, $\|C\| < 1$, $\|C - A\| < \varepsilon/2$, and $\langle e_j^*, x\rangle \neq 0$ for every norming vector $x \in S_{E_N}$ for C and all $j \in [0, N]$. We also fix a norming vector $x_1 \in S_{E_N}$ for C.

Our strategy is now to apply the result proved in Step 1 to the operator C^* acting on $E_N^* = [e_0^*, \dots, e_N^*] \subseteq \ell_{p'}(\mathbb{Z}_+)$, where $1/p + 1/p' = 1$. Since $1 < p < \infty$, we have $1 < p' < \infty$. We first observe that the operator C^* belongs to $\mathcal{E}_1(E_N^*)$, and that its norming vectors are the unimodular multiples of the vector $y_1^* := \mathbf{J}(Cx_1) \,/\, \|C\|^{p/p'}$. Indeed, if $x^* \in S_{E_N^*}$ is such that $\|C^*x^*\| = \|C^*\|$, there exists $x \in S_{E_N}$ such that $\langle C^*x^*, x\rangle = \|C^*\|$, *i.e.* $\langle x^*, Cx\rangle = \|Cx\|$. Hence x is a norming vector for C, and thus x belongs to $\mathbb{T}x_1$, and $|\langle x^*, Cx_1\rangle| = \|Cx_1\| = \|C\|$. It follows that x^* is a unimodular multiple of the vector $\mathbf{J}(Cx_1) \,/\, \|C\|^{p/p'} = y_1^*$, and so C^* is absolutely exposing. Since moreover $\|C^*\| = 1$ and

$1 < p' < \infty$, it is legitimate to apply the result proved in Step 1: there exists an operator $D \in \mathcal{E}_1(E_N^*)$ with $\|D - C^*\| < \varepsilon/2$ such that $\langle y^*, e_j\rangle \neq 0$ for every norming vector $y^* \in S_{E_N^*}$ for the operator D and for every $j \in [0, N]$. Let us fix a norming vector y_0^* for D.

Set $B := D^* \in \mathcal{B}_1(E_N)$. Then $\|B - A\| < \varepsilon$, and the same argument as above shows that $B \in \mathcal{E}_1(E_N)$, and that for any norming vector $x_2 \in S_{E_N}$ for B, the vector $\mathbf{J}(Bx_2)\,/\,\|B\|^{p/p'}$ is a unimodular multiple of y_0^*. So $\mathbf{J}(Bx_2)$ is a non-zero multiple of y_0^*, and $\langle \mathbf{J}(Bx_2), e_j\rangle \neq 0$ for every $j \in [0, N]$. It follows from the expression of $\mathbf{J}(Bx_2)$ that $\langle e_j^*, Bx_2\rangle \neq 0$ for all $j \in [0, N]$. Moreover, Lemma 5.19 implies that if $\varepsilon > 0$ is sufficiently small, the operator B is so close to C that $\langle e_j^*, x_2\rangle \neq 0$ for all $j \in [0, N]$. Hence B has all the required properties, and Proposition 5.20 is proved. □

We are now ready to prove Proposition 5.15.

Proof of Proposition 5.15. — Let $A \in \mathcal{B}_1(E_N)$. By Lemma 5.17 and Proposition 5.20, we can find an operator A' extremely close to it which is absolutely exposing, evenly distributed, and is such that $\|A'\| < 1$. So we may and do assume that A satisfies these additional properties.

Write E_{2N+1} as $E_{2N+1} = E_N \oplus G_N$, where $G_N := [\, e_j\ ;\ N+1 \leqslant j \leqslant 2N+1\,]$. Define $S_N : E_N \to G_N$ by setting

$$S_N x := \sum_{i=0}^{N} e_i^*(x) e_{N+1+i} \quad \text{for every } x \in E_N.$$

For every parameters $\eta, \delta > 0$, we define an operator $B_{\eta,\delta} \in \mathcal{B}(E_{2N+1})$ in the following way:

$$B_{\eta,\delta}(x + S_N y) := A(x + \delta y) + \eta\, S_N\, A(x + \delta y), \quad x, y \in E_N.$$

In matrix form with respect to the decomposition $E_{2N+1} = E_N \oplus G_N$,

$$B_{\eta,\delta} = \begin{pmatrix} A & \delta A \\ \eta A & \eta\delta A \end{pmatrix}.$$

For every $x, y \in E_N$, we have

$$\begin{aligned} \|B_{\eta,\delta}(x + S_N y)\| &= (1+\eta^p)^{1/p}\, \|A(x + \delta y)\| \\ &\leqslant (1+\eta^p)^{1/p}\, \|A\|\, \big(\|x\| + \delta\, \|y\|\big) \\ &\leqslant (1+\eta^p)^{1/p}\, (1+\delta^{p'})^{1/p'}\, \|A\| \cdot \Big(\, \|x\|^p + \|y\|^p\, \Big)^{1/p} \\ &= (1+\eta^p)^{1/p}\, (1+\delta^{p'})^{1/p'}\, \|A\| \cdot \|x + S_N y\|. \end{aligned}$$

Hence $\|B_{\eta,\delta}\| \leqslant (1+\eta^p)^{1/p}\, (1+\delta^{p'})^{1/p'}\, \|A\|$. We now claim that $\|B_{\eta,\delta}\|$ is exactly equal to $(1+\eta^p)^{1/p}\, (1+\delta^{p'})^{1/p'}\, \|A\|$. Indeed, let $x_0 \in S_{E_N}$ be a norming vector for A, and let u be the vector of E_{2N+1} defined as $u := x_0 + \delta^{p'-1} S_N x_0$. We set also $u_0 := u\,/\,\|u\|$. We have

$$\begin{aligned} \|B_{\eta,\delta}(u)\| &= (1+\eta^p)^{1/p}\, \|A(x_0 + \delta^{p'} x_0)\| = (1+\eta^p)^{1/p}\, (1+\delta^{p'})\, \|A x_0\| \\ &= (1+\eta^p)^{1/p}\, (1+\delta^{p'})\, \|A\| \end{aligned}$$

since $x_0 \in S_{E_N}$ is norming for A. Also $\|u\| = (1+\delta^{(p'-1)p})^{1/p} = (1+\delta^{p'})^{1/p}$, and hence

$$\|B_{\eta,\delta}(u_0)\| = (1+\eta^p)^{1/p}\, (1+\delta^{p'})^{1/p'}\, \|A\|.$$

This proves our claim. Moreover, we have also shown that

$$u_0 = \frac{x_0 + \delta^{p'-1} S_N x_0}{(1+\delta^{p'})^{1/p}}$$

is a norming vector for $B_{\eta,\delta}$.

Let us now check that $B_{\eta,\delta}$ is absolutely exposing. Let $v \in S_{E_{2N+1}}$ be such that $\|B_{\eta,\delta} v\| = \|B_{\eta,\delta}\|$. Write $v = x + S_N y$ with $x, y \in E_N$ and $\|x\|^p + \|y\|^p = 1$. Then $\|B_{\eta,\delta} v\| = (1+\eta^p)^{1/p} \|A(x+\delta y)\|$, so we have

$$\|A(x+\delta y)\| = \big(1+\delta^{p'}\big)^{1/p'} \|A\|.$$

It follows from this equality that x and y are necessarily non-zero. Since

$$\|x+\delta y\| \leqslant \|x\| + \delta\|y\| \leqslant \big(1+\delta^{p'}\big)^{1/p'} \big(\|x\|^p + \|y\|^p\big)^{1/p} = \big(1+\delta^{p'}\big)^{1/p'},$$

it also follows that $\|x+\delta y\| = \|x\| + \delta\|y\| = (1+\delta^{p'})^{1/p'}$. So there exists $\alpha > 0$ such that $y = \alpha x$. Finally, observe that $\frac{x+\delta y}{\|x+\delta y\|}$ is a norming vector for A, and hence is a unimodular multiple of x_0. We thus have $(1+\alpha\delta)x = (1+\delta^{p'})^{1/p'} \lambda x_0$ for some $\lambda \in \mathbb{T}$, so that

$$v = \frac{(1+\delta^{p'})^{1/p'}}{1+\alpha\delta}\, \lambda\, (x_0 + \alpha S_N x_0).$$

Since $\|v\| = 1$, we have $1 + \alpha\delta = (1+\delta^{p'})^{1/p'} (1+\alpha^p)^{1/p}$. By the equality case in Hölder's inequality, it follows that the vectors $\binom{1}{\delta^{p'}}$ and $\binom{1}{\alpha^p}$ are colinear in $\mathbb{R}^2$, which implies that $\alpha = \delta^{p'-1}$. Looking back at the definition of u_0, we thus see that v is a unimodular multiple of u_0. Hence $B_{\eta,\delta}$ is absolutely exposing, whatever the choices of the parameters η and δ.

The next step is to verify that $B_{\eta,\delta}$ is evenly distributed. With $u = x_0 + \delta^{p'-1} S_N x_0$ as above, it is enough to show that $\langle e_j^*, u\rangle \neq 0$ and $\langle e_j^*, B_{\eta,\delta} u\rangle \neq 0$ for all $j \in [0, 2N+1]$. We have $\langle e_j^*, u\rangle = \langle e_j^*, x_0\rangle$ if $j \in [0, N]$ and $\langle e_j^*, u\rangle = \delta^{p-1} \langle e_{j-N-1}^*, x_0\rangle$ if $j \in [N+1, 2N+1]$; so $\langle e_j^*, u\rangle \neq 0$ in both cases since x_0 is a norming vector for A and A is assumed to be evenly distributed. Likewise, since $B_{\eta,\delta}(u) = A(x_0 + \delta^{p'} x_0) + \eta\, S_N A(x_0 + \delta^{p'} x_0)$ we have $\langle e_j^*, B_{\eta,\delta} u\rangle = (1+\delta^{p'}) \langle e_i^*, Ax_0\rangle$ if $j \in [0, N]$ and $\langle e_j^*, B_{\eta,\delta} u\rangle = \eta\,(1+\delta^{p'}) \langle e_{j-N-1}^*, Ax_0\rangle$ if $j \in [N+1, 2N+1]$; so $\langle e_j^*, B_{\eta,\delta} u\rangle \neq 0$.

It remains to choose the parameters η and δ in such a way that $\|B_{\eta,\delta} P_N - A\| < \varepsilon$ and $\|B_{\eta,\delta}\| = 1$; and, moreover, to show that $\|B_{\eta,\delta} - A\| < \varepsilon$ if $\|A\|$ is close enough to 1. If $\eta > 0$ is chosen sufficiently small, then it follows from the definition of $B_{\eta,\delta}$ that $\|B_{\eta,\delta} P_N - A\| < \varepsilon$ whatever the choice of δ. Moreover, once η is chosen and is small enough, we can fix $\delta > 0$ in such a way that $\|B_{\eta,\delta}\| = 1$. This is possible since $\|A\| < 1$. Indeed, if $\eta > 0$ is such that $(1+\eta^p)^{1/p} \|A\| < 1$, then $\delta > 0$ is uniquely determined by the equation

$$(1+\eta^p)^{1/p} (1+\delta^{p'})^{1/p'} \|A\| = 1,$$

namely

$$\delta = \left[\left(\frac{1}{\|A\|\,(1+\eta^p)^{1/p}}\right)^{p'} - 1\right]^{1/p'} \leqslant \left(\frac{1}{\|A\|^{p'}} - 1\right)^{1/p'} =: \delta_A.$$

Finally, if $\|A\|$ is very close to 1 then δ_A is very small; so, looking at the definition of $B_{\eta,\delta}$, we see that $\|B_{\eta,\delta} - A\| < \varepsilon$ provided that $\|A\|$ is close enough to 1. $\square$

Remark 5.23. — The proof of Theorem 5.3 does not extend in a straightforward manner to the case where $X = c_0$. The fact that we are working on an ℓ_p-space with $p > 1$ is used in an important manner in the proof of Proposition 5.20 (both in the differentiability argument underlying the proof of Lemma 5.21, and in the duality argument used in Step 2 of the proof of Proposition 5.20), as well as in the proof of Proposition 5.16. Although it seems likely, in view of Proposition 5.5, that the statement of Theorem 5.3 also holds for $X = c_0$, we are not able to provide a proof of it.

5f. Additional results. — Theorem 5.3 rules out one trivial reason why a typical operator $T \in \mathcal{B}_1(\ell_p, \mathtt{SOT})$, $p > 2$ should have a non-trivial invariant subspace, but it does not bring us any nearer to an answer to this question.

In the Hilbertian case, here is a highly non-elementary proof of the fact that a typical $T \in (\mathcal{B}_1(H), \mathtt{SOT})$ has non-trivial invariant subspaces: by Theorem 3.4, we know that a typical $T \in (\mathcal{B}_1(H), \mathtt{SOT})$ is such that $\sigma(T) = \overline{\mathbb{D}}$; and by the Brown-Chevreau-Pearcy Theorem from [5], any $T \in \mathcal{B}_1(H)$ whose spectrum contains the unit circle $\mathbb{T}$ has a non-trivial invariant subspace.

It is tempting to try to generalize this argument on more general Banach spaces. The trouble is that there is no full analogue of the Brown-Chevreau-Pearcy Theorem in a Banach space setting. A deep result due to Ambrozie and Müller [2] states that if T is a *polynomially bounded* operator on X such that $\sigma(T)$ contains $\mathbb{T}$ and $\|T^n x\| \to 0$ for all $x \in X$, then T has a non-trivial invariant subspace. When $X = \ell_p$, $1 < p < \infty$, or $X = c_0$, we do know, thanks to Propositions 3.7 and 3.9, that a typical $T \in (\mathcal{B}_1(X), \mathtt{SOT})$ is such that $\sigma(T)$ contains $\mathbb{T}$ and that $\|T^n x\| \to 0$ for all $x \in X$. However, whereas any Hilbert space contraction is polynomially bounded by von Neumann's inequality, the next proposition shows that on $X = \ell_p$ with $1 \leqslant p < \infty$, $p \neq 2$, as well as on c_0, polynomial boundedness is *not* typical.

Proposition 5.24. — *Let $X = c_0$ or $X = \ell_p$ with $1 \leqslant p < \infty$ and $p \neq 2$. Then, the set of all polynomially bounded contractions is F_σ and meager in $(\mathcal{B}_1(X), \mathtt{SOT})$. In particular, a typical $T \in (\mathcal{B}_1(X), \mathtt{SOT})$ is not polynomially bounded.*

Proof. — The key point of the proof is the following well known fact, of which we give a proof for convenience of the reader.

Claim 5.25. — Let S be the unweighted forward shift on X. Then S is not polynomially bounded.

Proof of Claim 5.25. — We first consider the case $X = \ell_p$, $1 \leqslant p < 2$. For any polynomial $p(z) = \sum_{j=0}^d a_j z^j$, we have

$$\|p(S)e_0\| = \left(\sum_{j=0}^{d} |a_j|^p \right)^{1/p}.$$

For every $d \geqslant 0$, let p_d be the classical *Rudin-Shapiro polynomial* of degree d (see [36]). This polynomial has the following properties :

$$p_d(z) = \sum_{j=0}^{d} a_{j,d} z^j, \quad \text{with } a_{j,d} \in \{-1, 1\} \text{ for every } 0 \leqslant j \leqslant d, \quad \text{and} \quad \|p_d\|_{\infty,\mathbb{D}} \leqslant \sqrt{2(d+1)}.$$

So we have

$$\|p_d(S)\| \geqslant \|p_d(S)e_0\| = (d+1)^{1/p} \geqslant \frac{1}{\sqrt{2}}\,(d+1)^{\frac{1}{p}-\frac{1}{2}}\|p_d\|_{\infty,\mathbb{D}},$$

which shows that S is not polynomially bounded since $p<2$.

Essentially the same proof (looking at the vector e_{d+1} rather than at e_0) shows that the backward shift B is not polynomially bounded on ℓ_p, $1\leqslant p<2$. So the cases where $X=c_0$ and $X=\ell_p$, with $2<p<\infty$ can be handled by a duality argument. □

Let us denote by $\mathcal{G}$ the set of all $T\in\mathcal{B}_1(X)$ which are not polynomially bounded. We have to show that $\mathcal{G}$ is a dense G_δ subset of $(\mathcal{B}_1(X),\texttt{SOT})$.

The fact that $\mathcal{G}$ is $\texttt{SOT}$-G_δ is straightforward: indeed, if $T\in\mathcal{B}_1(X)$ then

$$T\in\mathcal{G}\iff\forall K\in\mathbb{N}\;\exists p\in\mathbb{C}[z]\;\exists x\in S_X\;;\;\|p(T)x\|>K\|p\|_{\infty,\mathbb{D}}.$$

Let us now show that $\mathcal{G}$ is $\texttt{SOT}$-dense in $\mathcal{B}_1(X)$. Let $A\in\mathcal{B}_1(X)$ be arbitrary, and set $T_N:=P_NA_{|E_N}\oplus S_N$, where S_N is the forward shift acting on F_N. Then $\|T_N\|\leqslant 1$, $T_N\xrightarrow{\texttt{SOT}}A$, and $T_N\in\mathcal{G}$ because S_N is not polynomially bounded on X by Claim 5.25. □

In the same circle of ideas, we mention the work of V. Müller, who proved in [30] the following result: Assume that X is a Banach space not containing an isomorphic copy of c_0, and let $T\in\mathcal{B}(X)$ be a power-bounded operator on X such that $1\in\sigma(T)$. Then, there exist $x_0\neq 0$ in X and $x_0^*\neq 0$ in X^* such that $\mathrm{Re}\,\langle x_0^*,T^nx_0\rangle\geqslant 0$ for all $n\in\mathbb{Z}_+$. Thus, T admits a non-trivial invariant closed convex cone, namely the set

$$L:=\{x\in X;\;\forall n\geqslant 0\;:\;\mathrm{Re}\,\langle x_0^*,T^nx\rangle\geqslant 0\}.$$

It follows that T has a (non-zero) non-supercyclic vector. Indeed, by a result of [26], any vector $x\in L$ is non-supercyclic if $\sigma_p(T^*)=\varnothing$; whereas if T^* has an eigenvalue, then T has a non-trivial invariant subspace and hence non-cyclic (non-zero) vectors. Since a typical $T\in(\mathcal{B}_1(X),\texttt{SOT})$ is such that $1\in\sigma(T)$ when $X=\ell_p$, $1\leqslant p<\infty$ or $X=c_0$ (by Proposition 3.9), we may therefore state

Proposition 5.26. — *Assume that $X=\ell_p$, $1<p<\infty$. Then a typical operator $T\in(\mathcal{B}_1(X),\texttt{SOT})$ admits a non-trivial invariant closed cone and (hence) a non-zero non-supercyclic vector.*

Müller's result from [30] cannot be applied when $X=c_0$. However, we are able to prove the existence of non-supercyclic vectors in the c_0-case as well.

Proposition 5.27. — *A typical $T\in(\mathcal{B}_1(c_0),\texttt{SOT})$ admits a non-zero non-supercyclic vector.*

Proof. — As in the proof of Theorem 5.2, we use the Banach-Mazur game. Let us denote by $\mathcal{A}\subseteq(\mathcal{B}_1(c_0),\texttt{SOT})$ the set of all $T\in\mathcal{B}_1(c_0)$ admitting a non-zero non-supercyclic vector. We will describe a strategy for player II in the Banach-Mazur game $\mathbf{G}(\mathcal{A})$, and show that this strategy is winning under a suitable assumption.

With the notations of the proof of Theorem 5.2, assume that player I has just played an open set of the form $\mathcal{U}_{2k}=\mathcal{U}(N_{2k},A_{2k},\varepsilon_{2k})$. Then, player II plays the open set

$\mathcal{U}_{2k+1} = \mathcal{U}(N_{2k+1}, A_{2k+1}, \varepsilon_{2k+1})$, where $N_{2k+1} := N_{2k} + 1$, $0 < \varepsilon_{2k+1} < \varepsilon_{2k}/2$, and $A_{2k+1} \in \mathcal{B}_1(E_{N_{2k+1}})$ is defined as follows:

$$\begin{cases} A_{2k+1}e_n := \big(1 - \frac{\varepsilon_{2k}}{2}\big)A_{2k}e_n & \text{for all } 0 \leqslant n \leqslant N_{2k}; \\ A_{2k+1}e_{N_{2k}+1} := e_{N_{2k}+1}. \end{cases}$$

This is indeed a legal move for II since $\varepsilon_{2k+1} < \varepsilon_{2k}/2$ and $\|A_{2k+1}e_n - A_{2k}e_n\| \leqslant \varepsilon_{2k}/2$ for all $0 \leqslant n \leqslant N_{2k}$ (which implies that $\mathcal{U}_{2k+1} \subseteq \mathcal{U}_{2k}$). Note that, by the definition of A_{2k+1}, we have

$$\langle e^*_{N_{2k}+1}, A_{2k+1}e_{N_{2k}+1}\rangle = 1. \tag{5.2}$$

The actual choice of ε_{2k+1} will be determined by some large positive integer L_{2k+1}, that player II selects in such a way that

$$\left(1 - \frac{\varepsilon_{2k}}{4}\right)^{L_{2k+1}} \leqslant \frac{1}{2^{k+1}}. \tag{5.3}$$

The following fact will be useful.

Fact 5.28. — Let $(\mathcal{U}_n)_{n\geqslant 0}$ be a run in the Banach-Mazur game where player II has followed the strategy described above. If $T \in \bigcap_{n\geqslant 0} \mathcal{U}_n$ then, for any $k \geqslant 0$, we have

$$\|e^*_{N_{2k}+1}TP_{\mathbb{N}\setminus\{N_{2k}+1\}}\| \leqslant \varepsilon_{2k+1}.$$

Proof of Fact 5.28. — Let $x \in c_0$ with $\|x\| \leqslant 1$. Set $y := e_{N_{2k}+1} + e^{i\theta}P_{\mathbb{N}\setminus\{N_{2k}+1\}}x$, where $\theta \in \mathbb{R}$ is such that $e^{i\theta}\langle e^*_{N_{2k}+1}, TP_{\mathbb{N}\setminus\{N_{2k}+1\}}x\rangle = |\langle e^*_{N_{2k}+1}, TP_{\mathbb{N}\setminus\{N_{2k}+1\}}x\rangle|$. Then $\|y\| \leqslant 1$. Hence,

$$\begin{aligned} 1 \geqslant |\langle e^*_{N_{2k}+1}, Ty\rangle| &\geqslant |e^{i\theta}\langle e^*_{N_{2k}+1}, TP_{\mathbb{N}\setminus\{N_{2k}+1\}}x\rangle + \langle e^*_{N_{2k}+1}, A_{2k+1}e_{N_{2k}+1}\rangle| \\ &\qquad - |\langle e^*_{N_{2k}+1}, (T - A_{2k+1})e_{N_{2k}+1}\rangle| \\ &\geqslant |\langle e^*_{N_{2k}+1}TP_{\mathbb{N}\setminus\{N_{2k}+1\}}, x\rangle| + 1 - \varepsilon_{2k+1} \qquad \text{by (5.2)}. \end{aligned}$$

So we get $|\langle e^*_{N_{2k}+1}TP_{\mathbb{N}\setminus\{N_{2k}+1\}}, x\rangle| \leqslant \varepsilon_{2k+1}$ for every $x \in B_{c_0}$. $\square$

We can now prove

Lemma 5.29. — *Set $L_{-1} := 0$. Assume that at each move, player* II *decides to choose $L_{2k+1} > L_{2k-1}$ satisfying* (5.3)*, and ε_{2k+1} small enough to ensure that*

$$L_{2k+1}(N_{2k} + 1)\varepsilon_{2k+1} \leqslant \frac{\varepsilon_{2k}}{4}\left(1 - \frac{\varepsilon_{2k}}{2}\right)^{L_{2k+1}}.$$

Then, the above strategy is winning for II.

Proof of Lemma 5.29. — Let $(\mathcal{U}_n)_{n\geqslant 0}$ be a run in the Banach-Mazur game where player II has followed his strategy, and let $T \in \bigcap_{n\geqslant 0} \mathcal{U}_n$. We are going to show that the vector

$$x := e_0 + \sum_{k=0}^{\infty} \frac{1}{2^{k+1}} e_{N_{2k}+1}$$

is a non-supercyclic vector for T. The key point is the following fact.

Fact 5.30. — For any $k \geqslant 0$ and all n with $L_{2k-1} \leqslant n < L_{2k+1}$, we have

$$\|T^n x\| \leqslant 8\,|\langle e^*_{N_{2k}+1}, T^n x\rangle|.$$

Taking Fact 5.30 for granted, it is easy to finish the proof of Lemma 5.29. Indeed, let $n \geqslant 0$, and choose k such that $L_{2k-1} \leqslant n < L_{2k+1}$. For any $\lambda \in \mathbb{C}$, we have

$$\begin{aligned}\|\lambda T^n x - e_0\| &\geqslant \max\Big(1 - |\lambda|\|T^n x\|, |\lambda|\,|\langle e^*_{N_{2k}+1}, T^n x\rangle|\Big)\\ &\geqslant \max\Big(1 - |\lambda|\|T^n x\|, |\lambda|\,\|T^n x\|/8\Big)\\ &\geqslant 1/9.\end{aligned}$$

So we see that $e_0 \notin \overline{\{\lambda T^n x;\ n \geqslant 0,\ \lambda \in \mathbb{C}\}}$, and hence that x is not a supercyclic vector for T. □

Proof of Fact 5.30. — This will follow from the next four claims.

Claim 5.31. — For any $k, n \geqslant 0$, we have

$$|\langle e^*_{N_{2k}+1}, T^n x\rangle| \geqslant \frac{1}{2^{k+1}} - 2n\,\varepsilon_{2k+1}.$$

Proof of Claim 5.31. — Let us fix $k \geqslant 0$. The result is clear for $n = 0$ since $\langle e^*_{N_{2k}+1}, x\rangle = \frac{1}{2^{k+1}}$ by the definition of x. Moreover, if $n \geqslant 1$ then

$$\begin{aligned}&|\langle e^*_{N_{2k}+1}, T^n x\rangle|\\ &\quad\geqslant |\langle e^*_{N_{2k}+1}, TP_{\{N_{2k}+1\}}T^{n-1}x\rangle| - |\langle e^*_{N_{2k}+1}, TP_{\mathbb{N}\setminus\{N_{2k}+1\}}T^{n-1}x\rangle|\\ &\quad\geqslant |\langle e^*_{N_{2k}+1}, A_{2k+1}P_{\{N_{2k}+1\}}T^{n-1}x\rangle| - |\langle e^*_{N_{2k}+1}, (T - A_{2k+1})P_{\{N_{2k}+1\}}T^{n-1}x\rangle|\\ &\qquad - |\langle e^*_{N_{2k}+1}, TP_{\mathbb{N}\setminus\{N_{2k}+1\}}T^{n-1}x\rangle|\\ &\quad\geqslant |\langle e^*_{N_{2k}+1}, T^{n-1}x\rangle| - \varepsilon_{2k+1}\|T^{n-1}x\| - \varepsilon_{2k+1}\|T^{n-1}x\| \qquad \text{by (5.2) and Fact 5.28}\\ &\quad\geqslant |\langle e^*_{N_{2k}+1}, T^{n-1}x\rangle| - 2\varepsilon_{2k+1}.\end{aligned}$$

So the claim follows by induction. □

Claim 5.32. — For any $k, n \geqslant 0$, we have

$$\|P_{(N_{2k},\infty)}T^n P_{[0,N_{2k}]}x\| \leqslant n(N_{2k}+1)\,\varepsilon_{2k+1}.$$

Proof of Claim 5.32. — Let us fix $k \geqslant 0$. The result is clearly true for $n = 0$ since $P_{(N_{2k},\infty)}T^0 P_{[0,N_{2k}]} = 0$. Moreover, if $n \geqslant 1$ then

$$\begin{aligned}&\|P_{(N_{2k},\infty)}T^n P_{[0,N_{2k}]}x\|\\ &\quad\leqslant \|P_{(N_{2k},\infty)}TP_{[0,N_{2k}]}T^{n-1}P_{[0,N_{2k}]}x\| + \|P_{(N_{2k},\infty)}TP_{(N_{2k},\infty)}T^{n-1}P_{[0,N_{2k}]}x\|\\ &\quad\leqslant \|P_{(N_{2k},\infty)}A_{2k+1}P_{[0,N_{2k}]}T^{n-1}P_{[0,N_{2k}]}x\|\\ &\qquad + \|P_{(N_{2k},\infty)}(T - A_{2k+1})P_{[0,N_{2k}]}T^{n-1}P_{[0,N_{2k}]}x\| + \|P_{(N_{2k},\infty)}T^{n-1}P_{[0,N_{2k}]}x\|\\ &\quad\leqslant 0 + (N_{2k}+1)\varepsilon_{2k+1} + \|P_{(N_{2k},\infty)}T^{n-1}P_{[0,N_{2k}]}x\|;\end{aligned}$$

and the claim follows by induction. □

Claim 5.33. — For any $k \geqslant 0$ and all $n \leqslant L_{2k+1}$, we have

$$\|T^n P_{[0,N_{2k}]}x\| \leqslant \Big(1 - \frac{\varepsilon_{2k}}{4}\Big)^n.$$

Proof of Claim 5.33. — This is clear if $n = 0$. Assume that $1 \leqslant n \leqslant L_{2k+1}$ and that the inequality has been proved for $n-1$. Then

$$\begin{aligned}
&\|T^n P_{[0,N_{2k}]}x\| \\
&\quad\leqslant \|TP_{[0,N_{2k}]}T^{n-1}P_{[0,N_{2k}]}x\| + \|TP_{(N_{2k},\infty)}T^{n-1}P_{[0,N_{2k}]}x\| \\
&\quad\leqslant \|A_{2k+1}P_{[0,N_{2k}]}T^{n-1}P_{[0,N_{2k}]}x\| \\
&\qquad + \|(T-A_{2k+1})P_{[0,N_{2k}]}T^{n-1}P_{[0,N_{2k}]}x\| + \|P_{(N_{2k},\infty)}T^{n-1}P_{[0,N_{2k}]}x\| \\
&\quad\leqslant \left(1-\frac{\varepsilon_{2k}}{2}\right)\|A_{2k}P_{[0,N_{2k}]}T^{n-1}P_{[0,N_{2k}]}x\| \\
&\qquad + (N_{2k}+1)\varepsilon_{2k+1} + (n-1)(N_{2k}+1)\varepsilon_{2k+1} \qquad \text{by Claim 5.32} \\
&\quad\leqslant \left(1-\frac{\varepsilon_{2k}}{2}\right)\|T^{n-1}P_{[0,N_{2k}]}x\| + L_{2k+1}(N_{2k}+1)\varepsilon_{2k+1} \qquad \text{since } n \leqslant L_{2k+1} \\
&\quad\leqslant \left(1-\frac{\varepsilon_{2k}}{2}\right)\left(1-\frac{\varepsilon_{2k}}{4}\right)^{n-1} + \frac{\varepsilon_{2k}}{4}\left(1-\frac{\varepsilon_{2k}}{2}\right)^{L_{2k+1}} \qquad \text{by assumption on } \varepsilon_{2k+1} \\
&\quad\leqslant \left(1-\frac{\varepsilon_{2k}}{2}\right)\left(1-\frac{\varepsilon_{2k}}{4}\right)^{n-1} + \frac{\varepsilon_{2k}}{4}\left(1-\frac{\varepsilon_{2k}}{2}\right)^{n-1} \\
&\quad= \left(1-\frac{\varepsilon_{2k}}{4}\right)^{n}.
\end{aligned}$$

This proves the claim. □

Claim 5.34. — We have $\|T^{L_{2k-1}}x\| \leqslant \frac{1}{2^{k-1}}$ for all $k \geqslant 0$.

Proof of Claim 5.34. — This is true if $k = 0$ since $\|T^{L_{-1}}x\| = \|x\| = 1 \leqslant 2$; so assume that $k \geqslant 1$, and let $k' := k-1$. Then

$$\begin{aligned}
\|T^{L_{2k-1}}x\| = \|T^{L_{2k'+1}}x\| &\leqslant \|T^{L_{2k'+1}}P_{[0,N_{2k'}]}x\| + \|T^{L_{2k'+1}}P_{(N_{2k'},\infty)}x\| \\
&\leqslant \|T^{L_{2k'+1}}P_{[0,N_{2k'}]}x\| + \|P_{(N_{2k'},\infty)}x\| \\
&= \|T^{L_{2k'+1}}P_{[0,N_{2k'}]}x\| + \frac{1}{2^{k'+1}} \\
&\leqslant \left(1-\frac{\varepsilon_{2k'}}{4}\right)^{L_{2k'+1}} + \frac{1}{2^{k'+1}} \qquad \text{by Claim 5.33} \\
&\leqslant \frac{1}{2^{k'+1}} + \frac{1}{2^{k'+1}} = \frac{1}{2^{k-1}}.
\end{aligned}$$

□

We can now conclude the proof of Fact 5.30, and hence that of Lemma 5.29. Let $k \geqslant 0$ and $L_{2k-1} \leqslant n < L_{2k+1}$. On the one hand, we have by Claim 5.31:

$$\begin{aligned}
|\langle e^*_{N_{2k}+1}, T^n x\rangle| &\geqslant \frac{1}{2^{k+1}} - 2L_{2k+1}\varepsilon_{2k+1} \\
&\geqslant \frac{1}{2^{k+2}} \qquad \text{since} \quad 2L_{2k+1}\varepsilon_{2k+1} \leqslant \tfrac{\varepsilon_{2k}}{2}\left(1-\tfrac{\varepsilon_{2k}}{2}\right)^{L_{2k+1}} \leqslant \tfrac{1}{2^{k+2}};
\end{aligned}$$

and on the other hand, by Claim 5.34 (and since $\|T\| \leqslant 1$):

$$\|T^n x\| \leqslant \|T^{L_{2k-1}}x\| \leqslant \frac{1}{2^{k-1}}.$$

Hence, $|\langle e^*_{N_{2k}+1}, T^n x\rangle| \geqslant \|T^n x\|/8$. □

By Lemma 5.29, the proof of Proposition 5.27 is now complete. □

6. Typical properties of "triangular plus 1" contractions

In this section, we study typical properties of contraction operators which are "triangular plus 1" with respect to some fixed basis of the underlying Banach space. One of the results proved in this section (namely, Lemma 6.2) will be used in the proof of Theorem 2.6, to be given in the forthcoming Section 7 (Theorem 7.5).

In the first part of this section, we place ourselves in the Hilbertian setting. Let H be a Hilbert space (complex, infinite-dimensional and separable), and let $(f_j)_{j\geqslant 0}$ be a fixed orthonormal basis of H. We denote by $\mathcal{T}_1(H)$ the set of all the operators $T\in\mathcal{B}_1(H)$ which are "triangular plus 1" with respect to the basis (f_j), with positive entries on the first subdiagonal:

$$\mathcal{T}_1(H)=\big\{T\in\mathcal{B}_1(H)\;;\;Tf_j\in[f_0,\dots,f_{j+1}]\text{ for all } j\quad\text{and}\quad\langle Tf_j,f_{j+1}\rangle>0\big\}.$$

Since any cyclic operator $T\in\mathcal{B}_1(H)$ is unitarily equivalent to some operator belonging to $\mathcal{T}_1(H)$, this is a rather natural class to consider. The following fact is easy to check.

Lemma 6.1. — *The set $\mathcal{T}_1(H)$ is a G_δ subset of $(\mathcal{B}_1(H),\texttt{SOT})$, hence of $(\mathcal{B}_1(H),\texttt{SOT}^*)$, and hence a Polish space with respect to both topologies. However, $\mathcal{T}_1(H)$ is nowhere dense in $(\mathcal{B}_1(H),\texttt{SOT})$ as well as in $(\mathcal{B}_1(H),\texttt{SOT}^*)$.*

Proof. — For any fixed $j\in\mathbb{Z}_+$, the condition "$Tf_j\in[f_0,\dots,f_{j+1}]$" defines a closed set in $(\mathcal{B}_1(H),\texttt{SOT})$ since it is equivalent to "$\langle Tf_j,f_n\rangle=0$ for all $n>j+1$", and the condition "$\langle Tf_j,f_{j+1}\rangle>0$" defines a G_δ set because $(0,\infty)$ is a G_δ subset of $\mathbb{C}$; so $\mathcal{T}_1(H)$ is a G_δ in $(\mathcal{B}_1(H),\texttt{SOT})$, hence in $(\mathcal{B}_1(H),\texttt{SOT}^*)$. Moreover, $\mathcal{T}_1(H)$ is contained in the $\texttt{SOT}$ - closed set

$$\mathcal{F}:=\big\{T\in\mathcal{B}_1(H)\;;\;Tf_j\in[f_0,\dots,f_{j+1}]\text{ for all } j\in\mathbb{Z}_+\big\},$$

whose complement is clearly dense in $(\mathcal{B}_1(H),\texttt{SOT}^*)$, hence in $(\mathcal{B}_1(H),\texttt{SOT})$. □

Also we have

Lemma 6.2. — *The set of operators $T\in\mathcal{B}_1(H)$ which are unitarily equivalent to some operator belonging to $\mathcal{T}_1(H)$ is comeager in $(\mathcal{B}_1(H),\texttt{SOT})$ and in $(\mathcal{B}_1(H),\texttt{SOT}^*)$.*

Proof of Lemma 6.2. — The orbit of the set $\mathcal{T}_1(H)$ under unitary equivalence is exactly the class $\mathcal{CY}_1(H)$ of all cyclic contractions on H. Let $(V_k)_{k\geqslant 1}$ be a basis of open sets for H, and let $(p_r)_{r\geqslant 1}$ be an enumeration of all complex polynomials with coefficients in $\mathbb{Q}+i\,\mathbb{Q}$. The class $\mathcal{CY}_1(H)$ contains the set $\mathcal{CY}_{d,1}(H)$ of cyclic contractions on H with a dense set of cyclic vectors. It follows easily from the Baire Category Theorem that

$$\mathcal{CY}_{d,1}(H)=\bigcap_{(k,l)\in\mathbb{N}^2}\;\bigcup_{r\geqslant 1}\big\{T\in\mathcal{B}_1(H)\;;\;p_r(T)^{-1}(V_k)\cap V_l\neq\varnothing\big\}\cdot$$

From this, we see that the set $\mathcal{CY}_{d,1}(H)$ is $\texttt{SOT}$ - G_δ in $\mathcal{B}_1(H)$, hence $\texttt{SOT}^*$- G_δ. So we only need to prove that $\mathcal{CY}_{d,1}(H)$ is $\texttt{SOT}^*$- dense in $\mathcal{B}_1(H)$. But this follows immediately from the fact, proved in [18, Corollary 2.12], that the *hypercyclic* operators are $\texttt{SOT}^*$- dense in $\mathcal{B}_M(H)$ for any $M>1$, since non-zero multiples of hypercyclic operators have a dense set of cyclic vectors. □

Since $\mathcal{T}_1(H)$ is nowhere dense in $(\mathcal{B}_1(H),\texttt{SOT})$, it does not follow immediately from Theorem 3.4 that a typical $T\in(\mathcal{T}_1(H),\texttt{SOT})$ is unitarily equivalent to B_∞. However, we will prove that it is indeed the case. This will follow from a more general result on preservation of comeagerness.

Let us denote $\mathcal{U}(H)$ the unitary group of H. This is a Polish group when endowed with $\texttt{SOT}$. In what follows, we will denote by $\mathcal{U}_{f_0}(H)$ the set of all $U \in \mathcal{U}(H)$ such that $Uf_0 = f_0$. This is an $\texttt{SOT}$ - closed subgroup of $\mathcal{U}(H)$, hence a Polish group. We say that a set $\mathcal{P} \subseteq \mathcal{B}_1(H)$ is $\mathcal{U}_{f_0}$ - *invariant* if $U\mathcal{P}U^{-1} \subseteq \mathcal{P}$ for all $U \in \mathcal{U}_{f_0}(H)$; equivalently, if $U\mathcal{P}U^{-1} = \mathcal{P}$ for all $U \in \mathcal{U}_{f_0}(H)$. The result we intend to prove reads as follows.

Theorem 6.3. — *Let $\mathcal{P} \subseteq \mathcal{B}_1(H)$ be $\mathcal{U}_{f_0}(H)$ - invariant. If the set $\mathcal{P}$ is comeager in $(\mathcal{B}_1(H), \texttt{SOT})$, then $\mathcal{P} \cap \mathcal{T}_1(H)$ is comeager in $(\mathcal{T}_1(H), \texttt{SOT})$.*

Note that we do not assume that $\mathcal{P}$ is G_δ in $(\mathcal{B}_1(H), \texttt{SOT})$. This is, however, not a real complication in view of the following folklore lemma. We thank C. Rosendal for showing us how to prove it.

Lemma 6.4. — *Let G be a Polish group acting continuously on a Polish space Z, and let P be a comeager and G - invariant subset of Z. Then P contains a dense, G - invariant G_δ set in Z.*

Proof. — Since P is comeager, it contains a dense G_δ set B. Consider the *Vaught transform* B^* of B, which is defined as follows:

$$B^* := \big\{z \in Z;\ g \cdot z \in B \text{ for a comeager set of } g \in G\big\}.$$

As shown in [38], the Vaught transform preserves the multiplicative Borel classes. Hence, B^* is a G_δ set. Since B is comeager in Z, it follows from the Kuratowski-Ulam Theorem (see [24]) that B^* is comeager as well. Moreover, B^* is easily seen to be G - invariant. Finally, B^* is contained in P because P is G - invariant. □

We now give the

Proof of Theorem 6.3. — Assume that $\mathcal{P}$ is comeager in $(\mathcal{B}_1(H), \texttt{SOT})$. By Lemma 6.4, we may assume that $\mathcal{P}$ is also G_δ. Then $\mathcal{P} \cap \mathcal{T}_1(H)$ is G_δ in $(\mathcal{T}_1(H), \texttt{SOT})$; so we just need to show that $\mathcal{P} \cap \mathcal{T}_1(H)$ is $\texttt{SOT}$ - dense in $\mathcal{T}_1(H)$.

Observe that for any $f \in H\backslash\{0\}$, the set

$$\mathcal{G}_f := \{T \in \mathcal{B}_1(H);\ T \text{ is cyclic with cyclic vector } f\}$$

is a dense G_δ subset of $(\mathcal{B}_1(H), \texttt{SOT})$. Indeed, the fact that $\mathcal{G}_f$ is G_δ is easy to check, and density follows for example from [18, Lemma 2.23]. So the set $\mathcal{G}_0 := \mathcal{P} \cap \mathcal{G}_{f_0}$ is a dense G_δ subset of $(\mathcal{B}_1(H), \texttt{SOT})$; in particular $\mathcal{G}_0$ is dense in $\mathcal{G}_{f_0}$. Note also that $\mathcal{T}_1(H) \subseteq \mathcal{G}_{f_0}$.

For any $T \in \mathcal{G}_{f_0}$, let us denote by $(f_j(T))_{j\geqslant 0}$ the orthonormal basis of H obtained by applying the Gram-Schmidt orthonormalization process to the sequence $(T^j f_0)_{j\geqslant 0}$ (which is linearly independent and spans a dense linear subspace of X since f_0 is a cyclic vector for T); and let $U(T) : H \to H$ be the associated "change of basis" operator, *i.e.* the unitary operator defined by $U(T)f_j = f_j(T)$, $j \geqslant 0$. Note that in fact $U(T)$ belongs to $\mathcal{U}_{f_0}(H)$. Writing down the orthonormalization process explicitly, one easily checks that the maps $T \mapsto f_j(T)$ are continuous from $(\mathcal{G}_{f_0}, \texttt{SOT})$ into H. Hence, the map $T \mapsto U(T)$ is $\texttt{SOT}$ - continuous from $\mathcal{G}_{f_0}$ into $\mathcal{U}_{f_0}(H)$; and since $(\mathcal{U}_{f_0}(H), \texttt{SOT})$ is a topological group, the map $T \mapsto U(T)^{-1}$ is also continuous.

For any $T \in \mathcal{G}_{f_0}$, we set $R(T) := U(T)TU(T)^{-1}$. It is easily checked that $R(T) \in \mathcal{T}_1(H)$. Moreover, the map $T \mapsto R(T)$ is $\texttt{SOT}$ - continuous from $\mathcal{G}_0$ into $\mathcal{T}_1(H)$, because the product map $(T, S) \mapsto TS$ is jointly continuous on $(\mathcal{B}_1(H), \texttt{SOT}) \times (\mathcal{B}_1(H), \texttt{SOT})$. Note also that $R(T) = T$ for every $T \in \mathcal{T}_1(H)$, by the very definition of $\mathcal{T}_1(H)$ and of the map R. Finally, we have $R(T) \in \mathcal{P}$ for any $T \in \mathcal{G}_0 = \mathcal{P} \cap \mathcal{G}_{f_0}$ because $\mathcal{P}$ is $\mathcal{U}_{f_0}(H)$ - invariant. So we have defined an $\mathtt{SOT}$-continuous retraction $R : \mathcal{G}_{f_0} \to \mathcal{T}_1(H)$ such that $R(\mathcal{G}_0) \subseteq \mathcal{P}$. Since $\mathcal{G}_0$ is dense in $\mathcal{G}_{f_0}$, it follows immediately that $\mathcal{P} \cap \mathcal{T}_1(H)$ is dense in $(\mathcal{T}_1(H), \mathtt{SOT})$. □

Remark 6.5. — Theorem 6.3 admits the following partial converse: if $\mathcal{P} \subseteq \mathcal{B}_1(H)$ is a $\mathcal{U}_{f_0}(H)$-invariant set such that $\mathcal{P} \cap \mathcal{T}_1(H)$ is comeager in $(\mathcal{T}_1(H), \mathtt{SOT})$, and if $\mathcal{P}$ is also $\mathtt{SOT}$-G_δ, then $\mathcal{P}$ is comeager in $(\mathcal{B}_1(H), \mathtt{SOT})$. Indeed, in this case we just have to check that $\mathcal{P}$ is $\mathtt{SOT}$-dense in $\mathcal{B}_1(H)$; and with the notation of the above proof, it is enough to show that the $\mathtt{SOT}$-closure of $\mathcal{P}$ in $\mathcal{B}_1(H)$ contains $\mathcal{G}_{f_0}$ since $\mathcal{G}_{f_0}$ is $\mathtt{SOT}$-dense in $\mathcal{B}_1(H)$. Let $T \in \mathcal{G}_{f_0}$ be arbitrary. Since $\mathcal{P} \cap \mathcal{T}_1(H)$ is $\mathtt{SOT}$-dense in $\mathcal{T}_1(H)$, one can find a sequence $(S_n) \subseteq \mathcal{P} \cap \mathcal{T}_1(H)$ such that $S_n \xrightarrow{\mathtt{SOT}} S(T) := U(T)^{-1}TU(T)$. Then $T_n := U(T)S_nU(T)^{-1} \in \mathcal{P}$ for all n since $\mathcal{P}$ is $\mathcal{U}_{f_0}(H)$-invariant, and $T_n \xrightarrow{\mathtt{SOT}} T$.

From Theorem 6.3, we immediately deduce

Corollary 6.6. — *A typical $T \in (\mathcal{T}_1(H), \mathtt{SOT})$ is unitarily equivalent to B_∞, and hence has all properties listed in Corollary 3.5. In particular, a typical $T \in (\mathcal{T}_1(H), \mathtt{SOT})$ has a non-trivial invariant subspace.*

If $X = c_0$ or ℓ_p, with canonical basis $(e_j)_{j\geqslant 0}$, one can define $\mathcal{T}_1(X)$ in the obvious way as

$$\mathcal{T}_1(X) = \big\{T \in \mathcal{B}(X)\ ;\ Te_j \in [\,e_0, \dots, e_{j+1}\,] \text{ and } \langle e_{j+1}^*, Te_j\rangle > 0 \text{ for every } j \geqslant 0\big\}$$

and the question of the existence of non-trivial invariant subspaces for a typical operator $T \in (\mathcal{T}_1(X), \mathtt{SOT})$ makes sense. Not surprisingly, this question has a positive answer when $X = \ell_1$.

Proposition 6.7. — *Assume that $X = \ell_1$. Then, a typical $T \in (\mathcal{T}_1(X), \mathtt{SOT})$ has all the properties listed in Theorem 4.1. In particular, a typical $T \in (\mathcal{T}_1(X), \mathtt{SOT})$ has a non-trivial invariant subspace.*

Proof. — By the proof of Theorem 4.1, it is enough to check that a typical $T \in (\mathcal{T}_1(X), \mathtt{SOT})$ has the following properties:

(i) T^* is an isometry;
(ii) $T - \lambda$ has dense range for any $\lambda \in \mathbb{D}$;
(iii) $\sigma(T) \supseteq \mathbb{D}$.

(i) With the notations of the proof of Theorem 4.1, we already know that $\mathcal{I}_* \cap \mathcal{T}_1(X)$ is a G_δ subset of $\mathcal{T}_1(X)$; so we just need to check that $\mathcal{I}_* \cap \mathcal{T}_1(X)$ is dense in $(\mathcal{T}_1(X), \mathtt{SOT})$.

As usual, denote by $P_N : X \to [e_0, \dots, e_N]$ the canonical projection map and set $F_N := [e_j,\ j > N]$. Let also $\phi : \mathbb{Z}_+ \to \mathbb{Z}_+$ be a map taking every value $j \in \mathbb{Z}_+$ infinitely many times and such that $\phi(k) \leqslant k$ for all $k \geqslant 0$.

Given an arbitrary $A \in \mathcal{T}_1(X)$ with $\|A\| < 1$, choose a sequence of positive real numbers $(\varepsilon_n)_{n\geqslant 0}$ such that $\|A\| + \varepsilon_n \leqslant 1$ for all n and $\varepsilon_n \to 0$ as $n \to \infty$, and define for each $N \geqslant 0$ an operator T_N as follows:

$$T_N := P_NAP_N + \varepsilon_0 e_N^* \otimes e_{N+1} + \widetilde{B}_N(I - P_N),$$

where $\widetilde{B}_N : F_N \to X$ is the operator defined by

$$\widetilde{B}_N e_{N+1+k} = (1 - \varepsilon_{1+k})e_{\phi(k)} + \varepsilon_{1+k}e_{N+2+k} \quad \text{for every } k \geqslant 0.$$

Then $T_N \in \mathcal{T}_1(X)$. Moreover, by the choice of the map ϕ and since we are working on ℓ_1, it is easy to check that $\|T_N^* x^*\| \geqslant \|x^*\|$ for every $x^* \in X^* = \ell_\infty$. So we have $T_N \in \mathcal{I}_* \cap \mathcal{T}_1(X)$; and since $T_N \xrightarrow{\texttt{SOT}} A$ as $N \to \infty$, it follows that $\mathcal{I}_* \cap \mathcal{T}_1(X)$ is dense in $(\mathcal{T}_1(X), \texttt{SOT})$.

(ii) The fact that a typical $T \in (\mathcal{T}_1(X), \texttt{SOT})$ is such that $T - \lambda$ has dense range for every $\lambda \in \mathbb{D}$ (in fact, for every $\lambda \in \mathbb{C}$) follows from the proof of Proposition 3.9.

(iii) For any set $E \subseteq \mathbb{C}$, let us denote by $\mathcal{G}_E$ the set of all $T \in \mathcal{B}_1(X)$ such that $T - \lambda$ is not bounded below for any $\lambda \in E$. To prove that a typical $T \in (\mathcal{T}_1(X), \texttt{SOT})$ is such that $\sigma(T) \supseteq \mathbb{D}$, it is obviously enough to show that $\mathcal{G}_{\mathbb{D}} \cap \mathcal{T}_1(X)$ is comeager in $(\mathcal{T}_1(X), \texttt{SOT})$.

By the proof of Proposition 3.9, we know that $\mathcal{G}_K$ is an $\texttt{SOT}$ - G_δ set for any compact set K. Therefore, by the Baire Category Theorem and since $\mathbb{D}$ is a countable union of compact sets, we just have to check that $\mathcal{G}_K \cap \mathcal{T}_1(X)$ is $\texttt{SOT}$ - dense in $\mathcal{T}_1(X)$ for any compact set $K \subseteq \mathbb{D}$. So let us fix such a compact set K.

For any $N \geqslant 0$, let us denote by B_N the canonical backward shift acting on F_N with respect to the basis $(e_j)_{j>N}$. For any $\lambda \in \mathbb{D}$, the operator $B_N - \lambda$ is a Fredholm operator on F_N with $\dim\ker(B_N - \lambda) = 1 = \operatorname{ind}(B_N - \lambda)$. By the standard perturbation theory for Fredholm operators (see for instance [27, Proposition 2.c.9]) and since K is a compact subset of $\mathbb{D}$, it follow that there exists $\delta > 0$ such that, for any $N \geqslant 0$, the following holds true: any operator $R \in \mathcal{B}(F_N)$ with $\|R - B_N\| \leqslant \delta$ is such that $R - \lambda$ is Fredholm with $\operatorname{ind}(R - \lambda) = 1$ for every $\lambda \in K$. In particular, any operator $R \in \mathcal{B}(F_N)$ with $\|R - B_N\| \leqslant \delta$ is such that $R - \lambda$ is not one-to-one and hence not bounded below for any $\lambda \in K$.

Now, given an arbitrary $A \in \mathcal{T}_1(X)$ with $\|A\| < 1$, define for each $N \geqslant 0$ an operator $T_N \in \mathcal{B}(X)$ as follows:

$$T_N := P_N A P_N + \eta e_N^* \otimes e_{N+1} + J_N\big((1 - \delta/2) B_N + \delta/2\, S_N\big)(I - P_N),$$

where $J_N : F_N \to X$ is the canonical inclusion, S_N is the canonical forward shift acting on F_N and $\eta > 0$ is such that $\|A\| + \eta \leqslant 1$. Then $T_N \in \mathcal{T}_1(X)$; and by the choice of δ, we also have $T \in \mathcal{G}_K$. Since $T_N \xrightarrow{\texttt{SOT}} A$ as $N \to \infty$, this shows that $\mathcal{G}_K \cap \mathcal{T}_1(X)$ is dense in $(\mathcal{T}_1(X), \texttt{SOT})$. $\square$

Corollary 6.6 and Proposition 6.7 leave open the question to know if on $X = c_0$ or $X = \ell_p$, $1 < p < \infty$, $p \neq 2$, a typical $T \in (\mathcal{T}_1(X), \texttt{SOT})$ has a non-trivial invariant subspace. It is possible to prove directly that a typical $T \in (\mathcal{T}_1(X), \texttt{SOT})$ satisfies $\sigma(T) = \overline{\mathbb{D}}$. When $X = \ell_p$, $1 < p < \infty$, the results of [30] imply again that a typical $T \in (\mathcal{T}_1(X), \texttt{SOT})$ has a non-trivial invariant closed cone.

7. $\texttt{SOT}^*$- typical contractions on ℓ_p, $1 < p < \infty$

Let $X = \ell_p$, $1 < p < \infty$. The properties of $\texttt{SOT}^*$- typical operators of $\mathcal{B}_1(X)$ may be very different from those of $\texttt{SOT}$ - typical operators. Basically, the main difference between the two topologies is that the map $T \mapsto T^*$ is $\texttt{SOT}^*$ - continuous (because X is reflexive) but not $\texttt{SOT}$ - continuous. This allows for example to show in a not too complicated way that for any $M > 1$, an $\texttt{SOT}^*$ - typical $T \in \mathcal{B}_M(X)$ is such that T^* is hypercyclic (see [18, Corollary 2.12]); which implies that T^* is not an isometry and T has no eigenvalue. By homogeneity, it follows that a typical $T \in (\mathcal{B}_1(X), \texttt{SOT}^*)$ has no eigenvalue. (In the Hilbertian case, this was also proved in [13, Proposition 6.3].) So we may state

Proposition 7.1. — *Let $X = \ell_p$, $1 < p < \infty$. A typical $T \in (\mathcal{B}_1(X), \texttt{SOT}^*)$ has no eigenvalue and T^* is not an isometry.*

In contrast, recall what we know for the topology $\texttt{SOT}$: an $\texttt{SOT}$-typical contraction on ℓ_2 has plenty of eigenvalues; an $\texttt{SOT}$-typical contraction on ℓ_p, $p > 2$ has no eigenvalue but this seems to be rather hard to prove; and we have no idea of what is the situation for $1 < p < 2$.

Among the results proved in Section 3, what remains valid for every ℓ_p-space in the $\texttt{SOT}^*$ setting is the following analogue of Proposition 3.9. In the Hilbertian case, this was proved in [13, Proposition 6.11].

Proposition 7.2. — *Let $X = \ell_p$, $1 < p < \infty$. A typical $T \in (\mathcal{B}_1(X), \texttt{SOT}^*)$ has the following properties: $T - \lambda$ has dense range for every $\lambda \in \mathbb{C}$, and $\sigma(T) = \sigma_{ap}(T) = \overline{\mathbb{D}}$.*

Proof. — Using the same notation as in the proof of Proposition 3.9, we observe that the sets $\mathcal{G}_1$ and $\mathcal{G}_2$, being $\texttt{SOT}$-G_δ, are also $\texttt{SOT}^*$-G_δ. The only issue is thus to prove that these two sets are $\texttt{SOT}^*$-dense in $\mathcal{B}_1(X)$. Since the set of hypercyclic operators is in fact $\texttt{SOT}^*$-dense in $\mathcal{B}_M(X)$ for any $M > 1$ by [18, Corollary 2.12], the same argument as in the proof of Proposition 3.9 shows that $\mathcal{G}_1$ is $\texttt{SOT}^*$-dense in $\mathcal{B}_1(X)$. As to the $\texttt{SOT}^*$-density of the set $\mathcal{G}_2$, the argument is exactly the same since the sequence of operators (T_N) associated in the proof to an arbitrary $A \in \mathcal{B}_1(X)$ actually tends to A for the topology $\texttt{SOT}^*$. ☐

Corollary 7.3. — *A typical $T \in (\mathcal{B}_1(\ell_2), \texttt{SOT}^*)$ has a non-trivial invariant subspace.*

Proof. — This follows immediately from Proposition 7.2 and the Brown-Chevreau-Pearcy Theorem from [5], which states that any Hilbert space contraction whose spectrum contains the unit circle has a non-trivial invariant subspace. ☐

Corollary 7.4. — *For any $1 < p < \infty$, a typical $T \in (\mathcal{B}_1(\ell_p), \texttt{SOT}^*)$ has a non-trivial invariant closed cone.*

Proof. — This follows directly from the fact that an $\texttt{SOT}^*$-typical $T \in \mathcal{B}_1(X)$ is such that $\sigma(T) = \overline{\mathbb{D}}$, together with the result of [30]; see the discussion before Proposition 5.26. ☐

We do not know of any substantially simpler way than using the Brown-Chevreau-Pearcy Theorem to prove that an $\texttt{SOT}^*$-typical contraction on the Hilbert space has a non-trivial invariant subspace. Therefore, it is natural to investigate whether an $\texttt{SOT}^*$-typical $T \in \mathcal{B}_1(H)$ could enjoy some other properties, which would imply in a more elementary way that T has a non-trivial invariant subspace. Theorem 2.6 answers one of the many questions one can ask in this vein: we show that an $\texttt{SOT}^*$-typical contraction on a (complex, separable, infinite-dimensional) Hilbert space does not commute with any non-zero compact operator on H. One can therefore not use this approach, via the Lomonosov Theorem, to show that an $\texttt{SOT}^*$-typical $T \in \mathcal{B}_1(H)$ has a non-trivial invariant subspace.

Theorem 7.5. — *A typical $T \in (\mathcal{B}_1(H), \texttt{SOT}^*)$ does not commute with any non-zero compact operator.*

Before embarking on the proof of this result, let us recall some standard notations. For every $T \in \mathcal{B}(H)$, we denote by $\{T\}'$ the commutant of T:

$$\{T\}' := \{A \in \mathcal{B}_1(H)\ ;\ AT = TA\}.$$

Also, we denote by $\|T\|_e$ the essential norm of T, that is the distance of T to the algebra $\mathcal{K}(H)$ of compact operators:

$$\|T\|_e = \inf_{K \in \mathcal{K}(H)} \|T - K\|;$$

and by $r_e(T)$ the essential spectral radius of T:

$$r_e(T) = \max\{|z|\ ;\ z \in \sigma_e(T)\}.$$

Our strategy for proving Theorem 7.5 is the following. Let us denote by $\mathcal{M}$ the set of all $T \in \mathcal{B}_1(H)$ which do not commute with a non-zero compact operator, and by $\mathcal{M}_e$ the set of all $T \in \mathcal{B}_1(H)$ such that $\|A\| = \|A\|_e$ for every $A \in \{T\}'$. Clearly $\mathcal{M}_e \subseteq \mathcal{M}$. We will show that:

- $\mathcal{M}_e$ is $\texttt{SOT*}$-dense in $\mathcal{B}_1(H)$,
- there is an $\texttt{SOT*}$-G_δ set $\mathcal{G} \subseteq \mathcal{B}_1(H)$ such that $\mathcal{M}_e \subseteq \mathcal{G} \subseteq \mathcal{M}$.

Standard examples of operators T such that $\|A\| = \|A\|_e$ for every $A \in \{T\}'$ are the multiplication operators on the Hardy space $H^2(\mathbb{D})$. For any function $\phi \in H^\infty(\mathbb{D})$, let M_ϕ be the associated multiplication operator acting on $H^2(\mathbb{D})$. Denoting by $\mathbf{z} \in H^\infty(\mathbb{D})$ the function $z \mapsto z$, the commutant of $M_{\mathbf{z}}$ is equal to $\{M_\phi;\ \phi \in H^\infty(\mathbb{D})\}$. Moreover, standard arguments (see for instance [29, Section 3.3]) show that $\|M_\phi\|_e = \|\phi\|_\infty = \|M_\phi\|$ for any $\phi \in H^\infty(\mathbb{D})$. With this example in mind, we present in Theorem 7.6 below a general condition on an operator T acting on a Hilbert space implying that $\|A\|_e = \|A\|$ for any $A \in \{T\}'$. The assumptions on T concern the eigenvectors of its adjoint T^*, and they permit to represent T as a multiplication operator on a certain Hilbert space of holomorphic functions on the unit disk $\mathbb{D}$ (see [21, Problem 85]).

Theorem 7.6. — *Let H be a Hilbert space, and let $T \in \mathcal{B}_1(H)$. Suppose that there exists a family $(f_w)_{w\in\mathbb{D}}$ of vectors of H satisfying the following properties:*

(1) *the map $w \mapsto f_w$ is holomorphic on $\mathbb{D}$;*
(2) *for every $w \in \mathbb{D}$, we have $T^* f_w = w\, f_w$;*
(3) *$[\, f_w\ ;\ w \in \mathbb{D}\,] = H$;*
(4) *there exists a cyclic vector $x_0 \in H$ for Tsuch that $\langle f_w, x_0\rangle = 1$ for every $w \in \mathbb{D}$.*

Then every operator $A \in \{T\}'$ is such that $\|A\|_e = \|A\|$.

Proof of Theorem 7.6. — To each vector $x \in H$, we associate the map $\widetilde{x} : \mathbb{D} \to \mathbb{C}$ defined by

$$\widetilde{x}(w) := \langle f_{\overline{w}}\,, x\rangle \qquad \text{for every } w \in \mathbb{D}.$$

Let $\mathcal{H} := \{\widetilde{x}\ ;\ x \in H\}$. Then $\mathcal{H}$ is a linear space of holomorphic functions on $\mathbb{D}$, by (1); and by property (3), the map $U : H \to \mathcal{H}$ defined by $Ux := \widetilde{x}$ is a linear isomorphism from H onto $\mathcal{H}$. Therefore, $\mathcal{H}$ becomes a Hilbert space when endowed with the scalar product defined by $\langle \widetilde{x}, \widetilde{y}\rangle := \langle x, y\rangle$, and $U : H \to \mathcal{H}$ is a unitary operator.

Next, we observe that $\mathcal{H}$ is a reproducing kernel Hilbert space, with reproducing kernels $\widetilde{f_{\overline{w}}}$, $w \in \mathbb{D}$: for every $w \in \mathbb{D}$ and every function $\widetilde{x} \in \mathcal{H}$, $\langle \widetilde{f_{\overline{w}}}\,, \widetilde{x}\rangle = \langle f_{\overline{w}}\,, x\rangle = \widetilde{x}(w)$. Observe also that $\widetilde{x_0} = \mathbf{1}$ by property (4), so that the constant function $\mathbf{1}$ belongs to $\mathcal{H}$; and that if we denote by $\mathbf{w}$ the function $w \mapsto w$, then $\widetilde{T^n x_0} = \mathbf{w}^n$ for all $n \in \mathbb{N}$, by property (2). Hence, $\mathcal{H}$ contains all polynomial functions. Moreover, since x_0 is a cyclic vector for T and $\langle \widetilde{x}, \mathbf{w}^n\rangle = \langle \widetilde{x}, \widetilde{T^n x_0}\rangle = \langle x, T^n x_0\rangle$ for every $\widetilde{x} \in \mathcal{H}$ and all $n \geqslant 0$, the polynomial functions are dense in $\mathcal{H}$.

Using property (2), it is easily checked that $\widetilde{Tx}(w) = w\,\widetilde{x}(w)$ for every $\widetilde{x} \in \mathcal{H}$ and all $w \in \mathbb{D}$. This means that the multiplication operator $M_{\mathbf{w}}$ is well defined and bounded on $\mathcal{H}$ and that $T = U^{-1} M_{\mathbf{w}} U$. So we have $\|M_{\mathbf{w}}\| \leqslant 1$, and T is unitarily equivalent to $M_{\mathbf{w}}$.

This description of T as a multiplication operator on $\mathcal{H}$ yields a description of the commutant of T as a certain family of multiplication operators. Indeed, let $A \in \{T\}'$. Then $B := UAU^{-1}$ commutes with $M_{\mathbf{w}}$ on $\mathcal{H}$. Set $\phi := B\mathbf{1} \in \mathcal{H}$. For every $n \geqslant 0$, we have

$$M_{\mathbf{w}}^n \phi = M_{\mathbf{w}}^n B\mathbf{1} = BM_{\mathbf{w}}^n \mathbf{1} = B(\mathbf{w}^n).$$

Hence $Bp = \phi p$ for any polynomial function $p \in \mathcal{H}$. Since the polynomial functions are dense in $\mathcal{H}$ and since the evaluation functionals $\widetilde{x} \mapsto \widetilde{x}(w)$ are continuous on $\mathcal{H}$, it follows that $B\widetilde{x} = \phi\widetilde{x}$ for every $\widetilde{x} \in \mathcal{H}$. Accordingly, we now write $B = M_\phi$.

Our goal is now to show that the function ϕ is bounded on $\mathbb{D}$, and that we have $\|M_\phi\| = \|\phi\|_\infty = \|M_\phi\|_e$.

On the one hand, the operator M_ϕ is bounded on $\mathcal{H}$, so that $\|M_\phi^* \widetilde{f_{\overline{w}}}\| \leqslant \|M_\phi\| \, \|\widetilde{f_{\overline{w}}}\|$ for every $w \in \mathbb{D}$. But $M_\phi^* \widetilde{f_{\overline{w}}} = \phi(w)\, \widetilde{f_{\overline{w}}}$ because $\widetilde{f_{\overline{w}}}$ is the reproducing kernel for $\mathcal{H}$ at w. So we get $|\phi(w)| \leqslant \|M_\phi\|$ for every $w \in \mathbb{D}$; and hence ϕ is bounded with $\|\phi\|_\infty \leqslant \|M_\phi\|$.

On the other hand, observe that $M_{\mathbf{w}}$ is a completely non-unitary contraction on $\mathcal{H}$. Indeed, for every $w \in \mathbb{D}$, $\|M_{\mathbf{w}}^{*n} \widetilde{f_w}\| = |w|^n \, \|\widetilde{f_w}\|$ tends to 0 as n tends to infinity; and since $[\, \widetilde{f_w} \ ; \ w \in \mathbb{D}\,] = \mathcal{H}$, it follows that $\|M_w^{*n} \widetilde{x}\| \to 0$ for every $\widetilde{x} \in \mathcal{H}$. Being completely non-unitary, $M_{\mathbf{w}}$ admits a continuous functional calculus from $(H^\infty(\mathbb{D}), w^*)$ into $(\mathcal{B}(\mathcal{H}), \texttt{SOT})$, and the von Neumann inequality extends to functions of $H^\infty(\mathbb{D})$ (see [37] or [3]). Hence $\|\phi(M_{\mathbf{w}})\| \leqslant \|\phi\|_\infty$. Now $\psi(M_{\mathbf{w}}) = M_\psi$ for every $\psi \in H^\infty(\mathbb{D})$, *i.e.* $\psi(M_{\mathbf{w}})\, \widetilde{x} = \psi\, \widetilde{x}$ for all $\widetilde{x} \in \mathcal{H}$. Indeed, this is clear if ψ is a polynomial function; and the general case follows from the continuity of the functional calculus, the w^*-density of the polynomial functions in $H^\infty(\mathbb{D})$, and the continuity of the evaluation functionals on $(H^\infty(\mathbb{D}), w^*)$ and on the space $\mathcal{H}$. In particular $\phi(M_{\mathbf{w}}) = M_\phi$, and hence $\|M_\phi\| \leqslant \|\phi\|_\infty$.

The last step is to prove that $\|M_\phi\|_e \geqslant \|\phi\|_\infty$. Since $M_\phi^* f_{\overline{w}} = \phi(w) f_{\overline{w}}$ for every $w \in \mathbb{D}$, we see that $\phi(\mathbb{D}) \subseteq \sigma(M_\phi^*)$, and hence $\overline{\phi(\mathbb{D})} \subseteq \sigma(M_\phi^*)$. Choose $\lambda_0 \in \overline{\phi(\mathbb{D})}$ such that $|\lambda_0| = \|\phi\|_\infty$. Since $\|M_\phi^*\| = \|M_\phi\| = \|\phi\|_\infty$, λ_0 is a boundary point of the spectrum $\sigma(M_\phi^*)$ of M_ϕ^*. Moreover, since $\phi(\mathbb{D})$ is an open subset of $\mathbb{C}$, λ_0 is not an isolated point of $\sigma(M_\phi^*)$. Thus, we see that $\lambda_0 \in \partial\sigma(M_\phi^*)$ and λ_0 is not an isolated point of $\sigma(M_\phi^*)$. These two properties imply that $\lambda_0 \in \sigma_e(M_\phi^*)$ (see e.g [9, Theorem XI.6.8]). So we have $r_e(M_\phi^*) \geqslant \|\phi\|_\infty$, and hence $\|M_\phi\|_e = \|M_\phi^*\|_e \geqslant \|\phi\|_\infty$.

Since the operator $A \in \{T\}'$ we started with is unitarily equivalent to $B = M_\phi$, we can now conclude that $\|A\|_e = \|A\|$. □

Proof of Theorem 7.5. — Let $(K_q)_{q\geqslant 1}$ be a sequence of non-zero compact operators which is dense in $\mathcal{K}(H)$ for the operator norm topology. For each $q \geqslant 1$, set

$$\mathfrak{B}_q := \overline{B}\,(K_q, \|K_q\|/3) \subseteq \mathcal{B}(H).$$

We first observe that

$$\mathcal{K}(H)\backslash\{0\} \subseteq \bigcup_{q\geqslant 1} \mathfrak{B}_q.$$

Indeed, let K be a non-zero compact operator on H, and let $\varepsilon > 0$ be such that $\|K\| > 4\varepsilon$. Choose $q \geqslant 1$ such that $\|K - K_q\| < \varepsilon$. Then $\|K_q\| > 3\varepsilon$, so that

$$K \in B(K_q, \varepsilon) \subseteq B(K_q, \|K_q\|/3) \subseteq \mathfrak{B}_q.$$

Next, for each $q \geqslant 1$, consider the set

$$\mathcal{F}_q := \{T \in \mathcal{B}_1(H) \ ; \ \exists\, A \in \mathfrak{B}_q \ ; \ AT = TA\}\cdot$$

We prove the following fact, where we use in a crucial way the topology $\texttt{SOT}^*$ (the argument would break down if we were to consider the topology $\texttt{SOT}$ instead).

Fact 7.7. — For each $q \geqslant 1$, the set $\mathcal{F}_q$ is closed in $(\mathcal{B}_1(H), \texttt{SOT}^*)$.

Proof of Fact 7.7. — Let

$$\mathfrak{F}_q := \{(T, A) \in \mathcal{B}_1(H) \times \mathfrak{B}_q \ ; \ AT = TA\},$$

so that $\mathcal{F}_q$ is the projection of $\mathfrak{F}_q$ along the second coordinate. Since $(\mathfrak{B}_q, \texttt{WOT})$ is compact, it is enough to show that $\mathfrak{F}_q$ is closed in $(\mathcal{B}_1(H), \texttt{SOT}^*) \times (\mathfrak{B}_q, \texttt{WOT})$.

Let (T_n, A_n) belong to $\mathfrak{F}_q$ for every $n \in \mathbb{N}$, and suppose that $T_n \xrightarrow{\texttt{SOT}^*} T$ and $A_n \xrightarrow{\texttt{WOT}} A$, with $(T, A) \in \mathcal{B}_1(H) \times \mathfrak{B}_q$. For every vectors $x, y \in H$ and every $n \geqslant 1$, we have

$$\langle y, A_n T_n x\rangle = \langle y, T_n A_n x\rangle, \quad \text{i.e.} \quad \langle A_n^* y, T_n x\rangle = \langle T_n^* y, A_n x\rangle.$$

On the one hand, $A_n^* y \to A^* y$ and $A_n x \to Ax$ weakly, and on the other hand, $T_n x \to Tx$ and $T_n^* y \to T^* y$ in norm. It follows that

$$\langle A_n^* y, T_n x\rangle \to \langle A^* y, Tx\rangle \quad \text{and} \quad \langle T_n^* y, A_n x\rangle \to \langle T^* y, Ax\rangle,$$

so that $\langle y, ATx\rangle = \langle y, TAx\rangle$. Thus $AT = TA$, which proves that $\mathfrak{F}_q$ is indeed closed in $(\mathcal{B}_1(H), \texttt{SOT}^*) \times (\mathfrak{B}_q, \texttt{WOT})$. $\square$

Let us now define

$$\mathcal{G} := \mathcal{B}_1(H) \backslash \bigcup_{q \geqslant 1} \mathcal{F}_q,$$

which is a G_δ subset of $(\mathcal{B}_1(H), \texttt{SOT}^*)$ by Fact 7.7.

Recall also that we denote by by $\mathcal{M}$ the set of all $T \in \mathcal{B}_1(H)$ which do not commute with a non-zero compact operator – that is, the set which we want to prove is $\texttt{SOT}^*$-comeager in $\mathcal{B}_1(H)$ – and that

$$\mathcal{M}_e = \{T \in \mathcal{B}_1(H) \ ; \ \forall\, A \in \{T\}', \ \|A\|_e = \|A\|\}.$$

Fact 7.8. — We have $\mathcal{M}_e \subseteq \mathcal{G} \subseteq \mathcal{M}$.

Proof of Fact 7.8. — By the very definition of $\mathcal{G}$ and since $\mathcal{K}(H) \backslash \{0\} \subseteq \bigcup_{q \geqslant 1} \mathfrak{B}_q$, it is clear that $\mathcal{G} \subseteq \mathcal{M}$. Moreover, an operator $T \in \mathcal{B}_1(H)$ belongs to $\mathcal{G}$ if and only if any operator $A \in \{T\}'$ satisfies $\|A - K_q\| > \|K_q\|/3$ for every $q \geqslant 1$. So, proving that $\mathcal{M}_e \subseteq \mathcal{G}$ amounts to showing that if $A \in \mathcal{B}(H)$ is such that $\|A\|_e = \|A\|$, then $\|A - K_q\| > \|K_q\|/3$ for every $q \geqslant 1$. Suppose that $\|A - K_q\| \leqslant \|K_q\|/3$ for some $q \geqslant 1$. If $\|A\| > \|K_q\|/3$, then $\|A\|_e > \|K_q\|/3$, and hence $\|A - K_q\| > \|K_q\|/3$, which is impossible. If $\|A\| \leqslant \|K_q\|/3$, then $\|A - K_q\| \geqslant 2\|K_q\|/3$; hence $\|K_q\|/3 \geqslant 2\|K_q\|/3$, which is impossible since $K_q \neq 0$. $\square$

By Fact 7.8, it now suffices to prove the following proposition in order to terminate the proof of Theorem 7.5.

Proposition 7.9. — *The set $\mathcal{M}_e$ is dense in $(\mathcal{B}_1(H), \texttt{SOT}^*)$.*

Proof of Proposition 7.9. — We are going to prove that the set $\mathcal{C}$ of operators $T \in \mathcal{B}_1(H)$ satisfying the assumptions of Theorem 7.6 is $\texttt{SOT}^*$-dense in $\mathcal{B}_1(H)$. Fix an orthonormal basis $(e_j)_{j \geqslant 0}$ of H, and consider the associated class $\mathcal{T}_1(H)$. Since $\mathcal{C}$ is stable under unitary equivalence, and since the orbit of $\mathcal{T}_1(H)$ under unitary equivalence is $\texttt{SOT}^*$-dense in $\mathcal{B}_1(H)$ by Lemma 6.2, it suffices to show that $\mathcal{C} \cap \mathcal{T}_1(H)$ is $\texttt{SOT}^*$-dense in $\mathcal{T}_1(H)$.

Let $A \in \mathcal{T}_1(H)$. We need to find $T \in \mathcal{C} \cap \mathcal{T}_1(H)$ such that T is $\texttt{SOT}^*$-close to A.

Let $N \in \mathbb{N}$, let $\eta > 0$, and let $B \in \mathcal{T}_1(H)$ be such that $\|(B-A)P_N\| < \eta$. We define an operator $T \in \mathcal{B}(H)$ as follows:

$$T := BP_N + S(I - P_N)\,,$$

where S is the canonical forward shift.

Since $B \in \mathcal{T}_1(H)$, we see that $T \in \mathcal{T}_1(H)$. Moreover, it is not hard to check that if N is large enough and η is small enough then, regardless of the choice of B such that $\|(B-A)P_N\| < \eta$, the operator T is $\mathtt{SOT}^*$- close to A. So we fix N large enough and η small enough, and our goal is now to choose B in such a way that $T \in \mathcal{C}$, *i.e.* there exists a family $(f_w)_{w\in\mathbb{D}}$ of vectors of H satisfying properties (1) – (4) of Theorem 7.6.

Let us denote by $B_N := P_N B P_N$ the compression of the operator B to the space $E_N = [e_0, \dots, e_N]$, and set $b_N := \langle Be_N, e_{N+1}\rangle$. Then

$$T^* e_j = \begin{cases} B_N^* e_j & \text{if } 0 \leqslant j \leqslant N, \\ b_N\, e_N & \text{if } j = N+1 \\ e_{j-1} & \text{if } j > N+1. \end{cases}$$

We now choose B in such a way that the following properties hold true.

- The eigenvalues of the matrix B_N^* are all distinct.
- Write $\sigma(B_N^*) = \{\lambda_0, \dots, \lambda_N\}$ and, for each $0 \leqslant n \leqslant N$, let $v_n \in E_N$ be an eigenvector of B_N^* associated to the eigenvalue λ_n (so that $(v_0, \dots, v_N)$ is a basis of E_N). Then $\langle v_n, e_0\rangle \neq 0$ for $n = 0, \dots, N$, and the vector e_N can be written as a linear combination $e_N = \sum_{n=0}^N \beta_n v_n$ where all the coefficients β_n are non-zero;

For every $w \in \mathbb{D}\backslash\sigma(B_N^*)$, we define

$$\begin{aligned} g_w :=& b_N\,(w - B_N^*)^{-1} e_N + \sum_{j\geqslant N+1} w^{j-(N+1)} e_j \\ =& \, b_N \sum_{n=0}^{N} \frac{\beta_n}{w - w_n}\, v_n + \sum_{j\geqslant N+1} w^{j-(N+1)} e_j. \end{aligned}$$

It is not difficult to check that $T^* g_w = w\, u_w$ for every $w \in \mathbb{D}\backslash\sigma(B_N^*)$. Now, we set

$$f_w := p(w) g_w \qquad \text{where} \quad p(w) := \prod_{n=0}^{N} (w - \lambda_n).$$

Then the map $w \mapsto f_w$ extends holomorphically to the whole disk $\mathbb{D}$, and we have $T^* f_w = w\, f_w$ for every $w \in \mathbb{D}$. So conditions (1) and (2) from Theorem 7.6 are satisfied. In order to check condition (3), suppose that $x \in H$, written as $x = \sum_{j\geqslant 0} x_j e_j$, is such that $\langle f_w, x\rangle = 0$ for every $w \in \mathbb{D}$. Then $\langle g_w, x\rangle = 0$ for every $w \in \mathbb{D}\backslash\{w_0, \dots, w_N\}$, *i.e.*

$$b_N \sum_{n=0}^{N} \frac{\beta_n \langle v_n, x\rangle}{w - \lambda_n} = - \sum_{j\geqslant N+1} \overline{x_j}\, w^{j-(N+1)} \quad \text{for every } w \in \mathbb{D}\backslash\{\lambda_0, \dots, \lambda_N\}.$$

Since the right hand side of this identity defines a holomorphic function on $\mathbb{D}$ and since $b_N > 0$ and $\beta_n \neq 0$ for every $0 \leqslant n \leqslant N$, it follows that $\langle v_n, x\rangle = 0$ for every $0 \leqslant n \leqslant N$. This implies on the one hand that $x_j = 0$ for every $0 \leqslant j \leqslant N$ (since the vectors v_n form a basis of E_N); and, on the other hand, that the holomorphic function $w \mapsto \sum_{j\geqslant N+1} \overline{x_j}\, w^{j-(N+1)}$ is identically zero on $\mathbb{D}$, and so $x_j = 0$ for every $j \geqslant N+1$. We have proved that $x = 0$, which yields condition (3).

Lastly, we have to prove condition (4), namely that there exists $x_0 \in H$ which is a cyclic vector for T and such that $\langle f_w, x_0\rangle = 1$ for every $w \in \mathbb{D}$. To do this, we note that

$$\langle f_w, e_{N+1}\rangle = p(w) \qquad \text{and} \qquad \langle f_w, e_0\rangle = p(w) \sum_{n=0}^{N} \frac{\beta_n \langle v_n, e_0\rangle}{w - \lambda_n} := q(w).$$

By the choice of p, we see that q is a polynomial. Explicitely:

$$q(w) = \sum_{n=0}^{N} \beta_n \langle v_n, e_0\rangle \prod_{\substack{k=0\\k\neq n}}^{N} (w - \lambda_k), \quad w \in \mathbb{D}.$$

For every $n = 0, \dots, N$, we have

$$q(\lambda_n) = \beta_n \langle v_n, e_0\rangle \prod_{\substack{k=0\\k\neq n}}^{N} (\lambda_n - \lambda_k).$$

Since $\beta_n \neq 0$, $\langle v_n, e_0\rangle \neq 0$ and $\lambda_0, \dots, \lambda_N$ are pairwise distinct, it follows that $q(\lambda_n) \neq 0$ for every $0 \leqslant n \leqslant N$. Since the roots of the polynomial p are exactly the numbers $\lambda_0, \dots, \lambda_N$, we thus see that the polynomials p and q have no common zeros; so there exist two polynomials r and s such that $rp + sq = 1$. Let $x_0 := r(T)e_{N+1} + s(T)e_0$. Then, we have for every $w \in \mathbb{D}$:

$$\begin{aligned}\langle f_w, x_0\rangle &= \langle r(T)^* f_w, e_{N+1}\rangle + \langle s(T)^* f_w, e_0\rangle\\ &= r(w)\langle f_w, e_{N+1}\rangle + s(w)\langle f_w, e_0\rangle\\ &= (rp + sq)(w) = 1.\end{aligned}$$

Also $\langle f_w, T^n x_0\rangle = w^n$ for all $n \geqslant 0$, and thus $\langle f_w, q(T)x_0\rangle = q(w) = \langle f_w, e_0\rangle$, *i.e.* $\langle f_w, q(T)x_0 - e_0\rangle = 0$ for every $w \in \mathbb{D}$. So $q(T)x_0 = e_0$ by (3), and since e_0 is cyclic for T, it follows that x_0 is cyclic as well. This proves condition (4), and terminates the proof that T satisfies the assumptions of Theorem 7.6. □

Proposition 7.9 is thus proved, and Theorem 7.5 follows. □

Remark 7.10. — It is also true that a typical $T \in (\mathcal{B}_1(H), \texttt{SOT})$ does not commute with any non-zero compact operator, but this is much easier to prove thanks to the Eisner-Mátrai Theorem. Indeed, the operator B_∞ on $\ell_2(\mathbb{Z}_+, \ell_2)$ being a co-isometry whose powers B_∞^n tend to zero for SOT, it does not commute with any non-zero compact operator.

8. Further remarks and questions

We collect in this final section some of the many questions which arise naturally in connection with the results presented above.

We know [13] that the orbit of the backward shift of infinite multiplicity B_∞ under unitary equivalence is comeager in $(\mathcal{B}_1(H), \texttt{SOT})$, where H is a Hilbert space. Since many of the properties of SOT-typical contractions on a Hilbert space are also true on $X = \ell_1$, it is natural to ask:

Question 8.1. — Let $X = \ell_1$. Does there exist an operator $T_0 \in \mathcal{B}_1(X)$ whose similarity orbit intersected with $\mathcal{B}_1(X)$ is SOT-comeager in $\mathcal{B}_1(X)$?

One can ask a similar question when $X = \ell_p$ for $p \neq 1$ and $p \neq 2$, both for the topology SOT and for the topology SOT*, although a positive answer does not seem very likely:

Question 8.2. — Let $X = \ell_p$, $1 < p < \infty$, $p \neq 2$. Does there exist an operator $T_0 \in \mathcal{B}_1(X)$ whose similarity orbit intersected with $\mathcal{B}_1(X)$ is SOT-comeager (resp. SOT*-comeager) in $\mathcal{B}_1(X)$? When $p = 2$, does there exist an operator $T_0 \in \mathcal{B}_1(H)$ whose orbit under unitary equivalence is SOT*-comeager in $\mathcal{B}_1(H)$?

In another direction, the proof of Theorem 4.1 suggests the following question.

Question 8.3. — Let X be a Banach space. Assume that a typical $T \in (\mathcal{B}_1(X), \texttt{SOT})$ is not one-to-one. Does it follow that a typical $T \in \mathcal{B}_1(X)$ is such that $\dim\ker(T) = \infty$?

The results of Sections 4 and 5 suggest of course the following questions:

Question 8.4. — Let $X = \ell_p$, $1 < p < \infty$, $p \neq 2$ or $X = c_0$. Does a typical operator $T \in (\mathcal{B}_1(X), \texttt{SOT})$ have a non-trivial invariant subspace? What about a typical operator $T \in (\mathcal{B}_1(X), \texttt{SOT}^*)$?

Question 8.5. — Let $X = \ell_p$, $1 < p < 2$ or $X = c_0$. Is it true that for any $M > 1$, a typical $T \in (\mathcal{B}_1(X), \texttt{SOT})$ is such that $(MT)^*$ is hypercyclic? In the ℓ_p-case, is it true at least that a typical $T \in \mathcal{B}_1(X)$ has no eigenvalue?

Question 8.6. — Let $X = \ell_p$, $1 < p < 2$. Is it true that every SOT*-comeager subset of $\mathcal{B}_1(X)$ is also SOT-comeager?

Question 8.7. — Let $X = c_0$. Does a typical $T \in (\mathcal{B}_1(X), \texttt{SOT})$ have a non-trivial invariant closed cone?

We have observed in Remark 7.10 (respectively proved in Theorem 7.5) that a typical $T \in (\mathcal{B}_1(H), \texttt{SOT})$ (resp. $T \in (\mathcal{B}_1(H), \texttt{SOT}^*)$) does not commute with a non-zero compact operator. This motivates several questions.

Question 8.8. — Let $X = \ell_p$, $1 < p < \infty$, $p \neq 2$. Is it true that a typical operator $T \in (\mathcal{B}_1(X), \texttt{SOT})$ does not commute with any non-zero compact operator? What about a typical $T \in (\mathcal{B}_1(X), \texttt{SOT}^*)$?

The Lomonosov Theorem implies that an operator commuting with a non-zero compact operator has a non-trivial invariant subspace, but the full statement of the Lomonosov Theorem is actually much stronger:

If $T \in \mathcal{B}_1(X)$ is such that its commutant $\{T\}'$ contains an operator A which is not a multiple of the identity operator and which commutes with a non-zero compact operator, then T has a non-trivial invariant subspace.

We call (LH) the hypothesis of the Lomonosov Theorem. It was proved by Hadwin, Nordgren, Radjavi, and Rosenthal in [20] that there exists an operator T on a complex separable Hilbert space H which does not satisfy (LH): for every $A \in \{T\}' \backslash \mathbb{C}Id$, one has $\{A\}' \cap \mathcal{K}(H) = \{0\}$. We do not know whether such operators are typical in $(\mathcal{B}_1(H), \texttt{SOT}^*)$:

Question 8.9. — Let H be a Hilbert space. Is it true that a typical $T \in (\mathcal{B}_1(H), \texttt{SOT}^*)$ does not satisfy (LH)?

Observe that $B_\infty \in \mathcal{B}_1(\ell_2(\mathbb{Z}_+, \ell_2))$ does satisfy (LH): this follows from a result of [10], which shows that the unweighted forward shift S on ℓ_2 commutes with a non-zero compact operator; as a consequence, a typical operator $T \in (\mathcal{B}_1(H), \texttt{SOT})$ satisfies (LH).

Question 8.10. — Let $X = \ell_p$, $1 < p < \infty$, $p \neq 2$. Is it true that a typical operator $T \in (\mathcal{B}_1(X), \mathtt{SOT})$ satisfies (LH)? What about a typical $T \in (\mathcal{B}_1(X), \mathtt{SOT}^*)$?

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