ON THE DYNAMICS OF COMPOSITION OPERATORS: SUPERCYCLICITY, ODOMETERS AND TRANSLATIONS

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Abstract. We study the dynamical properties of composition operators acting on Banach spaces of measurable functions. In particular, we study in some detail the composition operators induced by odometers, which allows us to give a variety of new examples and counter-examples. We also get general statements about supercyclicity and frequent hypercyclicity of composition operators on L_p - spaces.

1. Introduction

1.1. **Linear dynamics.** Linear dynamics, the study of the dynamical properties of linear operators, has been the object of extensive research over the past forty years. We refer to [6] and [21] for detailed presentations of the area, and to [18] for a survey of more recent developments. Here, we just recall the few basic definitions which will be needed in this paper. We fix a complex Banach space X and $T \in \mathfrak{L}(X)$, a bounded linear operator on X.

The first definitions are related to the behaviour of T with respect to the nonempty open subsets of X. We say that T is **topologically transitive** if for all nonempty open sets $U, V \subset X$, there exists $n \in \mathbb{N}$ such that $T^n(U) \cap V \neq \varnothing$; and that T is **topologically mixing** if for all $U, V \subset X$ nonempty open, there exists $n_0 \in \mathbb{N}$ such that $T^n(U) \cap V \neq \varnothing$ for all $n \geq n_0$.

The next definitions are related to the orbits of T. We say that T is **hypercyclic** if there exists $x \in X$ with a dense T - orbit, *i.e.* $O(x,T) := \{T^n x : \; n \geq 0\}$ is dense in X. This can happen only if X is separable, and when it is so, hypercyclicity is equivalent to topological transitivity. More generally, given a set $\Gamma \subset \mathbb{C}$, we say that T is **Γ - supercyclic** if there exists $x \in X$ such that $\Gamma \cdot O(x,T) = \{zT^n x : \; z \in \Gamma, \; n \geq 0\}$ is dense in X. The vector x is then called a Γ-supercyclic vector for T. So "$\{1\}$ - supercyclic" means "hypercyclic"; and $\mathbb{C}$ - supercyclic operators are simply called supercyclic.

Hypercyclic vectors have "big" orbits. At the other extreme, a vector $u \in X$ is called a **periodic point** for T if there exists $d \geq 1$ such that $T^d(u) = u$. The operator T is said to be **chaotic** if it is hypercyclic with a dense set of periodic points.

Finally, we say that T is **frequently hypercyclic** (resp. **$\mathcal{U}$ - frequently hypercyclic**) if there exists $x \in X$ such that, for every nonempty open set $V \subset X$, the set $\mathcal{N}_T(x,V) := \{n \in \mathbb{N} : \; T^n x \in V\}$ has positive lower density (resp. positive upper density).

Date: May 15, 2026.

An important line of investigation in linear dynamics is the study of some concrete classes of operators, with a twofold aim: to find examples and counterexamples for various dynamical properties and the possible implications between them, and to understand better these classes of operators. In this paper, we intend to follow this plan for *composition operators* acting on Banach spaces of measurable functions, with a strong emphasis on composition operators induced by *odometers*. The study of this class of operators has been pioneered in [9], and pursued very recently in [15]. Let us first introduce the general framework of composition operators on Lebesgue spaces.

1.2. **Composition operators.** Let $(\Omega, \mathcal{B}, \mu)$ be a σ-finite measure space (with $\mu \neq 0$), and let $\varphi : \Omega \to \Omega$ be a measurable transformation. We assume that the transformation φ is **nonsingular**, which means that $\mu(B) = 0$ implies $\mu(\varphi^{-1}(B)) = 0$, for any $B \in \mathcal{B}$; in other words, the image measure $\mu\varphi^{-1}$ is absolutely continuous with respect to μ. We say that $(\Omega, \mathcal{B}, \mu, \varphi)$ is a *nonsingular measurable system*.

Denoting by $L_0(\Omega, \mu)$ the linear space of all measurable functions $f : \Omega \to \mathbb{C}$, where two functions are identified if they are μ - almost everywhere equal, the nonsingularity assumption ensures that the operator $C_\varphi : L_0(\Omega, \mu) \to L_0(\Omega, \mu),\ f \mapsto f \circ \varphi$ is well defined. This operator C_φ is called the **composition operator** with symbol φ. It is well-known that C_φ is bounded on $L_p(\Omega, \mu)$, $1 \leq p < \infty$ if and only if there exists some constant $K < \infty$ such that

$$\mu(\varphi^{-1}(B)) \leq K\mu(B) \quad \text{for all } B \in \mathcal{B}. \tag{1.1}$$

In that case $\|C_\varphi\| = \|f_\varphi\|_\infty^{1/p}$, where f_φ is the Radon-Nykodim derivative of the measure $\mu\varphi^{-1}$ with respect to μ (see [28, Theorem 2.1.1]).

We warn the reader that the definition of nonsingularity varies in the literature. In the present paper, we have chosen the definition which is most commonly used in the study of composition operators, whereas in many references on dynamical systems, a transformation φ is said to be nonsingular when $\mu(\varphi^{-1}(B)) = 0$ *if and only if* $\mu(B) = 0$.

The nonsingular system $(\Omega, \mathcal{B}, \mu, \varphi)$ is said to be **invertible** if the transformation φ is bijective, and if φ^{-1} is measurable and nonsingular. (Note that if $(\Omega, \mathcal{B})$ is a standard Borel space, then the measurablility of φ^{-1} is for free.)

Definition 1.1. The transformation $\varphi : (\Omega, \mathcal{B}, \mu) \to (\Omega, \mathcal{B}, \mu)$ is said to be

- **conservative** if for each $B \in \mathcal{B}$ with $\mu(B) > 0$ there exists $n \geq 1$ such that $\mu(B \cap \varphi^{-n}(B)) > 0$;
- **dissipative** if there exists $W \in \mathcal{B}$ such that $\Omega = \bigcup_{n\in\mathbb{Z}} \varphi^n(W) \bmod \mu$ and W is a **wandering set** for φ, *i.e.* $\varphi^i(W) \cap \varphi^j(W) = \varnothing$ for all $i, j \in \mathbb{Z}$, $i \neq j$.

For example, if the measure μ is finite and φ is measure-preserving, then φ is conservative by Poincaré's recurrence Theorem; whereas if $\Omega = \mathbb{Z}$ and μ is the counting measure, then the shift map $\sigma(i) = i + 1$ is dissipative.

The importance of these notions stems from the fact that any invertible nonsingular system $(\Omega, \mathcal{B}, \mu, \varphi)$ can be decomposed into a conservative part and a dissipative part: there exist two (possibly empty) φ - invariant measurable sets C and D such that $C \cap D = \varnothing$, $\Omega = C \cup D$, $\varphi_{|C}$ is conservative and $\varphi_{|D}$ is dissipative. This is (a special case of) **Hopf's decomposition Theorem**; see e.g. [22, Theorem 2.19] or [23, Section 1.3].

As it turns out, the study of composition operators induced by a nonsingular transformation seems to be easier when the transformation φ is dissipative, especially if the technical assumption of being of *bounded distorsion* is added. The dynamical properties of C_φ are then comparable to that of an associated weighted shift; see e.g. [14] or [16]. In the present paper, we investigate the composition operators induced by well-known examples of *conservative* transformations, namely odometers. Before giving the relevant definitions, we briefly review some known results concerning the dynamics of general composition operators on L_p - spaces.

1.3. **Dynamical properties of composition operators.** Topologically transitive and mixing composition operators on $L_p(\Omega, \mu)$ have been completely characterized in [4] (see also [19]). These characterizations, which hold in fact in a context more general than L_p - spaces, will be recalled in Section 2.

Regarding *supercyclicity* of composition operators, the situation is less clear. Recall first that $\mathbb{R}$ - supercyclicity and $\mathbb{R}_+$- supercyclicity are always equivalent by [7]; and that for operators T such that T^* has no eigenvalue, supercyclicity and $\mathbb{R}_+$- supercyclicity are equivalent by [25]. In [17], $\mathbb{R}$ - supercyclic composition operators on L_p are characterized, and supercyclic composition operators on L_p are characterized in the dissipative case. In Section 2 of the present paper, we will give a characterization of supercyclic composition operators in a context more general than L_p - spaces. This will imply in particular that supercyclicity and $\mathbb{R}_+$- supercyclicity are always equivalent for composition operators on L_p, and that when $\mu(\Omega) < \infty$, supercyclicity on $L_p(\Omega, \mu)$ is in fact equivalent to hypercyclicity. The latter result has been obtained independently in [15].

As for *frequent hypercyclicity*, again the situation is well understood for composition operators induced by dissipative systems of bounded distorsion, see [16]. Moreover it is also shown in [16] that there is no hope to apply one of the main tools for proving frequent hypercyclicity (namely, the Frequent Hypercyclicity Criterion) in the conservative context, the case which interests us here. However, composition operators induced by odometers have plenty of periodic vectors (see Lemma 1.6 below); and as shown in [20], the periodic vectors can be very useful to check that a given operator is frequently hypercyclic or $\mathcal{U}$-frequently hypercyclic. We use this idea to give, in Sections 3 and 4, sufficient conditions for a general composition operator to be frequently hypercyclic or $\mathcal{U}$-frequently hypercyclic, which will prove to be rather efficient in the odometer context.

1.4. **Odometers.** Let $(m_i)_{i\geq 1}$ be a sequence of integers with $m_i \geq 2$ for all $i \geq 1$. Define $\Omega := \prod_{i=1}^{\infty} \Omega_i$, where $\Omega_i = \mathbb{Z}/m_i\mathbb{Z}$. We endow Ω with the product topology, which turns it into a compact metrizable space. One possible compatible metric is given by $d(x, y) := 2^{-i(x,y)}$, where $i(x, y)$ is the least index i such that $x_i \neq y_i$.

The space Ω is a topological abelian group when addition is performed coordinatewise. However, we will consider another operation on Ω, namely *addition with carrying to the right*, which will be denoted by $\boldsymbol{+}$ (boldface) and is defined as follows:

for $x, y \in \Omega$ and $i \geq 1$,

$$(x \boldsymbol{+} y)_i = x_i + y_i + \varepsilon_{i-1}\ [m_i]$$

where the sequence $(\varepsilon_i)_{i\geq 0}$ is defined inductively by $\varepsilon_0 = 0$ and, for $i \geq 1$,

$$\varepsilon_i = \begin{cases} 0 & \text{if } x_i + y_i + \varepsilon_{i-1} < m_i \\ 1 & \text{otherwise.} \end{cases}$$

Here, of course, we identify $\Omega_i = \mathbb{Z}/m_i\mathbb{Z}$ with $[\![0, m_i - 1]\!]$.

In other words, $(x \boldsymbol{+} y)_1 = x_1 + y_1\ [m_1]$ and, for $i \geq 2$, $(x \boldsymbol{+} y)_i = x_i + y_i\ [m_i]$ or $x_i + y_i + 1\ [m_i]$, depending on whether one has to perform a carry due to the computation of $(x \boldsymbol{+} y)_{i-1}$. This is indeed a (commutative) group law on Ω with neutral element $0 = (0, 0, 0, \dots)$. If $x \in \Omega$, then $y = -x$ can be computed inductively: $y_1 = 0$ if $x_1 = 0$ and $y_1 = m_1 - x_1$ if $x_1 > 0$; and for $i \geq 2$, $y_i = m_i - (x_i + 1)$ if the computation of $(x + y)_{i-1}$ has led to a carry, and otherwise $y_i = 0$ if $x_i = 0$ and $y_i = m_i - x_i$ if $x_i > 0$. The group operations are clearly continuous, so $(\Omega, \boldsymbol{+})$ is a compact abelian group.

We consider the map $\mathfrak{o} : \Omega \to \Omega$ defined as follows:

$$\mathfrak{o}(x) := x \boldsymbol{+} a \qquad \text{where } a = (1, 0, 0, 0, \dots).$$

In other words, $\mathfrak{o}$ is the translation by a in the group $(\Omega, \boldsymbol{+})$. By definition of the addition with carry $\boldsymbol{+}$, we have $\mathfrak{o}(m_1 - 1, m_2 - 1, \dots) = (0, 0, \dots)$ and, for any $x \neq (m_1 - 1, m_2 - 1, \dots)$,

$$(\mathfrak{o}(x))_i = \begin{cases} 0 & \text{if } i < l(x) \\ x_i + 1 & \text{if } i = l(x) \\ x_i & \text{if } i > l(x) \end{cases}$$

where $l(x)$ is the smallest index l such that $x_l \neq m_l - 1$. In other words, if $x = (m_1 - 1, \dots m_{l-1} - 1, x_l, x_{l+1}, x_{l+2}, \dots)$ with $x_l < m_l - 1$ (where the initial sequence of $m_i - 1$'s may be empty), then $\mathfrak{o}(x) = (0, \dots, 0, x_l + 1, x_{l+1}, x_{l+2}, \dots)$. The map $\mathfrak{o}$ is a homeomorphism of Ω; and in fact an isometry for the distance d defined above since for any $l \geq 1$, the first l coordinates of $\mathfrak{o}(x)$ depends only on the first l coordinates of x. For future reference, we quote the following basic fact concerning the iterates of $\mathfrak{o}$.

Fact 1.2. *Let $M_1 = 1$ and $M_i = m_1 \cdots m_{i-1}$ for $i \geq 2$. Any integer $k \geq 0$ can be uniquely written as a finite sum $k = \sum_{i\geq 1} k_i M_i$, where $0 \leq k_i \leq m_i - 1$ for all i; and with this notation, we have for all $x \in \Omega$:*

$$\mathfrak{o}^k(x) = x \boldsymbol{+} (k_1, k_2, \dots).$$

Now, for each $i \geq 1$, we consider a probability measure μ_i on Ω_i. Writing $\mu_i(j)$ instead of $\mu_i(\{j\})$, we assume that $\mu_i(j) > 0$ for all $j \in \Omega_i$, and we denote by μ the product measure $\otimes_{i\geq 1}\mu_i$. We will always assume that the measure μ is *non-atomic*, which means in the present context that $\mu(\{x\}) = 0$ for every $x \in \Omega$. Equivalently,

$$\prod_{i=1}^{\infty} \max\big\{\mu_i(j) :\ j \in \Omega_i\big\} = 0.$$

Denoting by $\mathcal{B}$ the Borel σ-algebra of Ω, the system $(\Omega, \mathcal{B}, \mu, \mathfrak{o})$ is called an **odometer**. It is well known that any odometer $(\Omega, \mathcal{B}, \mu, \mathfrak{o})$ is nonsingular, invertible and conservative; see [22, Theorem 9.9]. Since this is not obvious, and since nonsingularity is essential for the composition operator $C_{\mathfrak{o}}$ to be well-defined, we will outline a proof below.

Let us now discuss the boundedness of the composition operator $C_{\mathfrak{o}}$ on $L_p(\Omega, \mu)$, $1 \le p < \infty$.

Given $B_1 \subset \Omega_1, \dots, B_n \subset \Omega_n$, we denote by $[B_1, \dots, B_n]$ the **cylinder set**

$$[B_1, \dots, B_n] := \{x \in \Omega : \; x_1 \in B_1, \dots, x_n \in B_n\}.$$

If $B_1 = \{x_1\}, \dots, B_n = \{x_n\}$, we simply write $[x_1, \dots, x_n]$, and we say that $[x_1, \dots, x_n]$ is a **basic cylinder set**. Observe that if $[x_1, \dots, x_n]$ is a basic cylinder set and $(x_1, \dots, x_n) \neq (0, \dots, 0)$, then

$$\mathfrak{o}^{-1}([x_1, \dots, x_n]) = [m_1 - 1, \dots, m_{l(x)-1} - 1, x_{l(x)} - 1, x_{l(x)+1}, \dots, x_n]$$

where $l(x)$ is the smallest index l such that $x_l > 0$; whereas

$$\mathfrak{o}^{-1}([0, \dots, 0]) = [m_1 - 1, \dots, m_n - 1].$$

Hence, there exists some constant $K < \infty$ such that $\mu(\mathfrak{o}^{-1}(B)) \le K\mu(B)$ for all basic cylinder sets B if and only if

$$\sup_{l \ge 1} \left[\prod_{i=1}^{l-1} \frac{\mu_i(m_i - 1)}{\mu_i(0)} \times \sup_{j \in \Omega_l} \frac{\mu_l(j-1)}{\mu_l(j)} \right] < \infty, \tag{1.2}$$

where $j-1$ has to be understood as $m_l - 1$ if $j = 0$. Since every open subset of Ω is a countable disjoint union of basic cylinder sets, and since (for a given K) the family of all sets $B \in \mathcal{B}$ for which inequality (1.1) holds true is closed under decreasing countable intersections, it follows from the outer regularity of the measures μ and $\mu\mathfrak{o}^{-1}$ that (1.2) characterizes the boundedness of $C_{\mathfrak{o}}$ on $L_p(\Omega, \mu)$.

In Section 5, we study in some detail the dynamics of composition operators induced by odometers on L_p, in the spirit of [9] and [15]. Let us recall two of the main results of [9], which relate hypercyclicity and mixing of $C_{\mathfrak{o}}$ to the asymptotic behaviour of the sequence $(\eta_i)_{i \ge 1}$ defined by

$$\eta_i := \max\bigl\{\mu_i(j) : \; j \in \Omega_i\bigr\}.$$

Theorem 1.3. *Assume that $C_{\mathfrak{o}}$ is bounded on $L_p(\Omega, \mu)$, $1 \le p < \infty$.*

1. *If $\limsup_{i \to \infty} \eta_i = 1$, then $C_{\mathfrak{o}}$ is hypercyclic.*
2. *If $\eta_i \to 1$ as $i \to \infty$, then $C_{\mathfrak{o}}$ is topologically mixing; and the converse is true provided the sequence (m_i) is bounded.*

In the present paper, using probabilistic arguments, we shall provide another sufficient condition for hypercyclicity, which will allow us to completely characterize the hypercyclic composition operators induced by odometers with the same measure on each component. We will also characterize topological mixing for $C_{\mathfrak{o}}$ without any assumption on the sequence (m_i), and this will enable us to give interesting new examples. Finally we shall investigate frequent hypercyclicity, which will lead in particular to examples of composition operators which are frequently hypercyclic, chaotic and topologically mixing, as well as examples that are frequently hypercyclic and chaotic but *not* topologically mixing.

1.5. **Odometers, continued.** For the sake of completeness, we outline a proof of the fact that any odometer $(\Omega, \mathcal{B}, \mu, \mathfrak{o})$ with a non-atomic measure μ is nonsingular, invertible and conservative. More precisely, we prove the following lemma, where $0 = (0, 0, \dots) \in \Omega$.

Lemma 1.4. *Let μ be an arbitrary product probability measure on $\Omega = \prod_{i=1}^{\infty} \Omega_i$. The system $(\Omega, \mathcal{B}, \mu, \mathfrak{o})$ is nonsingular if and only if $\mu(\{0\}) > 0$ or $\mu(\{\mathfrak{o}^{-1}(0)\}) = 0$; and it is (nonsingular and) invertible if and only if $\mu(\{0\}) = 0 = \mu(\{\mathfrak{o}^{-1}(0)\}$ or $\mu(\{0\})\, \mu(\{\mathfrak{o}^{-1}(0)\}) > 0$. Finally, if the measure μ is non-atomic then $(\Omega, \mathcal{B}, \mu, \mathfrak{o})$ is conservative.*

Proof. If $\mu(\{0\}) = 0$ and $\mu(\{\mathfrak{o}^{-1}(0)\}) > 0$ then $(\Omega, \mathcal{B}, \mu, \mathfrak{o})$ is obviously singular. Conversely, assume that $\mu(\{\mathfrak{o}^{-1}(0)\}) = 0$ or $\mu(\{0\}) > 0$, and let us show that $(\Omega, \mathcal{B}, \mu, \mathfrak{o})$ is nonsingular. We have to prove that the measure $\mu\mathfrak{o}^{-1}$ is absolutely continuous with respect to μ, which we do by directly exhibiting the Radon-Nikodym derivative.

Let $\mathfrak{p} = \mathfrak{o}^{-1}$. For any $x \neq 0 = (0, 0, \dots)$ in Ω, denote again by $l(x)$ the smallest index l such that $x_l > 0$. Then

$$\mathfrak{p}(x) = (m_1 - 1, \dots m_{l(x)-1} - 1, x_{l(x)} - 1, x_{l(x)+1}, x_{l(x)+2}, \dots).$$

It follows that

$$h(x) := \prod_{i=1}^{\infty} \frac{\mu_i\big(\mathfrak{p}(x)_i\big)}{\mu_i(x_i)}$$

is well defined for all $x \neq 0$ since the product is finite in that case. It is also well-defined for $x = 0$ if $\mu(\{0\}) > 0$, namely $h(0) = \frac{\mu(\{\mathfrak{p}(0)\})}{\mu(\{0\})}$. In any case, $h(x)$ is well-defined μ-almost everywhere, and

$$\forall x \neq 0 \;:\; h(x) = \prod_{i=1}^{l(x)-1} \frac{\mu_i\big(m_i - 1\big)}{\mu_i(x_i)} \times \frac{\mu_{l(x)}\big(x_{l(x)} - 1\big)}{\mu_{l(x)}\big(x_{l(x)}\big)}\cdot$$

So h is a real-valued, nonnegative mesurable function. We show that $\mu\mathfrak{o}^{-1} = h\mu$.

We introduce some notation. For any $l \geq 1$, let $\Omega_{|l} = \prod_{i=1}^{l} \Omega_i$; and for $x \in \Omega$, let $x_{|l} = (x_1, \dots, x_l) \in \Omega_{|l}$. Since the l first coordinates of $\mathfrak{o}(x)$ or $\mathfrak{p}(x)$ depend only on the l first coordinates of x, the maps $\mathfrak{o}$ and $\mathfrak{p}$ induce bijections $\mathfrak{o}_l$ and $\mathfrak{p}_l$ of $\Omega_{|l}$ onto itself, namely $\mathfrak{o}_l(x_{|l}) = \mathfrak{o}(x)_{|l}$ and $\mathfrak{p}_l(x_{|l}) = \mathfrak{p}(x)_{|l}$ for all $x \in \Omega$. Moreover, if we define

$$E_l = \{x \in \Omega \;:\; l(x) \leq l\} = \{x \in \Omega \;:\; x_{|l} \neq (0, \dots 0)\},$$

then

$$h(x) = \prod_{i=1}^{l} \frac{\mu_i\big(\mathfrak{p}(x)_i\big)}{\mu_i(x_i)} = \frac{\mu_{|l}(\{\mathfrak{p}_l(x_{|l})\})}{\mu_{|l}(\{x_{|l}\})} \quad \text{for all } x \in E_l,$$

where $\mu_{|l} = \otimes_{i=1}^{l} \mu_i$. In particular, we may write $h(x) = h(x_{|l})$.

Now let us show that $\mu\mathfrak{o}^{-1} = h\mu$. For that, it is enough to show that if $f : \Omega \to \mathbb{R}$ is a nonnegative measurable function depending on finitely many coordinates, then

$$\int_\Omega fh\,d\mu = \int_\Omega f \circ \mathfrak{o}\,d\mu.$$

Assume that $f(x)$ depends only on the first N coordinates of x. Then, for any $l \geq N$, we may write $f(x) = f(x_{|l})$; and we have

$$\begin{aligned}\int_{E_l} fh\,d\mu &= \int_{E_l} f(x_{|l})h(x_{|l})\,d\mu(x)\\ &= \sum_{u\in\Omega_{|l}\setminus\{0_{|l}\}} f(u)h(u)\,\mu_{|l}(\{u\})\\ &= \sum_{u\in\Omega_{|l}\setminus\{0_{|l}\}} f(u)\,\mu_{|l}(\{\mathfrak{p}_l(u)\})\\ &= \sum_{v\in\Omega_{|l}\setminus\{\mathfrak{p}(0)_{|l}\}} f(\mathfrak{o}_l(v))\mu_{|l}(\{v\})\\ &= \int_{F_l} f\circ\mathfrak{o}\,d\mu,\end{aligned}$$

where $F_l = \mathfrak{p}(E_l) = \{y \in \Omega \,:\, y_{|l} \neq (m_1 - 1, \dots, m_l - 1)\}$.

Since the sequence (E_l) is increasing with $\bigcup_{l\geq1} E_l = \Omega \setminus \{0\}$, so that $\bigcup_{l\geq1} F_l = \Omega \setminus \{\mathfrak{p}(0)\}$, it follows that

$$\int_{\Omega\setminus\{0\}} fh\,d\mu = \int_{\Omega\setminus\{\mathfrak{p}(0)\}} f\circ\mathfrak{o}\,d\mu.$$

Now, recall that we are assuming that either $\mu(\{0\}) = 0 = \mu(\{\mathfrak{p}(0)\}$ or $\mu(\{0\}) > 0$. If $\mu(\{0\}) = 0 = \mu(\{\mathfrak{p}(0)\}$, we directly get $\int_\Omega fh\,d\mu = \int_\Omega f\circ\mathfrak{o}\,d\mu$. If $\mu(\{0\}) > 0$, then $\int_{\{0\}} fh\,d\mu = f(0)h(0)\mu(0) = (f\circ\mathfrak{o})(\mathfrak{p}(0))\mu(\{\mathfrak{p}(0)\}) = \int_{\{\mathfrak{p}(0)\}} f\circ\mathfrak{o}\,d\mu$; so we also get $\int_\Omega fh\,d\mu = \int_\Omega f\circ\mathfrak{o}\,d\mu$.

Since the map $\mathfrak{o} : \Omega \to \Omega$ is a homeomorphism, it is bimeasurable; and one shows in exactly the same way as above that $\mathfrak{p} = \mathfrak{o}^{-1}$ is nonsingular if $\mu(\{\mathfrak{o}^{-1}(0)\} > 0$ or $\mu(\{0\}) = 0$. Hence, $(\Omega, \mathcal{B}, \mu, \mathfrak{o})$ is nonsingular and invertible if $\mu(\{0\}) = 0 = \mu(\{\mathfrak{o}^{-1}(0)\}$ or $\mu(\{0\})\,\mu(\{\mathfrak{o}^{-1}(0)\}) > 0$.

Finally, let us show that $(\Omega, \mathcal{B}, \mu, \mathfrak{o})$ is conservative if the measure μ is non-atomic. Since the system is nonsingular and μ is non-atomic, it is enough to show that the transformation $\mathfrak{o}$ is *ergodic* with respect to μ, *i.e.* that any $\mathfrak{o}$ - invariant measurable function is μ - almost everywhere constant; see [1, Proposition 1.2.1].

For $n \geq 1$ and $x \in \Omega$, let us set

$$K_{n,x} = \sum_{i=1}^n M_i x_i,$$

where $M_1 = 1$ and $M_i = m_1 \cdots m_{i-1}$ if $i \geq 2$. Then $\mathfrak{o}^{-K_{n,x}}(x) = (0, \dots, 0, x_{n+1}, \dots)$ by Fact 1.2. It follows that if $f : \Omega \to \mathbb{R}$ is a measurable $\mathfrak{o}$ - invariant function, then $f(x) = f_n(\sigma^n(x))$ for all $x \in \Omega$, where $f_n(y) = f(0, \dots, 0, y_1, y_2, \dots)$ and $\sigma : \Omega \to \Omega$ is the backward shift. Hence, any $\mathfrak{o}$ - invariant measurable function is $\sigma^{-n}(\mathcal{B})$-measurable for all $n \geq 1$. Now, since μ is a product measure, we have $\bigcap_{n\geq 1} \sigma^{-n}(\mathcal{B}) = \{\varnothing, \mathcal{B}\}$ mod μ by Kolmogorov's 0-1 law. So every $\mathfrak{o}$-invariant measurable function is indeed μ-a.e. constant. □

Remark 1.5. In [9], the measure μ is not required to be non-atomic. We won't use this assumption either, but we prefer to keep it in order to emphasize that we are working with a conservative system. Note that in some of the examples given in [9], the measure μ does have atoms; see e.g. [9, Theorem 3.2]. We'll have to find similar examples with a non-atomic measure.

1.6. **Diagonal translation operators.** We shall also investigate the following variant of composition operators induced by odometers. We keep the same notation for $\Omega = \prod_{i\geq 1} \Omega_i$ and the product measure $\mu = \otimes_{i\geq 1}\mu_i$, but this time we consider the usual, coordinatewise addition on Ω, which we denote by $+$. We take $a := (1, 1, \dots)$ and consider the "diagonal" translation $\mathfrak{t} : \Omega \to \Omega$ defined by

$$\mathfrak{t}(x) = a + x = (x_i + 1)_{i\geq 1}.$$

The main difference with the odometer map $\mathfrak{o}$ is that the measure $\mu\mathfrak{t}^{-1}$ is again a product measure, namely $\mu\mathfrak{t}^{-1} = \otimes_{i\geq 1}\widetilde{\mu}_i$ where $\widetilde{\mu}_i(j) = \mu_i(j-1)$, $j \in \Omega_i$. By Kakutani's theorem [24], it follows that $\mathfrak{t} : (\Omega, \mathcal{B}, \mu) \to (\Omega, \mathcal{B}, \mu)$ is nonsingular if and only if

$$\prod_{i=1}^{\infty} \left(\sum_{j\in\Omega_i} \sqrt{\mu_i(j)\mu_i(j-1)} \right) > 0.$$

As for boundedness of $C_{\mathfrak{t}}$, note that if $[x_1, \dots, x_n]$ is a basic cylinder set, then

$$\mathfrak{t}^{-1}([x_1, \dots, x_n]) = [x_1 - 1, \dots, x_n - 1].$$

Hence, there exists some constant K such that $\mu(\mathfrak{t}^{-1}(B)) \leq K\mu(B)$ for all basic cylinder sets B if and only if

$$\prod_{i=1}^{\infty} \sup_{j\in\Omega_i} \frac{\mu_i(j-1)}{\mu_i(j)} < \infty; \tag{1.3}$$

and by the same argument as for odometers, this condition characterizes the boundedness of $C_{\mathfrak{t}}$ on $L_p(\Omega, \mu)$, $1 \leq p < \infty$. Composition operators $C_{\mathfrak{t}}$ will be called **diagonal translation operators**.

The dynamics of $C_{\mathfrak{t}}$ will be sometimes simpler to analyze than that of an odometer, because we do not have to handle carries (compare for instance the boundedness conditions (1.2) and (1.3)). Observe nevertheless that if the sequence (m_i) is bounded, then there exists an integer N such that $\mathfrak{t}^N = \mathrm{Id}$ and thus $C_{\mathfrak{t}}$ cannot be hypercyclic. Hence we need the sequence (m_i) to be unbounded if we want to get "interesting" examples.

1.7. **Periodic points.** It is easy to check directly (by looking at cylinder sets) that odometers $C_{\mathfrak{o}}$ and translation operators $C_{\mathfrak{t}}$ have dense sets of periodic points. This can also be deduced from the following lemma. Recall that a topological space is said to be zero-dimensional if it has a basis consisting of clopen sets.

Lemma 1.6. *Let Ω be a compact, zero-dimensional topological group, and let $a \in \Omega$. Let also μ be a Borel probability measure on Ω, and assume that the left translation $\tau_a(x) = ax$ induces a bounded composition operator on $L_p(\Omega, \mu)$, $1 \le p < \infty$. Then, the periodic points of C_{τ_a} are dense in $L_p(\Omega, \mu)$.*

Proof. Let us denote by $\mathcal{C}$ the family of all clopen subsets of Ω. Since Ω is zero-dimensional, $\mathbf{E} := \operatorname{span}\big(\mathbf{1}_C : \ C \in \mathcal{C}\big)$ is dense in $\mathcal{C}(\Omega)$, the space of continuous functions on Ω, by the Stone-Weierstrass Theorem. Hence, $\mathbf{E}$ is also dense in $L_p(\Omega, \mu)$. So we only need to show that any $f \in \mathbf{E}$ is a periodic point of C_{τ_a}; and for that, it is enough to prove the following: given $C_1, \dots, C_r \in \mathcal{C}$, one can find an integer $d \ge 1$ such that $a^{-d} C_j = C_j$ for $j = 1, \dots, r$. Moreover, replacing Ω by Ω^r and a by $(a, \dots, a) \in \Omega^r$ and considering $C_1 \times \cdots \times C_r$, we may in fact assume that $r = 1$, *i.e.* we have only one clopen set C.

Let us first observe that for any neighbourhood V of e, the unit element of Ω, one can find an integer $d \ge 1$ such that $a^{-d} \in V$. Indeed, the sequence $(a^n)_{n \ge 0}$ has an accumulation point by compactness of Ω, so one can find $n \ne n'$ such that $a^{n-n'} \in V \cap V^{-1}$.

Now, by a classical result of van Dantzig [29] (see [30] for a proof in English), Ω has a neighbourhood basis at e consisting of clopen subgroups. Hence, by compactness, the clopen set C is a finite union of right translates of clopen subgroups, say $C = \bigcup_{k=1}^r H_k b_k$. If we choose $d \ge 1$ such that $a^{-d} \in H := \bigcap_{k=1}^r H_k$, then $a^{-d} C = \bigcup_{k=1}^r a^{-d} H_k b_k = \bigcup_{k=1}^r H_k b_k = C$.

∎

1.8. **Organization of the paper.** In Section 2, we characterize the supercyclic composition operators in a general context, extending the results of [17], and we derive some consequences. In Sections 3 and 4, we obtain sufficient conditions for a composition operator to be frequently hypercyclic or $\mathcal{U}$-frequently hypercyclic. These conditions are based on the existence of many periodic vectors, in the spirit of [20]. Sections 5 and 6 are devoted to a thorough study of composition operators induced by odometers and of diagonal translation operators.

2. Supercyclic composition operators

2.1. **Framework.** In this section, we consider a σ-finite measure space $(\Omega, \mathcal{B}, \mu)$ and a Banach space $(\mathcal{X}, \|\cdot\|)$ with $\mathcal{X} \subset L_0(\Omega, \mu)$. We fix once and for all a nonsingular transformation $\varphi : (\Omega, \mathcal{B}, \mu) \to (\Omega, \mathcal{B}, \mu)$, and we assume that C_φ defines a bounded operator on $\mathcal{X}$.

In [4] and [19], a set of four natural conditions on $\mathcal{X}$ is introduced to study the dynamical properties of C_φ acting on $\mathcal{X}$. The framework of [19] is *a priori* more general than that of [4], so this is the one we will use. Note however that, as observed in [19], condition (C1) below is equivalent to condition (H3) in [4]; and that if $\mathcal{X}$ is a **lattice**, *i.e* ($|f| \le |g|$ and $g \in \mathcal{X}$) implies ($f \in \mathcal{X}$ and $\|f\| \le \|g\|$), then condition (C2) below is easily seen to be equivalent to condition (H4) in [4]. The four conditions given in [19] read as follows.

(H1) For any $E \in \mathcal{B}$ with finite measure, the indicator function $\mathbf{1}_E \in \mathcal{X}$.
(H2) $\operatorname{span}\big\{\mathbf{1}_E : E \in \mathcal{B},\ \mu(E) < \infty\big\}$ is dense in $\mathcal{X}$.

(C1) $\mu(|f| \geq 1) \to 0$ as $\|f\| \to 0$, $f \in \mathcal{X}$.
(C2) $\|\mathbf{1}_E\| \to 0$ as $\mu(E) \to 0$, $E \in \mathcal{B}$.

In more abstract terms, (C1) means that the inclusion map $\mathcal{X} \hookrightarrow L_0(\Omega, \mu)$ is continuous when $L_0(\Omega, \mu)$ is endowed with the topology of (global) convergence in measure. It can be formulated as follows: there is a function $\kappa_1 : \mathbb{R}_+ \to [0, \infty]$ such that $\mu(|f| \geq 1) \leq \kappa_1(\|f\|)$ for all $f \in \mathcal{X}$ and $\kappa_1(t) \to 0$ as $t \to 0$.

As for (C2), upon assuming (H1), it means that the map $E \mapsto \mathbf{1}_E$ from $\mathcal{B}_f := \{E \in \mathcal{B} : \mu(E) < \infty\}$ into $\mathcal{X}$ is continuous at $\varnothing$, when $\mathcal{B}_f$ is endowed with the (pseudo-)metric $d(A, B) = \mu(A \Delta B)$. Since $\|\mathbf{1}_A - \mathbf{1}_B\| \leq \|\mathbf{1}_{A \setminus B}\| + \|\mathbf{1}_{B \setminus A}\|$ for all $A, B \in \mathcal{B}_f$, this is equivalent to continuity on the whole of $\mathcal{B}_f$. Observe also that (C2) can be formulated as follows: there is a function $\kappa_2 : \mathbb{R}_+ \to [0, \infty]$ such that $\|\mathbf{1}_E\| \leq \kappa_2(\mu(E))$ for all $E \in \mathcal{B}_f$ and $\kappa_2(t) \to 0$ as $t \to 0$.

The following remark will not be needed, but it seems worth mentioning. Recall that a normed space $\mathcal{X} \subset L_0(\Omega, \mu)$ is said to have an *order-continuous norm* if, whenever $(f_n)_{n \in \mathbb{N}}$ is a pointwise decreasing sequence in $\mathcal{X}$ such that $f_n \to 0$ almost everywhere, it follows that $\|f_n\| \to 0$. For example, $L_p(\Omega, \mu)$ has an order continuous norm if $1 \leq p < \infty$, but $L_\infty(\Omega, \mu)$ does not.

Remark 2.1. Let $\mathcal{X} \subset L_0(\Omega, \mu)$ be a normed space, and assume that $\mathcal{X}$ is a lattice.

(1) If $\mathcal{X}$ is complete and $\mu(|f| \geq 1) < \infty$ for every $f \in \mathcal{X}$, then $\mathcal{X}$ satisfies (C1).
(2) If $\mathcal{X}$ satisfies (H1) and has an order-continuous norm, then $\mathcal{X}$ satisfies (C2).

Proof. (1) Towards a contradiction, assume that $\mathcal{X}$ does not satisfy (C1). Then, one can find a sequence $(f_k) \subset \mathcal{X}$ and $\delta > 0$ such that $\sum_{k=0}^\infty \|f_k\| < \infty$ and yet $\mu(|f_k| \geq 1) \geq \delta$ for all $k \geq 0$. Let us set $B_k = \{|f_k| \geq 1\}$, then $0 \leq \mathbf{1}_{B_k} \leq |f_k|$, so $\|\mathbf{1}_{B_k}\| \leq \|f_k\|$ by the lattice property, and hence the series $\sum \mathbf{1}_{B_k}$ converges to some $g \in \mathcal{X}$ since $\mathcal{X}$ is complete. By the lattice property again, we have $\sum_{k=0}^n \mathbf{1}_{B_k} \leq g$ for all $n \geq 0$: indeed, setting $g_p = \sum_{k=0}^p \mathbf{1}_{B_k}$, we have $g_p \geq g_n$ for all $p \geq n$, so $|g_p - g_n| = g_p - g_n$, hence $|g - g_n| = g - g_n$ since $g_p - g_n \to g - g_n$ and $|g_p - g_n| \to |g - g_n|$ in $\mathcal{X}$. Let $A = \bigcup_{k \geq 0} B_k$. Since $\mathbf{1}_{\bigcup_{k=0}^n B_k} \leq \sum_{k=0}^n \mathbf{1}_{B_k} \leq g$ for all $n \geq 0$, we have $\mathbf{1}_A \leq g$. Hence $\mathbf{1}_A \in \mathcal{X}$, so that $\mu(A) = \mu(\mathbf{1}_A \geq 1) < \infty$. Moreover, $\|\mathbf{1}_A\| \leq \|g\| \leq \sum_{k=0}^\infty \|\mathbf{1}_{B_k}\| \leq \sum_{k=0}^\infty \|f_k\|$. Similarly, if we set $A_n = \bigcup_{k \geq n} B_k$, then $\|\mathbf{1}_{A_n}\| \leq \sum_{k=n}^\infty \|f_k\|$ for all $n \geq 0$. By the lattice property (again), it follows that $\|\mathbf{1}_{\bigcap_{n \geq 0} A_n}\| = 0$, so that $\mu(\bigcap_{n \geq 0} A_n) = 0$. But $A_n \supset B_n = \{|f_n| \geq 1\}$, so $\mu(A_n) \geq \delta$ for all n. Since the sequence (A_n) is decreasing and $\mu(A_0) = \mu(A) < \infty$, this implies that $\mu(\bigcap_{n \geq 0} A_n) \geq \delta$, a contradiction.

(2) Assume that $\mathcal{X}$ does not satisfy (C2). Then, one can find $\delta > 0$ and a sequence $(E_k) \subset \mathcal{B}$ such that $\sum_{k=0}^\infty \mu(E_k) < \infty$ and $\|\mathbf{1}_{E_k}\| \geq \delta$ for all $k \geq 0$. For $n \geq 0$, let $F_n = \bigcup_{k \geq n} E_k$. Then $\mu(F_n) < \infty$, so $\mathbf{1}_{F_n} \in \mathcal{X}$ by (H1). Morevover, $F = \bigcap_{n \geq 0} F_n$ is such that $\mu(F) = 0$. By order-continuity, it follows that $\mathbf{1}_{F_n} \to 0$ in $\mathcal{X}$. But $\mathbf{1}_{F_n} \geq \mathbf{1}_{E_n}$, so $\|\mathbf{1}_{F_n}\| \geq \delta$ for all n by the lattice property, a contradiction □

In order to study supercyclicity of composition operators, we will need stronger versions of (C1) and (C2), that we call (C1') and (C2').

(C1') $\mu\big(|f|\geq 1\big)\,\mu\big(|g|\geq 1\big)\to 0$ as $\|f\|\|g\|\to 0$. Equivalently, there exists a function $\kappa_1:\mathbb{R}_+\to[0,\infty]$ such that $\mu\big(|f|\geq 1\big)\leq \kappa_1(\|f\|)$ for all $f\in\mathcal{X}$, and $\kappa_1(u)\kappa_1(v)\to 0$ whenever $uv\to 0$.

(C2') $\|\mathbf{1}_E\|\|\mathbf{1}_F\|\to 0$ as $\mu(E)\mu(F)\to 0$. Equivalently, there exists a function $\kappa_2:\mathbb{R}_+\to[0,\infty]$ such that $\|\mathbf{1}_E\|\leq\kappa_2\big(\mu(E)\big)$ for all $E\in\mathcal{B}_f$ and $\kappa_2(u)\kappa_2(v)\to 0$ as $uv\to 0$.

For example, $L_p(\Omega,\mu)$, $1\leq p<\infty$ satisfies (C1') with $\kappa_1(t)=t^p$ by Markov's inequality; and it satisfies (C2') with $\kappa_2(t)=t^{1/p}$. This implies in particular that if $\mathcal{X}$ embeds continuously in $L_p(\Omega,\mu)$ for some $1\leq p<\infty$, then $\mathcal{X}$ satisfies (C1') with $\kappa_1(t)=\big(\|\imath\|t\big)^p$, where $\imath$ is the inclusion map $\mathcal{X}\hookrightarrow L_p(\Omega,\mu)$; and that if $L_p(\Omega,\mu)$ embeds continuously in $\mathcal{X}$ for some $1\leq p<\infty$, then $\mathcal{X}$ satisfies (C2').

Remark 2.2. Assume that $\mu(\Omega)<\infty$. Then (C1) is equivalent to (C1'), and (C2) is equivalent to (C2') if $\mathcal{X}$ is a lattice satisfying (H1).

Proof. We have $\mu\big(|f|\geq 1\big)\mu\big(|g|\geq 1\big)\leq\mu(\Omega)\times\min\Big[\mu\big(|f|\geq 1\big),\mu\big(|g|\geq 1\big)\Big]$ for any $f,g\in\mathcal{X}$, so (C1) implies (C1'). And if $\mathcal{X}$ is a lattice, then $\|\mathbf{1}_E\|\,\|\mathbf{1}_F\|\leq\|\mathbf{1}\|\times\min\big(\|\mathbf{1}_E\|,\|\mathbf{1}_F\|\big)$ for any $E,F\in\mathcal{B}$, so (C2) implies (C2'). □

Example 2.3. Let $\Psi:\mathbb{R}_+\to\mathbb{R}_+$ be a nontrivial Orlicz function, and let $\mathcal{X}=L_\Psi(\Omega,\mathcal{B},\mu)$ be the associated Orlicz space, endowed with the Luxemburg norm

$$\|f\|=\inf\left\{a>0\,:\,\int_\Omega\Psi\left(\frac{|f|}{a}\right)d\mu\leq 1\right\}.$$

(i) The space $\mathcal{X}$ satisfies (C1) and (C2), hence (C1') and (C2') if $\mu(\Omega)<\infty$.

(ii) The space $\mathcal{X}$ satisfies (C1') if $\Psi(x)\Psi(y)\to\infty$ as $xy\to\infty$, and it satisfies (C2') if $\Psi^{-1}(x)\Psi^{-1}(y)\to\infty$ as $xy\to\infty$.

Proof. The nontriviality assumption means that there exists $\alpha\geq 0$ such that Ψ is strictly increasing on $[\alpha,\infty)$, so that $\Psi^{-1}:[0,\infty)\to[\alpha,\infty)$ is well defined. Note also that the second part of (i) follows from Remark 2.2.

If $f\in L_\Psi$ then, by Markov's inequality,

$$\mu\big(|f|\geq 1\big)\leq\mu\left[\Psi\left(\frac{|f|}{\|f\|}\right)\geq\Psi\left(\frac{1}{\|f\|}\right)\right]\leq\frac{1}{\Psi(1/\|f\|)}.$$

This shows that L_Ψ satisfies (C1); and that it satisfies (C1') if $\kappa_1(t)=1/\Psi(1/t)$ is such that $\kappa_1(u)\kappa_1(v)\to 0$ as $uv\to 0$, *i.e.* $\Psi(x)\Psi(y)\to\infty$ as $xy\to\infty$.

If $E\in\mathcal{B}$ has finite measure then, fo any $a>0$,

$$\int_\Omega\Psi\left(\frac{\mathbf{1}_E}{a}\right)d\mu=\Psi\left(\frac{1}{a}\right)\mu(E),$$

so that

$$\|\mathbf{1}_E\|=\frac{1}{\Psi^{-1}\big(1/\mu(E)\big)}.$$

This shows that L_Ψ satisfies (C2), and that it satisfies (C2') if $\Psi^{-1}(x)\Psi^{-1}(y)\to\infty$ as $xy\to\infty$. □

2.2. **Transitivity and mixing.** The characterizations of topological transitivity and mixing for C_φ obtained in [4] rely on "runaway like" properties of the transformation $\varphi : (\Omega, \mathcal{B}, \mu) \to (\Omega, \mathcal{B}, \mu)$. Following [19], let us say that φ satisfies the **Hypercyclic Runaway Condition** (HRC) if for any $A \in \mathcal{B}$ with finite measure and for every $\varepsilon > 0$, there exist $n \geq 1$ and $B \in \mathcal{B}$ such that $B \subset A$, $\mu(A \backslash B) < \varepsilon$, $\mu^*(\varphi^n(B)) < \varepsilon$ and $\mu(\varphi^{-n}(B)) < \varepsilon$. Here, $\mu^*(E)$ denotes the "outer measure" of a set $E \subset \Omega$, *i.e.* $\mu^*(E) = \inf \{\mu(F) : F \supset E, F \in \mathcal{B}\}$. Similarly, we say that φ satisfies the **Mixing Runaway Condition** (MRC) if for any $A \in \mathcal{B}$ with finite measure and for every $\varepsilon > 0$, there exists $n_0 \geq 1$ such that for all $n \geq n_0$, one can find $B \in \mathcal{B}$ such that $B \subset A$, $\mu(A \backslash B) < \varepsilon$, $\mu^*(\varphi^n(B)) < \varepsilon$ and $\mu(\varphi^{-n}(B)) < \varepsilon$. Finally, we say that $\varphi^{-1}(\mathcal{B})$ is **essentially equal** to $\mathcal{B}$, and we write $\varphi^{-1}(\mathcal{B}) =_{\text{ess}} \mathcal{B}$, if for any $A \in \mathcal{B}$, there exists $B \in \mathcal{B}$ such that $\mu\big(A \Delta \varphi^{-1}(B)\big) = 0$. The following results are proved in [4, 19].

Theorem 2.4. *Assume that the Banach space $\mathcal{X} \subset L_0(\Omega, \mu)$ satisfies conditions* (H1), (H2), (C1), (C2). *Then, $C_\varphi : \mathcal{X} \to \mathcal{X}$ is topologically transitive if and only if $\varphi^{-1}(\mathcal{B}) =_{\text{ess}} \mathcal{B}$ and φ satisfies* (HRC)*; and C_φ is topologically mixing if and only if $\varphi^{-1}(\mathcal{B}) =_{\text{ess}} \mathcal{B}$ and φ satisfies* (MRC).

When the measure μ is finite and the transformation φ is one-to-one and bimeasurable (*i.e.* $\varphi(B) \in \mathcal{B}$ for every $B \in \mathcal{B}$), these characterizations have the following simpler formulations.

Corollary 2.5. *Assume that $\mathcal{X}$ satisfies* (H1), (H2), (C1), (C2)*, that $\mu(\Omega) < \infty$, and that the transformation φ is one-to-one and bimeasurable.*

(1) *C_φ is topologically transitive if and only if φ satisfies any of the following three equivalent conditions:*
   (i) *for every $\varepsilon > 0$, there exist $n \in \mathbb{N}$ and $B \in \mathcal{B}$ such that $\mu(\Omega \backslash B) < \varepsilon$ and $B \cap \varphi^n(B) = \varnothing$;*
   (ii) *for every $\varepsilon > 0$, there exist $n \in \mathbb{N}$ and $B \in \mathcal{B}$ such that $\mu(B) < \varepsilon$ and $\mu(\Omega \backslash \varphi^{-n}(B)) < \varepsilon$;*
   (ii') *for every $\varepsilon > 0$, there exist $n \in \mathbb{N}$ and $B \in \mathcal{B}$ such that $\mu(\Omega \backslash B) < \varepsilon$ and $\mu(\varphi^n(B)) < \varepsilon$.*

(2) *C_φ is topologically mixing if and only if φ satisfies any of the following three equivalent conditions:*
   (i) *for every $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that, for all $n \geq n_0$, there exists $B \in \mathcal{B}$ with $\mu(\Omega \backslash B) < \varepsilon$ and $B \cap \varphi^n(B) = \varnothing$;*
   (ii) *for every $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that, for all $n \geq n_0$, there exists $B \in \mathcal{B}$ such that $\mu(B) < \varepsilon$ and $\mu(\Omega \backslash \varphi^{-n}(B)) < \varepsilon$;*
   (ii') *for every $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that, for all $n \geq n_0$, there exists $B \in \mathcal{B}$ such that $\mu(\Omega \backslash B) < \varepsilon$ and $\mu(\varphi^n(B)) < \varepsilon$.*

Proof. (1) The equivalence of (i) and topological transitivity is [4, Corollary 1.3]. The equivalence of (ii) and (ii') follows by taking $\widetilde{B} = \varphi^n(B)$.

The implication (i) $\Longrightarrow$ (ii') is obvious. Conversely if (ii) is satisfied, let $\varepsilon > 0$, and choose $n \in \mathbb{N}$ and $B \in \mathcal{B}$ such that $\mu(\Omega \backslash \varphi^{-n}(B)) < \varepsilon/2$ and $\mu(B) < \varepsilon/2$. Set

$\widetilde{B} := \varphi^{-n}(B)\backslash B$. Then $\mu\big(\Omega\backslash\widetilde{B}\big) < \varepsilon$ and $\varphi^n\big(\widetilde{B}\big)\cap\widetilde{B}\subset B\backslash B = \varnothing$. This shows that (ii) $\Longrightarrow$ (i).

(2) The equivalence of (i) and topological mixing is [4, Corollary 2.2]; and the equivalence of (i), (ii) and (ii') is proved as above. □

2.3. **Supercyclicity: preliminary facts.** To prove that supercyclic composition operators on L_p are in fact $\mathbb{R}_+$- supercyclic, we will need the following lemma.

Lemma 2.6. *Let X be a separable Banach space, let $T\in\mathfrak{L}(X)$ and let $\theta_0\in(0,\pi)$. Define $\Gamma_0 = \big\{z\in\mathbb{C}^*:\ \arg(z)\in[-\theta_0,\theta_0]\big\}$. If $x\in X$ is a supercyclic vector for T, then it is a Γ_0 - supercyclic vector for T.*

Proof. We shall use a variant of the classical Bourdon-Feldman theorem [13], which was proved in [6, Theorem 3.11]. Since T is (implicitely) assumed to be supercyclic, we have either $\sigma_p(T^*) = \varnothing$ or $\sigma_p(T^*) = \{\alpha\}$ for some $\alpha\in\mathbb{C}^*$. In the first case, we already know by [25] that x is $\mathbb{R}_+$- supercyclic and there is nothing to do. So let us assume that $\sigma_p(T^*) = \{\alpha\}$. Let

$$\begin{aligned}\mathcal{T} &= \{\lambda T^n:\ \lambda\in\Gamma_0,\ n\geq 0\},\\ \mathcal{S} &= \{P(T):\ P\in\mathbb{C}[X],\ P(\alpha)\neq 0\},\\ \mathcal{S}\cdot x &= \{P(T)x:\ P\in\mathbb{C}[X],\ P(\alpha)\neq 0\}.\end{aligned}$$

We observe that $\mathcal{T}\subset\mathcal{S}$, that each $S\in\mathcal{S}$ has dense range and commutes with every $A\in\mathcal{T}$, and that $\mathcal{S}\cdot x$ is connected and dense in X. Therefore, by [6, Theorem 3.11], to prove that $\overline{\Gamma_0\cdot O(x,T)} = X$, *i.e.* $\overline{\mathcal{T}\cdot x} = X$, we just have to show that $\overline{\Gamma_0\cdot O(x,T)}$ has nonempty interior. Now, it is clear that there exists some integer $r\geq 1$ such that

$$\mathbb{C}^*\cdot O(x,T) = \Gamma_0\cdot O(x,T)\cup(e^{2i\theta_0}\Gamma_0)\cdot O(x,T)\cup\cdots\cup(e^{2ir\theta_0}\Gamma_0)\cdot O(x,T).$$

Since $\overline{\mathbb{C}^*\cdot O(x,T)} = X$, it follows that $\overline{\Gamma_0\cdot O(x,T)}$ has nonempty interior. □

We will also need a geometrical lemma.

Lemma 2.7. *Let $\lambda\in\mathbb{C}^*$ with $\arg(\lambda)\in[-\pi/3,\pi/3]$ and let $z\in\mathbb{C}$.*

(a) *If $|\lambda z+4|\leq 1$, then $\Re e(z)\leq -1/|\lambda|$.*
(b) *If $|z-4|\leq 1$, then $\Re e(\lambda z)\geq|\lambda|$.*

Proof. In case (a), write $\lambda = |\lambda|e^{i\theta}$ with $\theta\in[-\pi/3,\pi/3]$, and $\lambda z+4 = \rho e^{i\alpha}$ with $\rho\in[0,1]$. Then

$$z = \frac{1}{|\lambda|}\big(-4e^{-i\theta}+\rho e^{i(\alpha-\theta)}\big),$$

so that

$$\Re e(z)\leq\frac{1}{|\lambda|}\times(-2+1) = -\frac{1}{|\lambda|}\cdot$$

The proof of (b) is similar. □

Finally, we will make use of the following version of the Supercyclicity Criterion. Recall that a subset $\mathcal{D}$ of a Banach space X is said to be *total* if $\operatorname{span}(\mathcal{D})$ is dense in X.

Lemma 2.8. *Let X be a separable Banach space, and let $T \in \mathfrak{L}(X)$. Assume that there exist two total sets $\mathcal{D}_1, \mathcal{D}_2 \subset X$, a sequence of integers $(n_k)_{k\geq 0}$ and, for each $k \geq 0$, maps $\alpha_k : \mathcal{D}_1 \to X$ and $\beta_k : \mathcal{D}_2 \to X$ such that*

- *$\alpha_k(x) \to x$ for all $x \in \mathcal{D}_1$ and $T^{n_k}\beta_k(y) \to y$ for all $y \in \mathcal{D}_2$;*
- *$\|T^{n_k}\alpha_k(x)\| \, \|\beta_k(y)\| \to 0$ for all $(x, y) \in \mathcal{D}_1 \times \mathcal{D}_2$.*

Then T is $\mathbb{R}_+$- supercyclic.

Proof. Assuming, as we may, that the sets $\mathcal{D}_1$ and $\mathcal{D}_2$ are linearly independent, we can extend the maps α_k and β_k by linearity to $\mathrm{span}(\mathcal{D}_1)$ and $\mathrm{span}(\mathcal{D}_2)$, and the extended maps still satisfy the above conditions. So, we may assume that $\mathcal{D}_1$ and $\mathcal{D}_2$ are in fact dense in X. Now, let U and V be two nonempty open sets in X. Choose $x \in \mathcal{D}_1 \cap U$ and $y \in \mathcal{D}_2 \cap V$. Since $\|T^{n_k}\alpha_k(x)\| \, \|\beta_k(y)\| \to 0$, we may find a sequence of positive real numbers (λ_k) such that $\lambda_k T^{n_k}\alpha_k(x) \to 0$ and $\lambda_k^{-1}\beta_k(y) \to 0$. Then $z_k := \alpha_k(x) + \lambda_k^{-1}\beta_k(y) \in U$ and $\lambda_k T^{n_k} z_k \in V$ if k is large enough. By Birkhoff's transitivity theorem, it follows that T is $\mathbb{R}_+$- supercyclic. □

Remark 2.9. Since $T \oplus T$ satisfies the hypotheses of the lemma if T does, the "correct" conclusion of the lemma should be that $T \oplus T$ is $\mathbb{R}_+$- supercyclic. As shown in [8], $T \oplus T$ is supercyclic if and only if T satisfies any of the known versions of the Supercyclicity Criterion.

2.4. **Supercyclicity: results.** From now on, we assume that the Banach space $\mathcal{X} \subset L_0(\Omega, \mu)$ is separable. Let us introduce the condition on φ which will characterize supercyclicity.

Definition 2.10. We say that the transformation $\varphi : (\Omega, \mathcal{B}, \mu) \to (\Omega, \mathcal{B}, \mu)$ satisfies the **Supercyclic Runaway Condition** (SRC) if for every $A \in \mathcal{B}$ with finite measure and for every $\varepsilon > 0$, there exist $B \in \mathcal{B}$ and $n \geq 1$ such that $B \subset A$, $\mu(A \backslash B) < \varepsilon$ and $\mu^*(\varphi^n(B))\mu(\varphi^{-n}(B)) < \varepsilon$.

We are now ready to prove one half of the characterization of supercyclic composition operators.

Theorem 2.11. *Suppose that $\mathcal{X}$ satisfies conditions* (H1) *and* (C1'). *If C_φ is supercyclic, then $\varphi^{-1}(\mathcal{B}) =_{\rm ess} \mathcal{B}$ and φ satisfies* (SRC).

Proof. Since C_φ is supercyclic, it has dense range and therefore $\varphi^{-1}(\mathcal{B}) =_{\rm ess} \mathcal{B}$, by [31, Lemma 1]. In [31], the result is proved for $\mathcal{X} = L_2$, but the proof relies only on the fact that any convergent sequence in $\mathcal{X}$ has a subsequence converging almost everywhere, which is guaranteed by (C1).

Let us fix $A \in \mathcal{B}$ with finite measure and $\varepsilon > 0$. We need to find $B \subset A$ and $n \in \mathbb{N}$ in accordance with (SRC).

Let $\delta \in (0, 1)$ to be chosen later. Since C_φ is supercyclic, it follows from Lemma 2.6 that there exist $f \in \mathcal{X}$, $n \in \mathbb{N}$ and $\lambda \in \mathbb{C}^*$ with $\arg(\lambda) \in [-\pi/3, \pi/3]$ such that

$$\|f - 4 \cdot \mathbf{1}_A\| < \delta \quad \text{ and } \quad \|\lambda f \circ \varphi^n + 4 \cdot \mathbf{1}_A\| < \delta.$$

We set

$$C := \{x \in \Omega : \ |f(x) - 4| < 1\} \, , \quad D := \{x \in \Omega : \ |\lambda f \circ \varphi^n(x) + 4| < 1\},$$

and

$$B := A \cap C \cap D.$$

Observe that

$$\begin{aligned} A\backslash B &\subset (A\backslash C) \cup (A\backslash D) \\ &\subset \big\{|f - 4\cdot \mathbf{1}_A| \geq 1\big\} \cup \big\{|\lambda f\circ\varphi^n + 4\cdot\mathbf{1}_A| \geq 1\big\}. \end{aligned}$$

Hence, if κ_1 is the function given by (C1'), which may be assumed to be nondecreasing, we get

$$\mu(A\setminus B) \leq 2\kappa_1(\delta).$$

We now show that

$$\varphi^{-n}(B) \subset \big\{\ |\lambda f\circ\varphi^n + 4\cdot\mathbf{1}_A| \geq |\lambda|\big\} \quad \text{and} \quad \varphi^n(B) \subset \big\{|f - 4\cdot\mathbf{1}_A| \geq 1/|\lambda|\big\}.$$

Indeed, if $x \in \varphi^{-n}(B)$, then $\varphi^n(x) \in C$, *i.e.* $|f\circ\varphi^n(x) - 4| < 1$. By Lemma 2.7, it follows that

$$\Re e(\lambda f\circ\varphi^n(x)) \geq |\lambda|,$$

so that

$$\begin{aligned} |\lambda f\circ\varphi^n(x) + 4\cdot\mathbf{1}_A(x)| &\geq \Re e(\lambda f\circ\varphi^n(x) + 4\cdot\mathbf{1}_A(x)) \\ &\geq \Re e(\lambda f\circ\varphi^n(x)) \\ &\geq |\lambda|. \end{aligned}$$

Similarly, if $y \in \varphi^n(B)$, then $y = \varphi^n(x)$ for some $x \in D$, so $|\lambda f(y) + 4| < 1$. Hence $\Re e(f(y)) \leq -1/|\lambda|$ by Lemma 2.7, so that

$$\begin{aligned} |f(y) - 4\cdot\mathbf{1}_A(y)| &\geq -\Re e(f(y) - 4\cdot\mathbf{1}_A(y)) \\ &\geq 1/|\lambda|. \end{aligned}$$

By (C1'), it follows that

$$\mu\big(\varphi^{-n}(B)\big)\, \mu^*\big(\varphi^n(B)\big) \leq \kappa_1\big(\delta/|\lambda|\big)\, \kappa_1\big(|\lambda|\,\delta\big);$$

and hence we get (SRC) for ε if δ is small enough. □

The converse of Theorem 2.11 reads as follows.

Theorem 2.12. *Suppose that $\mathcal{X}$ satisfies* (H2) *and* (C2'). *If $\varphi^{-1}(\mathcal{B}) =_{\rm ess} \mathcal{B}$ and φ satisfies* (SRC), *then $C_\varphi \oplus C_\varphi$ is $\mathbb{R}_+$- supercyclic.*

Proof. Let $\mathcal{D}$ be the (total) subset of $\mathcal{X}$ consisting of all functions $\mathbf{1}_D$ where $D \in \mathcal{B}$ has finite measure. We show that the assumptions of Lemma 2.8 are satisfied with $\mathcal{D}_1 = \mathcal{D} = \mathcal{D}_2$.

Let $(E_k)_{k\geq 0}$ be an increasing sequence of measurable sets with finite measure such that $\bigcup_{k\geq 0} E_k = \Omega$. By (SRC), one can find measurable sets $F_k \subset E_k$ and integers n_k such that $\mu(E_k \setminus F_k) < 2^{-k}$ and $\mu\big(\varphi^{-n_k}(F_k)\big)\mu^*\big(\varphi^{n_k}(F_k)\big) < 2^{-k}$. We also choose measurable sets $\widetilde{F}_k$ such that $\varphi^{n_k}(F_k) \subset \widetilde{F}_k$ and $\mu(\widetilde{F}_k) = \mu^*\big(\varphi^{n_k}(F_k)\big)$.

We observe that

$$\mu(S\setminus F_k) \to 0 \quad \text{for any } S \in \mathcal{B} \text{ with finite measure.} \tag{2.1}$$

Indeed, this is clear since $\mu(S\setminus F_k) = \mu\big(S\cap(E_k\setminus F_k)\big) + \mu(S\setminus E_k) \leq 2^{-k} + \mu(S\setminus E_k)$.

Now, we define maps $\alpha_k, \beta_k : \mathcal{D} \to \mathcal{X}$ as follows.

- For any $\mathbf{1}_A \in \mathcal{D}$, we set
$$\alpha_k(\mathbf{1}_A) = \mathbf{1}_{A\cap F_k}.$$
- For any $\mathbf{1}_B \in \mathcal{D}$, we choose a sequence $(B_k) \subset \mathcal{B}$ such that $\mathbf{1}_B = \mathbf{1}_{\varphi^{-n_k}(B_k)}$ almost everywhere for every $k \geq 0$, which is possible since $\varphi^{-1}(\mathcal{B}) =_{\rm ess} \mathcal{B}$, and we set
$$\beta_k(\mathbf{1}_B) = \mathbf{1}_{B_k\cap \widetilde{F}_k}.$$

Let $\kappa_2 : \mathbb{R}_+ \to [0,\infty]$ be the function given by (C2'). We may assume that κ_2 is nondecreasing.

For any $\mathbf{1}_A \in \mathcal{D}$, we have
$$\|\mathbf{1}_A - \alpha_k(\mathbf{1}_A)\| = \|\mathbf{1}_{A\setminus F_k}\| \leq \kappa_2\big(\mu(A\setminus F_k)\big),$$
so that $\alpha_k(\mathbf{1}_A) \to \mathbf{1}_A$ as $k\to\infty$ by (2.1).

For any $\mathbf{1}_B \in \mathcal{D}$, we have
$$C_\varphi^{n_k}\beta_k(\mathbf{1}_B) = \mathbf{1}_{\varphi^{-n_k}(B_k)\cap\varphi^{-n_k}(\widetilde{F}_k)} = \mathbf{1}_{B\cap\varphi^{-n_k}(\widetilde{F}_k)}.$$
Since $\varphi^{-n_k}(\widetilde{F}_k) \supset F_k$ and κ_2 is nondecreasing, it follows that
$$\|\mathbf{1}_B - C_\varphi^{n_k}\beta_k(\mathbf{1}_B)\| \leq \kappa_2\big(\mu(B\setminus F_k)\big);$$
and hence $C_\varphi^{n_k}\beta_k(\mathbf{1}_B) \to \mathbf{1}_B$ by (2.1).

Finally, for any $\mathbf{1}_A, \mathbf{1}_B \in \mathcal{D}$, we have
$$\begin{aligned}\|C_\varphi^{n_k}\alpha_k(\mathbf{1}_A)\|\,\|\beta_k(\mathbf{1}_B)\| &= \|\mathbf{1}_{\varphi^{-n_k}(A\cap F_k)}\|\,\|\mathbf{1}_{B_k\cap\widetilde{F}_k}\|\\ &\leq \kappa_2\big(\mu(\varphi^{-n_k}(F_k)\big)\kappa_2\big(\mu(\widetilde{F}_k)\big)\\ &= \kappa_2\big(\mu(\varphi^{-n_k}(F_k)\big)\kappa_2\big(\mu^*(\varphi^{n_k}(F_k)\big),\end{aligned}$$
so that $\|C_\varphi^{n_k}\alpha_k(\mathbf{1}_A)\|\,\|\beta_k(\mathbf{1}_B)\| \to 0$.

By Lemma 2.8, the proof of Theorem 2.12 is now complete. □

Remark 2.13. Assuming that $\mathcal{X}$ satisfies (H2) and (C2), it is shown in [19] that if $\varphi^{-1}(\mathcal{B}) =_{\rm ess} \mathcal{B}$ and φ satisfies the Hypercyclic Runaway Condition (HRC), then C_φ is weakly mixing. This result can be proved in exactly the same way as Theorem 2.12, by showing that C_φ satisfies the appropriate version of the Hypercyclicity Criterion, *i.e.* Lemma 2.8 where the condition "$\|T^{n_k}\alpha_k(x)\|\,\|\beta_k(y)\| \to 0$" is replaced by "$\|T^{n_k}\alpha_k(x)\| \to 0$ and $\|\beta_k(y)\| \to 0$".

Putting the two previous theorems together, we have obtained the following result.

Corollary 2.14. *Suppose that $\mathcal{X}$ satisfies* (H1), (H2), (C1'), (C2'). *Then, the following are equivalent:*

(i) *C_φ is supercyclic.*
(ii) *$C_\varphi \oplus C_\varphi$ is $\mathbb{R}_+$- supercyclic.*
(iii) *$\varphi^{-1}(\mathcal{B}) =_{\rm ess} \mathcal{B}$ and φ satisfies* (SRC).

Remark 2.15. By [25], another way of showing that C_φ is $\mathbb{R}_+$- supercyclic as soon as it is supercyclic would be to prove that C_φ^* has no eigenvalue. We don't know how to do that.

2.5. **Examples.** We first show that on a finite measure space, supercyclic and hypercyclic composition operators are the same. For L_p - spaces, this was obtained independently in [15].

Corollary 2.16. *Assume that $\mu(\Omega) < \infty$ and that $\mathcal{X}$ satisfies* (H1), (H2), (C1), (C2)*. Then, the following are equivalent:*

(i) *C_φ is supercyclic.*
(ii) *C_φ is weakly mixing.*
(iii) *$\varphi^{-1}(\mathcal{B}) =_{\rm ess} \mathcal{B}$ and for every $\varepsilon > 0$, there exists $B \in \mathcal{B}$ and $n \in \mathbb{N}$ such that $\mu(\Omega \backslash B) < \varepsilon$ and $B \cap \varphi^n(B) = \varnothing$.*

Proof. Note that (iii) clearly implies the Hypercyclic Runaway Condition (HRC). So one just has to show that (i) $\Longrightarrow$ (iii), since $\mathcal{X}$ satisfies (H2) and (C2) – see Remark 2.13.

Assume that C_φ is supercyclic, and let $\varepsilon > 0$. Since $\mathcal{X}$ satisfies (H1) and (C1), which is equivalent to (C1') because $\mu(\Omega) < \infty$, the transformation φ must satisfy (SRC). So there exist $B' \in \mathcal{B}$ and $n \in \mathbb{N}$ such that $\mu(\Omega \backslash B') < \varepsilon/2$ and $\mu^*(\varphi^n(B'))\mu(\varphi^{-n}(B')) < \varepsilon^2/4$. Choose a measurable set C such that $\varphi^n(B') \subset C$ and $\mu(C)\mu(\varphi^{-n}(B')) < \varepsilon^2/4$. Then either $\mu(C) < \varepsilon/2$ or $\mu(\varphi^{-n}(B')) < \varepsilon/2$. Let $B = B' \backslash C$ in the first case, and $B = B' \backslash \varphi^{-n}(B')$ in the second case. Then $\mu(\Omega \backslash B) < \varepsilon$ and $B \cap \varphi^n(B) = \varnothing$. ☐

We can also easily deduce the results of Salas [27] on supercyclicity of weighted shifts. Let $I = \mathbb{Z}$ or $\mathbb{Z}_+$, let $(\nu_i)_{i\in I}$ be a sequence of positive numbers such that $\sup_{i\in I} \nu_i/\nu_{i+1} < \infty$, and let $\mathbf{B} : \ell_p(I, \nu) \to \ell_p(I, \nu)$ be the backward shift acting on the weighted space $\ell_p(I, \nu) = \{x \in \mathbb{C}^I : \ \|x\|^p := \sum_{i\in I} |x_i|^p \nu_i < \infty\}$. When $I = \mathbb{Z}$, it is shown in [27] that $\mathbf{B}$ is supercyclic if and only if for all $i \in \mathbb{Z}$, there exists an increasing sequence of integers (n_k) such that $\nu_{i+n_k}\nu_{i-n_k} \to 0$. When $I = \mathbb{Z}_+$, $\mathbf{B}$ is always supercyclic, regardless of the sequence (ν_i). Since $\ell_p(I, \nu) = L_p(\Omega, \mu)$ where $\Omega = I$, $\mathcal{B} = \mathcal{P}(I)$, $\mu = \sum_{i\in I} \nu_i \, \delta_{\{i\}}$ and $\mathbf{B} = C_\sigma$ where $\sigma : \Omega \to \Omega$ is the shift map, $\sigma(i) = i + 1$, these results can be put in a slightly more general framework.

Corollary 2.17. *Let Ω be a countable (infinite) set, $\mathcal{B} = \mathcal{P}(\Omega)$, and let μ be a positive measure on $(\Omega, \mathcal{B})$ such that $0 < \mu(i) < \infty$ for all $i \in \Omega$. Let also $\varphi : \Omega \to \Omega$ be one-to-one, and assume that $\sup_{i\in\Omega} \mu(i)/\mu(\varphi(i)) < \infty$. The following are equivalent.*

(i) *C_φ is supercyclic on $L_p(\Omega, \mu)$, $1 \le p < \infty$.*
(ii) *For any $i, j \in \Omega$, one can find an increasing sequence of integers (n_k) such that $\mu\varphi^{n_k}(i)\, \mu\varphi^{-n_k}(j) \to 0$.*

Moreover, if the φ - orbits are totally ordered by inclusion, this is also equivalent to

(ii') *For any $i \in \Omega$, one can find an increasing sequence of integers (n_k) such that $\mu\varphi^{n_k}(i)\, \mu\varphi^{-n_k}(i) \to 0$.*

Proof. The condition $C := \sup_{i\in\Omega} \mu(i)/\mu(\varphi(i)) < \infty$ ensures that C_φ is bounded on $L_p(\Omega, \mu)$.

To prove that (i) $\Longrightarrow$ (ii), we apply (SRC) with $A := \{i, j\}$. If $\varepsilon > 0$ is less than $\min(\mu(i), \mu(j))$ and if B and n are given by (SRC), then we must have $B = A$, so that $\mu\varphi^n(\{i, j\})\, \mu\varphi^{-n}(\{i, j\}) < \varepsilon$. This gives (ii).

Conversely, assume that (ii) is satisfied. By a diagonal argument, one can find a single sequence (n_k) such that $\mu\varphi^{n_k}(i)\,\mu\varphi^{-n_k}(j)\to 0$ for all $i,j\in\Omega$. Then $\mu\varphi^{n_k}(B)\,\mu\varphi^{-n_k}(B)\to 0$ for every finite set $B\subset\Omega$, and it follows that φ satisfies (SRC).

Finally, assume that the φ-orbits are totally ordered by inclusion. Let $i,j\in\Omega$. By assumption, we have either $j=\varphi^m(i)$ or $i=\varphi^m(j)$ for some $m\geq 0$. In the first case, we may write $\varphi^n(i)=\varphi^{n-m}(j)$ for all $n\geq m$, so $\mu\varphi^n(i)=\mu\varphi^{-m}(\varphi^n(j))\leq C^m\mu\varphi^n(j)$ and hence $\mu\varphi^n(i)\,\mu\varphi^{-n}(j)\leq C^m\mu\varphi^n(j)\,\mu\varphi^{-n}(j)$. In the second case, we get in the same way $\mu\varphi^n(i)\,\mu\varphi^{-n}(j)\leq C^m\mu\varphi^n(i)\,\mu\varphi^{-n}(i)$ for all $n\geq m$. This shows that (ii') $\Longrightarrow$ (ii). □

3. Frequently hypercyclic composition operators

The easiest way to prove that an operator is frequently hypercyclic is to use the so-called Frequent Hypercyclicity Criterion (see e.g. [6, Theorem 6.18]). This criterion gives an extremely strong conclusion since any operator satisfying it is frequently hypercyclic, chaotic and topologically mixing. In the context of nonsingular measurable systems, a detailed study of which composition operators satisfy the Frequent Hypercyclicity Criterion has been made in [16] (see e.g. Theorem 3.2). In particular it is shown that if $(\Omega,\mathcal{B},\mu,\varphi)$ is a nonsingular system such that C_φ satisfies the Frequent Hypercyclicity Criterion on L_p, $p\geq 2$, then φ has to be dissipative. Since, as observed in the introduction, odometers are never dissipative, it follows that there is no hope to apply the Frequent Hypercyclicity Criterion in the odometer setting. However, we have also observed that odometers have a wealth of periodic points; and this will allow us to apply another criterion for frequent hypercyclicity, which appears in [20, Theorem 5.35]. Here and afterwards, we denote by $\mathrm{Per}(T)$ the set of all periodic points of an operator T, and by $\mathrm{per}(u)$ the period of a periodic point u, *i.e.* the least $d\geq 1$ such that $T^d u=u$.

Lemma 3.1. *Let X be a separable Banach space and let $T\in\mathfrak{L}(X)$. Assume that there exist a dense linear subspace X_0 of X contained in $\mathrm{Per}(T)$ with $T(X_0)\subset X_0$ and a constant $\kappa\in(0,1)$ such that the following holds true: for every $u\in X_0$, every $\varepsilon>0$ and every integer $d_0\geq 1$ which is the period of some vector in X_0, there exist $v\in X_0$ and integers $n,d\geq 1$ such that*

(a) *d is a multiple of d_0 and of $\mathrm{per}(v)$;*
(b) *$\|T^k v\|\leq\varepsilon$ for every $0\leq k\leq\kappa d$;*
(c) *$\|T^{n+k}v-T^k u\|\leq\varepsilon$ for every $0\leq k\leq\kappa d$.*

Then T is frequently hypercyclic.

Using this criterion, we now give a "runaway like" sufficient condition for a composition operator to be frequently hypercyclic. To put the result into a rather general framework, we need a new assumption. We still consider a σ-finite measure space $(\Omega,\mathcal{B},\mu)$ and a Banach space $(\mathcal{X},\|\cdot\|)$ with $\mathcal{X}\subset L_0(\Omega,\mu)$. We consider the following condition on $\mathcal{X}$:

(C3) $L_\infty(X,\mu)\cdot\mathcal{X}\subset\mathcal{X}$ and the map $L_\infty\times\mathcal{X}\to\mathcal{X},\ (f,g)\mapsto fg$ is continuous.

Again, the spaces $L_p(X,\mu)$, $1\leq p<\infty$, satisfy (C3); more generally, if $\mathcal{X}$ is a lattice then it satisfies (C3). In fact, (C3) means that there is an equivalent norm $|||\cdot|||$ on $\mathcal{X}$ such that $(\mathcal{X}, |||\cdot|||)$ is a lattice. Note also that if every convergent sequence in $\mathcal{X}$ has a subsequence converging almost everywhere, for example if $\mathcal{X}$ satisfies (C1), then the continuity of the map $(f,g)\mapsto fg$ follows from the inclusion $L_\infty(X,\mu)\cdot\mathcal{X}\subset\mathcal{X}$ by the closed graph theorem.

Theorem 3.2. *Let $(\Omega,\mathcal{B},\mu)$ be a separable measure space with $\mu(\Omega)<\infty$, let $\mathcal{X}\subset L_0(\Omega,\mu)$ be a Banach space satisfying* (H1), (C2) *and* (C3), *and let $\varphi:(\Omega,\mathcal{B},\mu)\to(\Omega,\mathcal{B},\mu)$ be a nonsingular transformation such that C_φ is bounded on $\mathcal{X}$. Assume that there exist $\mathcal{C}\subset\mathcal{B}$ and $\kappa\in(0,1)$ such that* $\operatorname{span}(\mathbf{1}_B:\ B\in\mathcal{C})$ *is dense in $\mathcal{X}$ and condition* (H_κ) *below is satisfied.*

(H_κ) *For every $\varepsilon>0$, every $m\in\mathbb{N}$ and any $B_1,\dots,B_r\in\mathcal{C}$, the following holds true: there exist $d\geq m$, $n\in\mathbb{N}$ and $B\in\mathcal{B}$ such that $\varphi^{-d}(B_j)=B_j$ for $j=1\dots,r$, $\varphi^{-d}(B)=B$ and*

$$\forall 0\leq k\leq\kappa d\ :\ \mu(\varphi^{-k}(B))\leq\varepsilon\quad\text{ and }\quad\mu\big(\Omega\setminus\varphi^{-(n+k)}(B)\big)\leq\varepsilon.$$

Then C_φ is frequently hypercyclic on $\mathcal{X}$.

Proof. Note first that $\mathcal{X}$ is separable. We shall apply Lemma 3.1 with $X_0:=\operatorname{span}(\mathbf{1}_B:\ B\in\mathcal{C})$, which is contained in $\operatorname{Per}(T)$ by (H_κ). Without loss of generality, we may assume that $\mathcal{C}$ is stable under finite intersections: indeed, if we denote by $\widetilde{\mathcal{C}}$ the family of all finite intersections of sets from $\mathcal{C}$, then (H_κ) is satisfied with $\widetilde{\mathcal{C}}$ in place of $\mathcal{C}$ since $\varphi^{-d}(B_1\cap B_2)=B_1\cap B_2$ if $\varphi^{-d}(B_1)=B_1$ and $\varphi^{-d}(B_2)=B_2$. So (we may assume that) X_0 is stable under products. Note also that $X_0\subset L_\infty(\Omega,\mu)$.

Let $0\neq f\in X_0$, let $\varepsilon>0$ and let d_0 be the period of some vector in X_0. By (C2), we may choose $\delta>0$ such that $\|1_E\|_{\mathcal{X}}\leq\varepsilon/(M\|f\|_\infty)$ for all $E\in\mathcal{B}$ with $\mu(E)\leq\delta$, where M is a constant, given by (C3), such that $\|hu\|_{\mathcal{X}}\leq M\|h\|_\infty\|u\|_{\mathcal{X}}$ for all $(h,u)\in L_\infty\times\mathcal{X}$.

Let $m\in\mathbb{N}$ be large. By (H_κ), one can find $d\geq m$ which is a multiple of d_0 and of $\operatorname{per}(f)$, $n\in\mathbb{N}$ and $B\in\mathcal{B}$ with $\varphi^{-d}(B)=B$ such that

$$\forall 0\leq k\leq\kappa d\ :\ \mu(\varphi^{-k}(B))\leq\delta\quad\text{ and }\quad\mu\big(\Omega\setminus\varphi^{-(n+k)}(B)\big)\leq\delta.$$

Note that d is a multiple of d_0 and of $\operatorname{per}(\mathbf{1}_Bf)$. Moreover, replacing κ by $\kappa/2$, taking m large enough and replacing n by the smallest multiple of the period of f greater than n, we may assume that n is a multiple of the period of f.

Let $g=\mathbf{1}_Bf$, so that $g\in X_0$ and d is a multiple of the period of g. By definition, we have

$$C_\varphi^kg=C_\varphi^kf\times\mathbf{1}_{\varphi^{-k}(B)}\quad\text{for all }k\geq 0.$$

In particular, for $k\in[0,\kappa d]$,

$$\|C_\varphi^kg\|_{\mathcal{X}}\leq M\|f\|_\infty\,\|\mathbf{1}_{\varphi^{-k}(B)}\|_{\mathcal{X}}\leq\varepsilon.$$

On the other hand, since n is a multiple of the period of f, we have $C_\varphi^{n+k}g=C_\varphi^kf\times\mathbf{1}_{\varphi^{-(n+k)}(B)}$; and since for $k\in[0,\kappa d]$,

$$\|\mathbf{1}-\mathbf{1}_{\varphi^{-(n+k)}(B)}\|_{\mathcal{X}}=\|\mathbf{1}_{\Omega\setminus\varphi^{-(n+k)}(B)}\|_{\mathcal{X}}\leq\varepsilon/(M\|f\|_\infty),$$

it follows that

$$\forall k\in[0,\kappa d]\ :\ \|C_\varphi^{n+k}g-C_\varphi^kf\|\leq\varepsilon.$$

By Lemma 3.1, we conclude that C_φ is frequently hypercyclic. ∎

Remark 3.3. We may exchange the roles played by $\varphi^{-k}(B)$ and $\Omega\setminus\varphi^{-k}(B)$ in (H_κ), by considering $g=(\mathbf{1}-\mathbf{1}_B)f$ instead of $g=\mathbf{1}_Bf$ in the above proof. Hence we may assume in (H_κ) that

$$\forall 0\le k\le \kappa d\ :\ \mu(\Omega\setminus\varphi^{-k}(B))\le\varepsilon\quad\text{and}\quad \mu\big(\varphi^{-(n+k)}(B)\big)\le\varepsilon.$$

Condition (H_κ) above is perhaps a bit cumbersome. However, here is a consequence of Theorem 3.2 whose statement may be slightly easier to grasp.

Corollary 3.4. *Let $(\Omega,\mathcal{B},\mu)$ be a separable measure space with $\mu(\Omega)<\infty$, let $\mathcal{X}\subset L_0(\Omega,\mu)$ be a Banach space satisfying (C2) and (C3), and let $\varphi:(\Omega,\mathcal{B},\mu)\to(\Omega,\mathcal{B},\mu)$ be a nonsingular transformation such that C_φ is bounded on $\mathcal{X}$. Assume that $\mathrm{span}(\mathbf{1}_B:\ B\in\mathcal{C})$ is dense in $\mathcal{X}$, where $\mathcal{C}=\{B\in\mathcal{B}\,:\,\exists d\ge 1\,,\ \varphi^{-d}(B)=B\}$. Moreover, assume that there exists $\kappa\in(0,1)$ such that, for every $\varepsilon>0$ and every $a\in\mathbb{N}$, there exist a multiple d of a, $n\in\mathbb{N}$ and a set $B\in\mathcal{B}$ with $\varphi^{-d}(B)=B$ such that*

$$\forall 0\le k\le\kappa d\ :\ \mu(\varphi^{-k}(B))\le\varepsilon\quad\text{and}\quad\mu\big(\Omega\setminus\varphi^{-(n+k)}(B)\big)\le\varepsilon.$$

Then C_φ is frequently hypercyclic on $\mathcal{X}$.

Proof. We have to check Condition (H_κ), which is easy: given $\varepsilon>0$, $m\in\mathbb{N}$ and $B_1,\dots,B_r\in\mathcal{C}$, choose $d_1,\dots,d_r\ge 1$ such that $\varphi^{-d_j}(B_j)=B_j$, and use the assumption of Corollary 3.4 with $a:=md_1\cdots d_r$. □

One can check that Corollary 3.4 applies to $\Omega=\mathbb{N}$ or $\mathbb{Z}$ and the shift map $\varphi(i)=i+1$; and this yields a new proof of a very well-known fact, namely that if $(\nu_i)_{i\in\Omega}$ is a sequence of positive numbers such that $\sup_{i\in\Omega}\nu_i/\nu_{i+1}<\infty$ and $\sum_{i\in\Omega}\nu_i<\infty$, then the backward shift $\mathbf{B}$ acting on $\ell_p(\Omega,\nu)$ is frequently hypercyclic. More generally, we have the following.

Corollary 3.5. *Let Ω be a countable (infinite) set, $\mathcal{B}=\mathcal{P}(\Omega)$, and let μ be a positive measure on $(\Omega,\mathcal{B})$ such that $\mu(\Omega)<\infty$ and $\mu(i)>0$ for all $i\in\Omega$. Let also $\varphi:\Omega\to\Omega$ be one-to-one and such that $\sup_{i\in\Omega}\mu(i)/\mu(\varphi(i))<\infty$. Moreover, assume φ has no periodic point. Then, C_φ is frequently hypercyclic on $L_p(\Omega,\mu)$, $1\le p<\infty$.*

Proof. We will need the following fact.

Fact 3.6. *For any finite sets $E,F\subset\Omega$, we have $\varphi^n(E)\cap F=\varnothing$ for all $n\in\mathbb{Z}$ such that $|n|$ is large enough.*

Proof. Otherwise one could find $x\in E$ and $y\in F$ such that $\varphi^n(x)=y$ for infinitely many $n\ge 1$, or $\varphi^n(y)=x$ for infinitely many $n\ge1$. If, for example, $\varphi^n(x)=y=\varphi^{n'}(x)$ for some $1\le n<n'$, then $\varphi^{n'-n}(x)=x$ by injectivity of φ, a contradiction since φ has no periodic point. □

Let us first check that $\mathrm{span}(\mathbf{1}_B:\ B\in\mathcal{C})$ is dense in $L_p(\Omega,\mu)$, where

$$\mathcal{C}=\big\{B\subset\Omega\,:\,\exists d\ge1\,,\ \varphi^{-d}(B)=B\big\}.$$

It is enough to show that for any finite set $E\subset\Omega$, one can approximate $\mathbf{1}_E$ as close as we wish by $\mathbf{1}_B$, for some $B\in\mathcal{C}$. Let $\varepsilon>0$, and choose a finite set $F\subset\Omega$ such that $\mu(\Omega\setminus F)<\varepsilon$. By Fact 3.6, there exists $d\ge1$ such that $\varphi^n(E)\cap F=\varnothing$ for all $n\in\mathbb{Z}$

such that $|n| \geq d$. Then $B = \bigcup_{l\in\mathbb{Z}} \varphi^{-ld}(E)$ belongs to $\mathcal{C}$ and $E \subset B \subset E \cup (\Omega \setminus F)$, so $\|\mathbf{1}_B - \mathbf{1}_E\| < \varepsilon^{1/p}$.

Now, we fix κ such that $0 < \kappa < 1/6$, and we show that the assumption of Corollary 3.4 is satisfied.

Let $\varepsilon > 0$, and choose a finite set $F \subset \Omega$ such that $\mu\big(\Omega \setminus F\big) < \varepsilon$. It is enough to show that if $d \in \mathbb{N}$ is large enough, then one can find $B \subset \Omega$ with $\varphi^{-d}(B) = B$ and $n \in \mathbb{N}$ such that

$$\forall 0 \leq k \leq \kappa d \,:\, \varphi^{-k}(B) \cap F = \varnothing \quad \text{and} \quad \varphi^{-n-k}(B) \supset F.$$

Let $d \in \mathbb{N}$ be large, and let $n \in \mathbb{N}$ be such that $3\kappa d \leq n \leq 4\kappa d$. If d is large enough then, by Fact 3.6 and since $0 < \kappa < 1/6$, we have $\varphi^{i+n}(F) \cap F = \varnothing$ for all i such that $|i| \leq 2\kappa d$ or $|i| \geq (1 - 2\kappa)d$. We fix d large enough and n such that $3\kappa d \leq n \leq 4\kappa d$, and we define

$$E = \bigcup_{0\leq k\leq \kappa d} \varphi^{k+n}(F)$$

and

$$B = \bigcup_{l\in\mathbb{Z}} \varphi^{-ld}(E) \in \mathcal{C}.$$

Let us fix $0 \leq k \leq \kappa d$. We show that $\varphi^{-k}(B) \cap F = \varnothing$ and $\varphi^{-n-k}(B) \supset F$.
First, by definition of B and E,

$$\varphi^{-n-k}(B) \supset \varphi^{-n-k}(E) \supset F.$$

Next,

$$\begin{aligned}
\varphi^{-k}(B) &= \varphi^{-k}(E) \cup \bigcup_{l\neq 0} \varphi^{-ld-k}(E) \\
&= \bigcup_{0\leq k'\leq \kappa d} \varphi^{n+k'-k}(F) \cup \bigcup_{\substack{0\leq k'\leq \kappa d \\ l\neq 0}} \varphi^{n+k'-k-ld}(F) \\
&\subset \bigcup_{|i|\leq 2\kappa d} \varphi^{n+i}(F) \cup \bigcup_{|j|\geq (1-2\kappa)d} \varphi^{n+j}(F) \\
&\subset \Omega \setminus F.
\end{aligned}$$

This concludes the proof. □

Remark 3.7. Under the assumptions of Corollary 3.5, it is not hard to check that in fact, C_φ satisfies the Frequent Hypercyclicity Criterion, so that it is also chaotic and topologically mixing. On the other hand, if φ has a periodic point, then C_φ cannot be hypercyclic. Indeed, assume that $\varphi^d(i_0) = i_0$ for some $i_0 \in \Omega$ and some $d \geq 1$. Then, for any $f \in L_p(\Omega, \mu)$, we have $C_\varphi^{dn} f(i_0) = f(i_0)$ for all $n \in \mathbb{N}$. It follows that C_φ^d is not hypercyclic, and hence C_φ is not hypercyclic either by Ansari's Theorem [2].

When Ω is a zero-dimensional compact group and φ is a translation, Theorem 3.2 implies the following statement.

Corollary 3.8. *Let Ω be a metrizable, compact and zero-dimensional topological group, and let μ be a Borel probability measure on Ω. Let also $a \in G$ be such that the composition operator C_{τ_a} associated to the left translation $\tau_a(x) = ax$ is bounded on $L_p(\Omega, \mu)$. Finally, let $(d_i)_{i\in\mathbb{N}}$ be an increasing sequence of integers such that $a^{d_i} \to e$, the unit element of Ω. Assume that there exists $\kappa > 0$ such that, for every $\varepsilon > 0$ the following holds true: for all $i_0 \in \mathbb{N}$, there exist $i \geq i_0$, $n \in \mathbb{N}$ and a Borel set $B \subset \Omega$ such that $a^{-d_i}B = B$ and*

$$\forall 0 \leq k \leq \kappa d_i \,:\, \mu(a^{-k}B) \leq \varepsilon \quad \text{and} \quad \mu(a^{-(n+k)}B) \geq 1-\varepsilon.$$

Then C_{τ_a} is frequently hypercyclic on $L_p(\Omega, \mu)$, $1 \leq p < \infty$.

Proof. Let $\mathcal{B}$ be the Borel σ-algebra of Ω. Since Ω is a Polish space, the probability space $(\Omega, \mathcal{B}, \mu)$ is separable.

Let us denote by $\mathcal{C}$ the family of all clopen subsets of Ω. Since Ω is zero-dimensional, $\operatorname{span}(\mathbf{1}_C : C \in \mathcal{C})$ is dense in $L_p(\Omega, \mu)$. We want to apply Theorem 3.2 with the family $\mathcal{C}$, so we need to check that Condition (H_κ) is satisfied. For that, we only need to show that if $C \subset \Omega$ is a clopen set, then $a^{-d_i}C = C$ provided $i \in \mathbb{N}$ is large enough; which follows from the proof of Lemma 1.6. ☐

4. $\mathcal{U}$-frequently hypercyclic operators

4.1. **A criterion for $\mathcal{U}$-frequent hypercyclicity.** In this section, we state a general result allowing to prove that an operator with many periodic vectors is $\mathcal{U}$-frequently hypercyclic. We even extend it to a more restrictive property, namely *hereditary* $\mathcal{U}$-frequent hypercyclicity. An operator $T \in \mathfrak{L}(X)$ is said to be hereditarily $\mathcal{U}$-frequently hypercyclic if for any countable family $(V_q)_{q\in\mathbb{N}}$ of nonempty open subsets of X, for any family $(B_q)_{q\in\mathbb{N}}$ of subsets of $\mathbb{N}$ with positive upper density, there exists $x \in X$ such that $\mathcal{N}_T(x, V_q) \cap B_q$ has positive upper density for all $q \in \mathbb{N}$. This notion was introduced and studied in [5].

Proposition 4.1. *Let X be a separable Banach space and let $T \in \mathfrak{L}(X)$ be such that $\mathrm{Per}(T)$ is dense in X.*

(a) *T is $\mathcal{U}$-frequently hypercyclic if and only if, for any neighbourhood W of 0 in X, there exists $\alpha > 0$ such that the following holds true: for every open set $\mathcal{O} \neq \varnothing$ in X and any $N \in \mathbb{N}$, one can find $x \in \mathcal{O}$ and an integer $m \geq N$ such that $\#\mathcal{N}_T(x, W) \cap [\![1, m]\!] \geq \alpha m$.*

(b) *The following assertions are equivalent:*

  (i) *For any nonempty open set $V \subset X$ and any $B \subset \mathbb{N}$ with positive upper density, there exists $\alpha_{V,B} > 0$ such that*
  $$\big\{u \in X :\, \overline{\mathrm{dens}}\big(\mathcal{N}_T(u, V) \cap B\big) \geq \alpha_{V,B}\big\}$$
  is dense in X.

  (ii) *For any set $A \subset \mathbb{N}$ with positive upper density and any neighbourhood W of 0 in X, there exists $\alpha > 0$ such that: for every open set $\mathcal{O} \neq \varnothing$ in X and any $N \in \mathbb{N}$, one can find $x \in \mathcal{O}$ and $m \geq N$ such that $\#\mathcal{N}_T(x, W) \cap A \cap [\![1, m]\!] \geq \alpha m$.*

Moreover, these assertions imply that T is hereditarily $\mathcal{U}$-frequently hypercyclic.

Proof. (a) Assume that T is $\mathcal{U}$-frequently hypercyclic and let y be a $\mathcal{U}$-frequently hypercyclic vector for T. Let W be a neighbourhood of 0. Let $\beta > 0$ be such that $\overline{\text{dens}}\big(\mathcal{N}_T(y, W)\big) > \beta$ and set $\alpha = \beta/2$. Let $\mathcal{O} \neq \varnothing$ be an open subset of X and let $N \in \mathbb{N}$. There exists $n \in \mathbb{N}$ such that $T^n y \in \mathcal{O}$. Set $x = T^n y$. There exists $m_1 > \max(N + n, 2n/\beta)$ such that

$$\#\left\{k \in [\![1, m_1]\!] :\ T^k y \in W\right\} \geq \beta m_1.$$

Then, if we set $m = m_1 - n$,

$$\begin{aligned}\#\left\{k \in [\![1, m]\!] :\ T^k x \in W\right\} &\geq \beta m_1 - n\\ &\geq \alpha m.\end{aligned}$$

To prove the converse, note first that, by the Baire category theorem and since any set of the form $\{x \in X : \#\mathcal{N}_T(x, W) \cap [\![1, m]\!] \geq \alpha m\}$ is open in X, the assumption concerning W and α implies that the set $\{x \in X : \overline{\text{dens}}\big(\mathcal{N}_T(x, W)\big) \geq \alpha\}$ is dense in X.

It suffices to prove that for any nonempty open set V, there exists $\alpha_V > 0$ such that the G_δ set

$$G_V := \left\{u \in X :\ \overline{\text{dens}}\big(\mathcal{N}_T(u, V)\big) \geq \alpha_V\right\}$$

is dense in X. Indeed, picking (V_p) a countable basis of nonempty open subsets of X, any vector in the dense G_δ set $\bigcap_p G_{V_p}$ will be a $\mathcal{U}$-frequently hypercyclic vector for T.

So let V be a nonempty open subset of X. Let $v \in V$ be a periodic point of T with period d, and let W_0 be a neighbourhood of 0 such that $v + W_0 \subset V$. Choose a neighbourhood W of 0 such that $T^i(W) \subset W_0$ for $i = 0, \dots, d$, and let $\alpha > 0$ be associated with this choice of W. Finally, let $\alpha_V := \alpha/d$.

Let U be a nonempty open subset of X: we need to find $u \in U$ such that $\overline{\text{dens}}\big(\mathcal{N}_T(u, V)\big) \geq \alpha_V$. By assumption, one can find x in the nonempty open set $U - v$ such that $\overline{\text{dens}}\big(\mathcal{N}_T(x, W)\big) \geq \alpha$. Then, one can find $i \in [\![1, d]\!]$ such that $\overline{\text{dens}}\big(\mathcal{N}_T(x, W) \cap (d\mathbb{N} - i)\big) \geq \alpha_V$, so that $\overline{\text{dens}}\big((\mathcal{N}_T(x, W) + i) \cap d\mathbb{N}\big) \geq \alpha_V$. Now, if $k \in (\mathcal{N}_T(x, W) + i) \cap d\mathbb{N}$ then, writing $x = u - v$ with $u \in U$ and $k = l + i$ with $l \in \mathcal{N}_T(x, W)$, we get

$$T^k u = T^i(T^l x) + v \in W_0 + v \subset V.$$

So $\overline{\text{dens}}\big(\mathcal{N}_T(u, V)\big) \geq \alpha_V$, as required.

(b) That (i) implies hereditary $\mathcal{U}$-frequent hypercyclicity follows from the Baire category theorem. Moreover, it is clear that (i) implies (ii). Conversely, assume that (ii) holds true. As above, note that the assumption concerning A, W and α in (ii) implies that the set $\{x \in X : \overline{\text{dens}}\big(\mathcal{N}_T(x, W) \cap A\big) \geq \alpha\}$ is dense in X.

Let $V \subset X$ be nonempty and open, let $B \subset \mathbb{N}$ with positive upper density and let v be a periodic vector belonging to V. Let W be a neighbourhood of 0 such that $v + W \subset V$ and let d be the period of v. For $i = 0, \dots, d - 1$, let $B_i = \{n \in B :\ n \equiv i\ [d]\}$. Then at least one B_i, that we will call B_p, has positive upper density. Let us take $A = B_p$ and apply (ii) to get some $\alpha = \alpha_{V,B} > 0$.

Let U be a nonempty open subset of X. By assumption, one can find $x \in U - T^{d-p} v$ such that $\overline{\text{dens}}\big(\mathcal{N}_T(x, W) \cap B_p\big) \geq \alpha_{V,B}$. Write $x = u - T^{d-p} v$ with

$u \in U$. Then, for any $k \in \mathcal{N}_T(x, W) \cap B_p$, we get

$$T^k u = T^k x + v \in v + W \subset V,$$

so that $\overline{\mathrm{dens}}\big(\mathcal{N}_T(u, V) \cap B\big) \geq \alpha_{V,B}$. This shows that (i) is satisfied. □

Remark 4.2. The proof of (i) $\Longrightarrow$ (ii) in (b) as well as the "only if" implication in (a) do not use the density of $\mathrm{Per}(T)$.

4.2. **Composition operators.** From Proposition 4.1, we now deduce a result regarding $\mathcal{U}$-frequent hypercyclicity of general composition operators.

Theorem 4.3. *Let $(\Omega, \mathcal{B}, \mu)$ be a separable measure space with $\mu(\Omega) < \infty$, let $\mathcal{X} \subset L_0(\Omega, \mu)$ be a Banach space satisfying* (H1), (C2) *and* (C3), *and let $\varphi : (\Omega, \mathcal{B}, \mu) \to (\Omega, \mathcal{B}, \mu)$ be a nonsingular transformation such that C_φ is bounded on $\mathcal{X}$. Let $\mathcal{C} := \{B \in \mathcal{B} :\ \exists d \geq 1,\ \varphi^{-d}(B) = B\}$ and let us assume that* $\mathrm{span}(\mathbf{1}_B :\ B \in \mathcal{C})$ *is dense in $\mathcal{X}$.*

1. *Assume that there exists $\alpha > 0$ such that the following holds true: for any $\varepsilon > 0$ and $N \in \mathbb{N}$, there exist $B \in \mathcal{B}$ and an integer $m \geq N$ such that $\mu(B) \leq \varepsilon$ and $\#\big\{k \in [\![1, m]\!] :\ \mu(\Omega \backslash \varphi^{-k}(B)) \leq \varepsilon\big\} \geq \alpha m$. Then C_φ is $\mathcal{U}$-frequently hypercyclic.*
2. *Assume that for any set $A \subset \mathbb{N}$ with positive upper density, there exists $\alpha > 0$ such that: for every $\varepsilon > 0$ and $N \in \mathbb{N}$, there exist $B \in \mathcal{B}$ and $m \geq N$ such that $\mu(B) \leq \varepsilon$ and $\#\big\{k \in [\![1, m]\!] :\ k \in A \text{ and } \mu(\Omega \backslash \varphi^{-k}(B)) \leq \varepsilon\big\} \geq \alpha m$. Then C_φ is hereditarily $\mathcal{U}$-frequently hypercyclic.*

Proof. We only prove (1), the proof of (2) being completely similar.

Observe first that $\mathrm{Per}(C_\varphi)$ is dense in $\mathcal{X}$. We show that the assumption of Proposition 4.1 (a) is satisfied with the same $\alpha > 0$ working for any W, namely α given by (1) above. Let W be a neighbourhood of 0 in $\mathcal{X}$ and $\delta > 0$ such that $W \supset \overline{B}(0, \delta)$. Let also $\mathcal{O}$ be a nonempty open subset of $\mathcal{X}$ and let $N \in \mathbb{N}$. We need to find $g \in \mathcal{O}$ and $m \geq N$ such that $\#\big\{k \in [\![1, m]\!] : \|C_\varphi^k g\| \leq \delta\big\} \geq \alpha m$.

Let $0 \neq f \in \mathcal{O} \cap L_\infty$, and choose $\eta > 0$ such that $\overline{B}(f, \eta) \subset \mathcal{O}$. Let also $M > 0$ be such that $\|hu\|_{\mathcal{X}} \leq M \|h\|_\infty \|u\|_{\mathcal{X}}$ for any $(h, u) \in L_\infty \times \mathcal{X}$. We consider $\varepsilon > 0$ such that $\|\mathbf{1}_E\| < \min(\delta, \eta)/(M\|f\|_\infty)$ for all $E \in \mathcal{B}$ with $\mu(E) \leq \varepsilon$.

By assumption one may find $B \in \mathcal{B}$ and $m \geq N$ such that $\mu(B) \leq \varepsilon$ and

$$\#\big\{k \in [\![1, m]\!] :\ \mu(\Omega \backslash \varphi^{-k}(B)) \leq \varepsilon\big\} \geq \alpha m.$$

We set $g := f(\mathbf{1} - \mathbf{1}_B)$ and first observe that

$$\|g - f\|_{\mathcal{X}} \leq M\|f\|_\infty \cdot \|\mathbf{1}_B\|_{\mathcal{X}} < M\|f\|_\infty \times \frac{\eta}{M\|f\|_\infty} = \eta.$$

Hence, $g \in \mathcal{O}$. Moreover, for any $k \geq 0$ such that $\mu(\Omega \backslash \varphi^{-k}(B)) \leq \varepsilon$, we have

$$\|C_\varphi^k g\|_{\mathcal{X}} \leq M \|C_\varphi^k f\|_\infty \, \|\mathbf{1} - \mathbf{1}_{\varphi^{-k}(B)}\|_{\mathcal{X}} \leq M \leq \delta.$$

Hence, $\#\left\{k \in [\![1, m]\!] :\ \|C_\varphi^k g\| \leq \delta\right\} \geq \alpha m$. □

Remark 4.4. We could exchange the roles of B and $\varphi^{-k}(B)$ in (1) by requiring that $\mu(\Omega \backslash B) \leq \varepsilon$ and $\#\big\{k \in [\![1, m]\!] :\ \mu(\varphi^{-k}(B)) \leq \varepsilon\big\} \geq \alpha m$. The proof is almost identical, taking now $g := f\mathbf{1}_B$.

When we work with a translation in a zero-dimensional compact group, the statement admits a simplified form: it is not necessary to assume that $\mathrm{span}(\mathbf{1}_B:\ B\in\mathcal{C})$ is dense in $L_p(\Omega,\mu)$, since this is automatically true (see the proof of Lemma 1.6).

Corollary 4.5. *Let Ω be a metrizable, compact zero-dimensional group, and let μ be a Borel probability measure on Ω. Let also $a\in G$, and assume that the composition operator C_{τ_a} associated to the left translation $x\mapsto ax$ is bounded on $L_p(\Omega,\mu)$, $1\le p<\infty$.*

(1) *Suppose that there exists $\alpha>0$ such that the following holds true: for every $\varepsilon>0$, one can find a Borel set $B\subset\Omega$ and an arbitrarily large integer m such that $\mu(B)\le\varepsilon$ and $\#\big\{k\in[\![1,m]\!]:\ \mu(a^{-k}B)\ge 1-\varepsilon\big\}\ge\alpha m$. Then C_{τ_a} is $\mathcal{U}$-frequently hypercyclic.*
(2) *Suppose that for any set $A\subset\mathbb{N}$ with $\overline{\mathrm{dens}}(A)>0$, there exists $\alpha>0$ such that: for every $\varepsilon>0$, one can find $B\subset\Omega$ and an arbitrarily large m such that $\mu(B)\le\varepsilon$ and $\#\big\{k\in[\![1,m]\!]:\ k\in A \text{ and } \mu(a^{-k}B)\ge 1-\varepsilon\big\}\ge\alpha m$. Then C_{τ_a} is hereditarily $\mathcal{U}$-frequently hypercyclic.*

One may ask if there is a converse to Theorem 4.3 under some assumptions of $\mathcal{X}$. We do not know if this is true, but one can observe the following.

Remark 4.6. Assume that $\mu(\Omega)<\infty$ and that $\mathcal{X}$ satisfies (H1) and (C1). If C_φ is $\mathcal{U}$-frequently hypercyclic then, for every $\varepsilon>0$, there exists $\alpha>0$ such that: for any $\delta>0$, there exists $B\in\mathcal{B}$ such that

$$\mu(B)\le\delta \qquad\text{and}\qquad \overline{\mathrm{dens}}\big(\{k\in\mathbb{N}:\ \mu(\Omega\backslash\varphi^{-k}(B))\le\varepsilon\}\big)\ge\alpha.$$

Proof. Let $\varepsilon>0$, and choose $\eta>0$ such that $\|\psi\|\le\eta\implies\mu\big(|\psi|\ge 1\big)<\varepsilon$. Let f be a $\mathcal{U}$-frequently hypercyclic vector for C_φ, let $V:=\{g\in\mathcal{X}:\ \|g-\mathbf{4}\|<\eta\}$ and let $\alpha:=\overline{\mathrm{dens}}\big(\mathcal{N}_{C_\varphi}(f,V)\big)$. Let $\delta>0$ and let $\eta'>0$ be such that $\|\psi\|\le\eta'\implies\mu\big(|\psi|\ge 1\big)<\delta$. Replacing f by some $f\circ\varphi^k$, which does not change the value of α, we may assume $\|f-\mathbf{2}\|<\eta'$. We then set $B:=\{x\in\Omega:\ |f(x)-4|<1\}$. Since $B\subset\{x:\ |f(x)-2|\ge 1\}$, we get $\mu(B)\le\delta$. Moreover, if $k\in\mathcal{N}_{C_\varphi}(f,V)$ then, since

$$\Omega\backslash\varphi^{-k}(B)\subset\big\{x\in\Omega:\ |f\circ\varphi^k(x)-4|\ge 1\big\}\qquad\text{and}\qquad \|f\circ\varphi^k-\mathbf{4}\|<\eta,$$

we get $\mu(\Omega\backslash\varphi^{-k}(B))\le\varepsilon$. So $\overline{\mathrm{dens}}\big(\{k\in\mathbb{N}:\ \mu(\Omega\backslash\varphi^{-k}(B))\le\varepsilon\}\big)\ge\alpha$. ☐

4.3. **$\mathcal{F}$-hypercyclicity.** In this section, we prove an abstract version of Proposition 4.1. To state it, we need to recall a few definitions. In what follows, we denote by $2^{\mathbb{N}}$ the space of all subsets of $\mathbb{N}$, identified with the Cantor space.

A *Furstenberg family* is a family $\mathcal{F}\subset 2^{\mathbb{N}}$ which is hereditary upward for inclusion and such that all sets in $\mathcal{F}$ are nonempty. An operator $T\in\mathfrak{L}(X)$ is said to be **$\mathcal{F}$-hypercyclic** if there exists $x\in X$ such that $\mathcal{N}_T(x,V)\in\mathcal{F}$ for every open set $V\neq\varnothing$. *Hereditary* $\mathcal{F}$-hypercyclicity is defined in the obvious way.

A Furstenberg family $\mathcal{F}\subset 2^{\mathbb{N}}$ is said to be *right translation-invariant* if $A\in\mathcal{F}\implies A+n\in\mathcal{F}$ for every $n\in\mathbb{N}$; and *left translation-invariant* if $A\in\mathcal{F}\implies A-n\in\mathcal{F}$ for every $n\in\mathbb{N}$, where $A-n=\{j\in\mathbb{N}:\ j+n\in A\}$.

Note that any Furstenberg family $\mathcal{F}\subset 2^{\mathbb{N}}$ can be written in an artificial way as a union of G_δ (in fact, closed) Furstenberg families; namely, $\mathcal{F}=\bigcup_{A\in\mathcal{F}}\mathcal{F}_A$ with $\mathcal{F}_A:=\{B\in 2^{\mathbb{N}}:\ B\supset A\}$. A Furstenberg family is **upper** in the sense of [11] precisely when it can be written as a union of left translation-invariant G_δ Furstenberg families. For example, the family $\mathcal{F}$ of all sets with positive upper density is upper, since $\mathcal{F} = \bigcup_{r>0} \mathcal{F}_r$ where $\mathcal{F}_r = \{A \subset \mathbb{N} : \overline{\mathrm{dens}}(A) \geq r\}$.

A Furstenberg family $\mathcal{F}$ is said to be *partition-regular* if, whenever $A_1, \dots, A_d \subset \mathbb{N}$ are such that $A_1 \cup \dots \cup A_d \in \mathcal{F}$, it follows that at least one $A_i \in \mathcal{F}$; and $\mathcal{F}$ is said to partition-regular *with respect to arithmetic progressions* if this holds at least when the sets A_i are arithmetic progressions of the same step d (for any d).

Proposition 4.7. *Let $T \in \mathfrak{L}(X)$ be such that the periodic vectors of T are dense in X. Let also $\mathcal{F} \subset 2^{\mathbb{N}}$ be a Furstenberg family written as $\mathcal{F} = \bigcup_{r\in R} \mathcal{F}_r$, where each $\mathcal{F}_r$ is a G_δ Furstenberg family.*

1. *Assume that each family $\mathcal{F}_r$ is right translation-invariant and that for any neighbourhood W of 0 in X and every $d \geq 1$, there exists $r \in R$ such that: for every nonempty open set $U \subset X$, one can find $x \in U$ and $i \in [\![0, d[\![$ such that $\mathcal{N}_T(x, W) \cap (i + d\mathbb{Z}_+) \in \mathcal{F}_r$. Then T is $\mathcal{F}$-hypercyclic.*
2. *Assume that $\mathcal{F}$ is partition regular with respect to arithmetic progressions and that, for any neighbourhood W of 0 in X and every $A \in \mathcal{F}$, there exists r such that $\{x \in X : \mathcal{N}_T(x, W) \cap A \in \mathcal{F}_r\}$ is dense in X. Then T is hereditarily $\mathcal{F}$-hypercyclic.*

Proof. (1) It suffices to prove that for any nonempty open set V, there exists $r = r_V$ such that the G_δ set

$$G_V := \{u \in X : \ \mathcal{N}_T(u, V) \in \mathcal{F}_r\}$$

is dense in X. Indeed, picking (V_p) a basis of nonempty open subsets of X, any vector in the dense G_δ set $\bigcap_p G_{V_p}$ will be a $\mathcal{F}$-hypercyclic vector for T.

So let V be a nonempty open subset of X. Let $v \in V$ be a periodic point of T with period d, and let W_0 be a neighbourhood of 0 such that $v + W_0 \subset V$. Choose a neighbourhood W of 0 such that $T^i(W) \subset W_0$ for $i = 1, \dots, d$, and let $r = r_V$ be associated with this choice of W.

Let U be a nonempty open subset of X: we need to find $u \in U$ such that $N_T(u, V) \in \mathcal{F}_r$. By assumption, one can find $i \in [\![1, d]\!]$ and x in the nonempty open set $U - v$ such that $\mathcal{N}_T(x, W) \cap (d\mathbb{N} - i) \in \mathcal{F}_r$. Then $(\mathcal{N}_T(x, W) + i) \cap d\mathbb{N} \in \mathcal{F}_r$ by right-translation-invariance of $\mathcal{F}_r$. Now, if $k \in (\mathcal{N}_T(x, W) + i) \cap d\mathbb{N}$ then, writing $x = u - v$ with $u \in U$ and $k = l + i$ with $l \in \mathcal{N}_T(x, W)$, we get

$$T^k u = T^i(T^l x) + v \in W_0 + v \subset V.$$

So $\mathcal{N}_T(u, V) \in \mathcal{F}_r$, as required.

(2) It is enough to show that for any nonempty open set V and for any $B \in \mathcal{F}$, there exists $r = r_{V,B} > 0$ such that the G_δ set

$$G_{V,B} := \{u \in X : \ \mathcal{N}_T(u, V) \cap B \in \mathcal{F}_r\}$$

is dense in X. Picking any sequence (V_q) of nonempty open subsets of X and any seuence $(B_q) \subset \mathcal{F}$, every vector x in the dense G_δ-set $\bigcap_q G_{V_q, B_q}$ will then be such that $\mathcal{N}(x, V_q) \cap B_q \in \mathcal{F}$ for all $q \in \mathbb{N}$.

So let V and B as above and let v be a periodic vector belonging to V. Let W be a neighbourhood of 0 such that $v + W \subset V$, and let d be the period of v. For $i = 0, \dots, d-1$, let $B_i := B \cap (i + d\mathbb{Z}_+)$. Since $\mathcal{F}$ is partition regular with respect to arithmetic progressions, there exists some i_0 such that $B_{i_0} \in \mathcal{F}$. We apply the assumption with $A := B_{i_0}$ to get some $r = r_{V,B}$.

Let U be any nonempty open subset of X; we need to find $u \in U$ such that $N_T(u,V) \cap B \in \mathcal{F}_r$. By assumption, one can find $x \in U - T^{d-i_0}v$ such that $\mathcal{N}_T(x,W) \cap B_{i_0} \in \mathcal{F}_r$. Now, writing $x = u - T^{d-i_0}v$ with $u \in U$, we have $\mathcal{N}_T(x,W) \cap B_{i_0} \subset \mathcal{N}_T(u,V)$: indeed, if $k \in \mathcal{N}_T(x,W) \cap B_{i_0}$, then

$$T^k u = T^k x + T^{k-i_0+d}v = T^k x + v \in v + W \subset V.$$

So we get $\mathcal{N}_T(u,V) \cap B_{i_0} \in \mathcal{F}_r$, and hence $\mathcal{N}_T(u,V) \cap B \in \mathcal{F}_r$. ☐

Proposition 4.1 is a special case of Proposition 4.7. Indeed, let $\mathcal{F}$ be the family of all sets $A \subset \mathbb{N}$ with positive upper density. Then $\mathcal{F} = \bigcup_{r>0} \mathcal{F}_r$ where each $\mathcal{F}_r := \{A \subset \mathbb{N} : \overline{\mathrm{dens}}(A) \geq r\}$ is G_δ and translation-invariant. Moreover, $\mathcal{F}$ is partition-regular by subadditivity of upper density. Let us show that (a) in Proposition 4.1 follows from (1) above. Given W and d, choose $\alpha > 0$ associated with W according to the assumption in (a), and let $r := \alpha/d$. By the Baire category theorem and since any set of the form $\{x \in X : \#\mathcal{N}_T(x,W) \cap [\![0,m]\!] \geq \alpha m\}$ is open in X, the set $\{x \in X : \overline{\mathrm{dens}}\big(\mathcal{N}_T(x,W)\big) \geq \alpha\}$ is dense in X. So we are done by subadditivity of upper density. Similarly, (b) in Proposition 4.1 follows from (1) above.

Here is another consequence of Proposition 4.7. This is not new, see [20].

Corollary 4.8. *Let $T \in \mathfrak{L}(X)$ be such that* $\mathrm{Per}(T)$ *is dense in X.*

- (a) *Assume that for any neighbourhood W of 0 and any open set $U \neq \varnothing$, we have $\mathcal{N}_T(U,W) \neq \varnothing$. Then T is hypercyclic.*
- (b) *Assume that for any neighbourhood W of 0 and any open set $U \neq \varnothing$, the set $\mathcal{N}_T(U,W)$ is cofinite. Then T is topologically mixing.*

Proof. (a) Apply (1) above with $\mathcal{F} := 2^{\mathbb{N}} \setminus \{\varnothing\}$.

(b) We apply (2) above with $\mathcal{F} = [\mathbb{N}]^\infty$, the family of all infinite subsets of $\mathbb{N}$, which is G_δ and partition regular. Note that topological mixing is equivalent to hereditary $\mathcal{F}$-hypercyclicity. Let W be a neighbourhood of 0 and let $A \in [\mathbb{N}]^\infty$. Towards a contradiction with (2), assume that there exists a nonempty open set U_0 such that $\mathcal{N}_T(x,W) \cap A$ is finite for all $x \in U_0$. Let $(F_n)_{n\in\mathbb{N}}$ be an enumeration of all finite subsets of $\mathbb{N}$. Then, the sets $\mathbf{F}_n := \{x \in X : \mathcal{N}_T(x,W) \cap A \subset F_n\}$ are closed in X, and $U_0 \subset \bigcup_{n\in\mathbb{N}} \mathbf{F}_n$. By the Baire category theorem, it follows that one can find a finite set F and a nonempty open set $U \subset U_0$ such that $\mathcal{N}_T(x,W) \cap A \subset F$ for all $x \in U$. In other words, $\mathcal{N}_T(U,V) \cap A \subset F$, a contradiction since A is infinite and $\mathcal{N}_T(U,V)$ is cofinite. ☐

Remark 4.9. Part (a) can be slightly strengthened, as follows: T is hypercyclic as soon as (i) $\mathcal{N}_T(U,W) \neq \varnothing$ for any U, W as in (a), and (ii) for any open set $V \neq \varnothing$, one can find $z \in V$ such that $\mathcal{N}_T(z,V)$ has bounded gaps (we do not assume that $\mathrm{Per}(T)$ is dense).

Proof. Let U, V be nonempty open sets in X. Choose a nonempty open set $V' \subset V$ and a neighbourhood W of 0 such that $V' + W \subset V$. By (ii), one can find $z \in V'$ and $d \in \mathbb{N}$ such that $\mathcal{N}_T(z,V')$ has gaps of length at most d. Let $U' := U - z$, and let W' be a neighbourhood of 0 such that $T^i(W') \subset W$ for $i = 0, \dots, d$. By (i), one can find $x \in U'$ and $n \in \mathbb{N}$ such that $T^n x \in W'$. Taking $i \le d$ such that $i+n \in \mathcal{N}_T(z, V')$, we see that $u := z + x \in U$ and $T^{i+n}u \in V' + T^i W' \subset V$. Hence $\mathcal{N}_T(U,V) \neq \varnothing$, and T is topologically transitive. ☐

5. Composition operators induced by odometers

5.1. **Notation.** In this section, we use the notation of Subsection 1.4. We fix once and for all an odometer $(\Omega, \mathcal{B}, \mu, \mathfrak{o})$, where

$$\Omega = \prod_{i \ge 1} \mathbb{Z}/m_i\mathbb{Z} =: \prod_{i\ge 1} \Omega_i,$$

$\mathcal{B}$ is the Borel σ-algebra of Ω and $\mu = \otimes_{i\ge 1}\mu_i$ is a product probability measure. We consider the operator $C_\mathfrak{o}$ acting on $L_p(\Omega,\mu)$, $1 \le p < \infty$; so we are assuming that the boundedness condition (1.2) is satisfied. We also assume that the measure μ is non-atomic, which means that

$$\prod_{i=1}^\infty \eta_i = 0$$

where

$$\eta_i = \max\bigl(\mu_i(j)\ :\ j \in \Omega_i\bigr).$$

Finally, we recall that the periodic points of $C_\mathfrak{o}$ are dense in $L_p(\Omega,\mu)$, so that $C_\mathfrak{o}$ is chaotic as soon as it is hypercyclic: this follows from Lemma 1.6; or, in a more elementary way, one just need to observe that if $C = [C_1, \dots, C_n] \subset \Omega$ is an arbitrary cylinder set and $d := \prod_{i=1}^n m_i$, then $\mathfrak{o}^{-d}(C) = C$ since $\mathfrak{o}^d(x)_i = x_i$ for every $x \in \Omega$ and $i = 1, \dots, n$ by Fact 1.2.

5.2. **Hypercyclicity.** The next theorem gives a simple sufficient condition for hypercyclicity of $C_\mathfrak{o}$. The proof relies on a probabilistic argument. For $i \ge 1$, let us set

$$\theta_i := \sup\bigl\{\mu_i(D) - \mu_i(D+k) :\ D \subset \Omega_i,\ k \in \mathbb{Z}\bigr\},$$

where addition is understood in $\Omega_i = \mathbb{Z}/m_i\mathbb{Z}$.

Theorem 5.1. *Assume that there exists an increasing sequence of positive integers $(i_s)_{s\ge 1}$ such that*

(a) $\displaystyle\sum_{s=1}^\infty \prod_{i=i_s+1}^{i_{s+1}-1} \mu_i(m_i - 1) < \infty$;

(b) $\displaystyle\frac{1}{n}\left(\sum_{s=1}^n \theta_{i_s}\right)^2 \to \infty$ *as* $n \to \infty$.

Then $C_\mathfrak{o}$ is hypercyclic on $L_p(\Omega,\mu)$, $1 \le p < \infty$.

Proof. Let $\varepsilon > 0$. Since for all $N \ge 0$,

$$\frac{1}{n}\left(\sum_{s=N+1}^{N+n} \theta_{i_s}\right)^2 \xrightarrow{n\to\infty} \infty,$$

we can assume, forgetting the first N terms of (i_s) for some large enough N, that we have selected an integer $n \geq 1$ such that

$$\sum_{s=1}^{n-1} \prod_{i=i_s+1}^{i_{s+1}-1} \mu_i(m_i - 1) < \varepsilon \qquad \text{and} \qquad \exp\left[-\frac{2}{9n}\left(\sum_{s=1}^{n} \theta_{i_s}\right)^2\right] < \varepsilon.$$

The reason for putting the strange factor $2/9$ will become clear in a few lines.

For $s = 1, \dots, n$, choose $D_{i_s} \subset \Omega_{i_s}$ and $k_{i_s} \in [\![0, m_{i_s} - 1]\!]$ such that

$$\theta_{i_s} = \mu_{i_s}(D_{i_s}) - \mu_{i_s}(D_{i_s} + k_{i_s}).$$

We consider the random variables X_s and Y_s defined on $(\Omega, \mathcal{B}, \mu)$ as follows:

$$X_s(x) := \begin{cases} 1 & \text{if } x_{i_s} \in D_{i_s}, \\ 0 & \text{otherwise} \end{cases} \qquad \text{and} \qquad Y_s(x) := \begin{cases} 1 & \text{if } x_{i_s} \in D_{i_s} + k_{i_s}, \\ 0 & \text{otherwise.} \end{cases}$$

Note that $X_1, \dots, X_n$ are independent Bernoulli variables with $\mathbb{E}(X_s) = \mu_{i_s}(D_{i_s})$; and $Y_1, \dots, Y_n$ are independent Bernoulli variables with $\mathbb{E}(Y_s) = \mu_{i_s}(D_{i_s} + k_{i_s})$.

Let us define

$$B_X := \left\{x \in \Omega :\ \sum_{s=1}^{n} X_s(x) \geq \sum_{s=1}^{n} \left(\mu_{i_s}(D_{i_s}) - \frac{\theta_{i_s}}{3}\right)\right\},$$

$$B_Y := \left\{x \in \Omega :\ \sum_{s=1}^{n} Y_s(x) \leq \sum_{s=1}^{n} \left(\mu_{i_s}(D_{i_s} + k_{i_s}) + \frac{\theta_{i_s}}{3}\right)\right\}$$

and, for $s = 1, \dots, n-1$,

$$E_s := \{x \in \Omega :\ x_i = m_i - 1 \text{ for } i = i_s + 1, \dots, i_{s+1} - 1\}.$$

Let finally

$$B := (B_X \cap B_Y) \setminus \bigcup_{s=1}^{n-1} E_s\,,$$

$$\text{and} \qquad k := \sum_{s=1}^{n} k_{i_s} \prod_{i=1}^{i_s - 1} m_i\,.$$

We claim that $\mu(B) > 1 - 3\varepsilon$ and that $B \cap \mathfrak{o}^k(B) = \varnothing$, which will show that $C_{\mathfrak{o}}$ is hypercyclic by Corollary 2.5.

Let $x \in B$. By Fact 1.2, we have $\mathfrak{o}^k(x) = x + (0, \dots, k_{i_1}, 0 \dots, k_{i_2}, 0 \dots, k_{i_n}, 0 \dots)$. Hence $(\mathfrak{o}^k(x))_{i_s} = x_{i_s} + k_{i_s} \pmod{m_{i_s}}$ for $s = 1$; and this remains true for $s = 2, \dots, n$ since x does not belong to E_{s-1}: indeed, there exists $i \in (i_{s-1}, i_s)$ such that $x_i \neq m_i - 1$, which prevents the occurence of a carry at the i_s-th position. So we have $Y_s(\mathfrak{o}^k(x)) = X_s(x)$ for $s = 1, \dots, n$. Since $x \in B_X$, it follows that

$$\begin{aligned} \sum_{s=1}^{n} Y_s(\mathfrak{o}^k(x)) &= \sum_{s=1}^{n} X_s(x) \\ &\geq \sum_{s=1}^{n} \left(\mu_{i_s}(D_{i_s}) - \frac{\theta_{i_s}}{3}\right) \end{aligned}$$

$$> \sum_{s=1}^{n}\left(\mu_{i_s}(D_{i_s}+k_{i_s})+\frac{\theta_{i_s}}{3}\right),$$

so that $\mathfrak{o}^k(x)\notin B_Y$. This shows that $B\cap\mathfrak{o}^k(B)=\varnothing$.

By definition of B_X, we have

$$\begin{aligned}\mu(B_X)&=1-\mu\left(\sum_{s=1}^{n}X_s<\sum_{s=1}^{n}\left(\mu_{i_s}(D_{i_s})-\frac{\theta_{i_s}}{3}\right)\right)\\&=1-\mu\left(\sum_{s=1}^{n}\big(X_s-\mathbb{E}(X_s)\big)<-\frac{1}{3}\sum_{s=1}^{n}\theta_{i_s}\right).\end{aligned}$$

Since $X_1,\dots,X_n$ are independent Bernoulli variables, it follows, by Hoeffding's inequality (see e.g. [12]), that

$$\mu(B_X)\geq 1-\exp\left(-\frac{2}{9n}\left(\sum_{s=1}^{n}\theta_{i_s}\right)^2\right);$$

and the choice of n ensures that $\mu(B_X)>1-\varepsilon$. Similarly, $\mu(B_Y)>1-\varepsilon$. Since

$$\sum_{s=1}^{n-1}\mu(E_s)=\sum_{s=1}^{n-1}\prod_{i=i_s+1}^{i_{s+1}-1}\mu_i(m_i-1)<\varepsilon,$$

we conclude that $\mu(B)>1-3\varepsilon$. □

Theorem 5.1 allows us to prove a strengthened version of [9, Corollary 3.5]. For $i\geq 1$, we set

$$\delta_i:=\min\{\mu_i(j):\ j\in\Omega_i\},$$

and we recall that

$$\eta_i=\max\{\mu_i(j):\ j\in\Omega_i\}.$$

Observe that

$$\theta_i\geq\eta_i-\delta_i.$$

Indeed, assuming that $\eta_i-\delta_i>0$, choose $j,k\in\Omega_i$ such that $\mu_i(j)=\eta_i$ and $\mu_i(j+k)=\delta_i$, and take $D=\{j\}$ in the definition of θ_i.

Corollary 5.2. *If* $\limsup\,(\eta_i-\delta_i)>0$, *then* $C_{\mathfrak{o}}$ *is hypercyclic.*

Proof. By assumption, there exist $c>0$ and an increasing sequence of integers $(i_s)_{s\geq 1}$ such that $\eta_{i_s}-\delta_{i_s}\geq c$ for all $s\geq 1$. Then, Condition (b) of Theorem 5.1 is clearly satisfied since $\theta_{i_s}\geq\eta_{i_s}-\delta_{i_s}\geq c$ for all $s\geq 1$. Moreover, since $\prod_{i=1}^{\infty}\eta_i=0$, we may assume, upon extracting a subsequence of (i_s), that $\prod_{i=i_s+1}^{i_{s+1}-1}\eta_i\leq 2^{-s}$ for all $s\geq 1$, so that Condition (a) is satisfied as well. □

In particular, we may apply Corollary 5.2 as soon as $\limsup\,\eta_i>1/2$ (whereas in [9] hypercyclicity was only proved when $\limsup\,\eta_i=1$).

We may also conclude that $C_{\mathfrak{o}}$ is hypercyclic as soon as $\limsup\eta_i>0$ and $\delta_i\to 0$. For example, this holds for the so-called *Ornstein odometer*.

Example 5.3. Let $(\Omega, \mathcal{B}, \mu, \mathfrak{o})$ be the Ornstein odometer, defined by $\Omega_i = [\![0, i]\!]$, $\mu_i(0) = 1/2$ and $\mu_i(j) = 1/2i$ for $1 \leq j \leq i$. Then $C_\mathfrak{o}$ is (bounded and) hypercyclic on $L_p(\Omega, \mu)$.

Finally, Corollary 5.2 allows us to completely characterize the hypercyclic composition operators induced by odometers with the same measure on each component.

Corollary 5.4. *Let $N \geq 2$, and le ν be a probability measure on $[\![0, N-1]\!]$. Assume that $\Omega_i = [\![0, N-1]\!]$ for all $i \geq 1$, and that all measures μ_i are equal to ν. Then $C_\mathfrak{o}$ is bounded on L_p if and only if $\nu(0) \geq \nu(N-1)$; and when this condition is satisfied, $C_\mathfrak{o}$ is hypercyclic on $L_p(\Omega, \mu)$ if and only if ν is not the uniform distribution on $[\![0, N-1]\!]$, i.e. μ is not the Haar measure of the compact group $(\Omega, +)$.*

Proof. Since $\mu_i = \nu$ for all i, the boundedness condition (1.2) for $C_\mathfrak{o}$ reduces to

$$\sup_{l \geq 1} \left(\frac{\nu(N-1)}{\nu(0)} \right)^{l-1} < \infty,$$

i.e. $\nu(N-1) \leq \nu(0)$. If ν is the uniform distribution, then $\mathfrak{o}$ is measure-preserving, so $C_\mathfrak{o}$ is an isometry and hence it is not hypercyclic. Otherwise, $\eta_i - \delta_i$ is positive and does not depend on i, so we may apply Corollary 5.2. ☐

Let us now discuss an example of a binary odometer with $\lim \eta_i = 1/2 = \lim \delta_i$.

Example 5.5. Let $\alpha > 0$. Let us assume that $\Omega_i = \{0, 1\}$ for all $i \geq 1$, and that $\mu_i(0) = \frac{1}{2} + \frac{1}{i^\alpha}$.

- if $\alpha \in (0, 1/2)$, then $C_\mathfrak{o}$ is hypercyclic;
- if $\alpha \in (1, \infty)$, then $C_\mathfrak{o}$ is not hypercyclic.

Proof. We first observe that $C_\mathfrak{o}$ is easily seen to be bounded on L_p, for any $\alpha > 0$.

Suppose that $\alpha \in (0, 1/2)$. Choose $\beta > 1$ such that $\alpha\beta < 1/2$, and let $i_s = \lfloor s^\beta \rfloor$ for all $s \geq 1$. Then $i_{s+1} - i_s \sim \beta s^{\beta - 1}$ as $s \to \infty$, so Condition (a) in Theorem 5.1 is satisfied since $\mu_i(m_i - 1) = \mu_i(1) \leq 1/2$ for all $i \geq 1$ and the series $\sum_s 2^{-cs^{\beta-1}}$ is convergent for any $c > 0$. Moreover, since $\varepsilon_i = \frac{2}{i^\alpha}$ for all $i \geq 1$, we see that

$$\sum_{s=1}^{n} \theta_{i_s} = \sum_{s=1}^{n} \frac{2}{i_s^\alpha} \sim c_\beta n^{1-\alpha\beta} \quad \text{as } n \to \infty,$$

which allows us to conclude that Condition (b) in Theorem 5.1 is satisfied since $1 - \alpha\beta > 1/2$.

Suppose that $\alpha > 1$. Let us observe that for all $i \geq 1$,

$$\frac{\eta_i}{\delta_i} = \frac{\frac{1}{2} + \frac{1}{i^\alpha}}{\frac{1}{2} - \frac{1}{i^\alpha}} = 1 + \frac{4}{i^\alpha} + o\left(\frac{1}{i^\alpha}\right) \quad \text{as } i \to \infty.$$

In particular, the infinite product $\prod_{i \geq 1} (\eta_i / \delta_i)$ is convergent, which prevents $C_\mathfrak{o}$ from being hypercyclic by [9, Theorem 3.8]. ☐

Question 5.6. What happens in Example 5.5 for $\alpha \in [1/2, 1]$?

Question 5.7. A look at the proof of [9, Theorem 3.8] reveals that when the infinite product $\prod_{i\geq 1}(\eta_i/\delta_i)$ is convergent, the operator $C_{\mathfrak{o}}$ is in fact power-bounded, *i.e.* $\sup_{n\in\mathbb{N}}\|C_{\mathfrak{o}}^n\|<\infty$ is bounded. Is it true that $C_{\mathfrak{o}}$ is hypercyclic as soon as it is not power-bounded?

5.3. **Mixing.** In [9], it is shown that $C_{\mathfrak{o}}$ is topologically mixing if $\lim_{i\to\infty}\eta_i=1$, and that the converse is true provided the sequence (m_i) is bounded. In this section, our first aim is to get a characterization working even for unbounded sequences (m_i). For this, we need to introduce a new sequence: for $i\geq 1$, we set

$$\kappa_i := \inf_{j\in[\![1,m_i-1]\!]} \sup\bigl\{\mu_i(D):\ D\subset[\![0,m_i-1]\!] \text{ and } (D+j)\cap D=\varnothing\bigr\}.$$

Here (and in the proof of the next theorem), addition is understood as addition in $\mathbb{Z}_+$, not in $\mathbb{Z}/m_i\mathbb{Z}$.

Theorem 5.8. *$C_{\mathfrak{o}}$ is topologically mixing if and only if* $\lim\limits_{i\to\infty}\kappa_i=1$.

Proof. The proof relies on Corollary 2.5.

Assume first that $C_{\mathfrak{o}}$ is mixing. Given $\varepsilon>0$, we need to find i_0 such that $\kappa_i\geq 1-\varepsilon$ for all $i\geq i_0$. By Corollary 2.5, there exists $k_0\in\mathbb{N}$ such that, for all $k\geq k_0$, we may find $B\in\mathcal{B}$ with $\mu(B)>1-\varepsilon/2$ and $\mathfrak{o}^k(B)\cap B=\varnothing$. Let $i_0\geq 1$ be such that $m_1\cdots m_{i_0-1}>k_0$; we check that i_0 works.

Let us fix $i\geq i_0$, and let $j\in[\![1,m_i-1]\!]$. We set $k:=jm_1\cdots m_{i-1}$ and we choose $B\in\mathcal{B}$ such that $\mu(B)>1-\varepsilon/2$ and $\mathfrak{o}^k(B)\cap B=\varnothing$.

For $a\in\Omega_i$, we set $B_a:=\{x\in B:\ x_i=a\}$, and we denote by $\widehat{B_a}$ the projection of B_a onto $\prod_{r\neq i}\Omega_r$. Let us also denote by $\widehat{\mu_i}$ the measure $\otimes_{r\neq i}\mu_r$. We observe that if $0\leq a\leq m_i-1-j$, then $\widehat{B_a}\cap\widehat{B_{a+j}}=\varnothing$. Indeed, otherwise there exists $(x_r)_{r\neq i}\in\prod_{r\neq i}\Omega_r$ such that

$$x=(x_1,\dots,x_{i-1},a,x_{i+1},\dots)\in B\quad\text{ and }\quad x'=(x_1,\dots,x_{i-1},a+j,x_{i+1},\dots)\in B;$$

but $\mathfrak{o}^k(x)=x'$ and this contradicts $\mathfrak{o}^k(B)\cap B=\varnothing$.

We now define

$$D:=\bigl\{a\in[\![0,m_i-1]\!]:\ \widehat{\mu_i}\bigl(\widehat{B_a}\bigr)>1/2\bigr\}.$$

Note that if $a,a'\in D$, then $\widehat{B_a}\cap\widehat{B_{a'}}\neq\varnothing$. Hence $D\cap(D+j)=\varnothing$, since otherwise there exists $a\in D\cap[\![0,m_i-1-j]\!]$ such that $a'=a+j\in D$ and we know that $\widehat{B_a}\cap\widehat{B_{a+j}}=\varnothing$. Moreover,

$$\begin{aligned}1-\frac{\varepsilon}{2}\leq\mu(B)&=\sum_{a\in\Omega_i}\mu_i(a)\widehat{\mu_i}\bigl(\widehat{B_a}\bigr)\\&\leq\mu_i(D)+\sum_{a\in\Omega_i\setminus D}\mu_i(a)\,\widehat{\mu_i}\bigl(\widehat{B_a}\bigr)\\&\leq 1-\mu_i(\Omega_i\setminus D)+\frac{1}{2}\,\mu_i(\Omega_i\setminus D).\end{aligned}$$

So we get $\mu_i(\Omega_i\setminus D)\leq\varepsilon$, *i.e.* $\mu_i(D)\geq 1-\varepsilon$. Thus, for any $j\in[\![1,m_i-1]\!]$ we can find a set $D\subset[\![0,m_i-1]\!]$ such that $D\cap(D+j)=\varnothing$ and $\mu_i(D)\geq 1-\varepsilon$; in other words, $\kappa_i\geq 1-\varepsilon$.

Conversely, assume that $\lim_{i\to\infty}\kappa_i = 1$. This time, given $\varepsilon > 0$, we need to find $k_0 \in \mathbb{N}$ such that, for any $k \geq k_0$, there exists $B \subset \Omega$ such that $\mu(\Omega \setminus B) \leq \varepsilon$ and $\mathfrak{o}^k(B) \cap B = \varnothing$.

By assumption, there exists $i_0 \geq 1$ such that the following holds true: for every $i \geq i_0$ and all $j \in [\![1, m_{i_0} - 1]\!]$, there exists $D_{i,j} \subset \Omega_i$ with $\mu_i(D_{i,j}) \geq 1 - \varepsilon/3$ and $(D_{i,j} + j) \cap D_{i,j} = \varnothing$. We show that $k_0 := \sum_{i=1}^{i_0} m_1 \cdots m_{i-1}$ works.

Let $k \geq k_0$. By the choice of k_0, one can write $k = \sum_{i=1}^{l} k_i m_1 \cdots m_{i-1}$, where $l \geq i_0$, $k_i \in [\![0, m_i - 1]\!]$ for all i and $k_l \neq 0$. By our choice of i_0, one can find $D'_l, D''_l \subset \Omega_l$ such that $\mu_l(D'_l), \mu_l(D''_l) \geq 1 - \varepsilon/3$ and $(D'_l + k_l) \cap D'_l = \varnothing = (D''_l + (k_l + 1)) \cap D''_l$ (if $k_l = m_l - 1$, just take $D''_l = \Omega_l$). We then set $D_l := D'_l \cap D''_l$ and observe that $\mu_l(D_l) \geq 1 - 2\varepsilon/3$. We also choose $D_{l+1} \subset \Omega_{l+1}$ such that $(D_{l+1} + 1) \cap D_{l+1} = \varnothing$ and $\mu_{l+1}(D_{l+1}) \geq 1 - \varepsilon/3$. Now, we define

$$B := [\Omega_1, \dots \Omega_{l-1}, D_l, D_{l+1}],$$

and observe that $\mu(B) \geq (1-2\varepsilon/3)(1-\varepsilon/3) \geq 1-\varepsilon$. We claim that $\mathfrak{o}^k(B) \cap B = \varnothing$.

Let $x \in B$: we need to show that $\mathfrak{o}^k(x) \notin B$. Recall that, by Fact 1.2, we have

$$\mathfrak{o}^k(x) = x \mathbin{\pmb{+}} (k_1, \dots, k_l, 0, 0, \dots).$$

We distinguish several cases.

- if $x_l \geq m_l - k_l$, then $(\mathfrak{o}^k(x))_{l+1} = x_{l+1} + 1$ (there is a carry at the $(l+1)$-th position) and $x_{l+1} + 1 \notin D_{l+1}$, so $\mathfrak{o}^k(x) \notin B$.
- if $x_l = m_l - k_l - 1$ and if there is a carry at the l-th position when we compute $\mathfrak{o}^k(x)$, then we are in the same situation.
- if $x_l = m_l - k_l - 1$ and if there is no carry at the l-th position when we compute $\mathfrak{o}^k(x)$, then $(\mathfrak{o}^k(x))_l = x_l + k_l \notin D_l$ because $x_l \in D_l$ and $D_l \cap (D_l + k_l) = \varnothing$, so $\mathfrak{o}^k(x) \notin B$.
- if $x_l < m_l - k_l - 1$, then $(\mathfrak{o}^k(x))_l = x_l + k_l$ or $x_l + k_l + 1$ depending on whether there is a carry over at the l-th position of not. In both cases, $\mathfrak{o}^k(x) \notin B$.

So we have shown that $\mathfrak{o}^k(B) \cap B = \varnothing$, which concludes the whole proof. □

Remark 5.9. From the definition of κ_i, it is clear that $\kappa_i \geq \eta_i$ for all $i \geq 1$. Moreover, it can also be seen from the definition of κ_i that if the sequence (m_i) is bounded and if $\kappa_i \to 1$ as $i \to \infty$, then $\eta_i \to 1$. So Theorem 5.8 is coherent with (and in fact, formally implies) Theorem 1.3 recalled in Section 1.4. On the other hand, if (m_i) is unbounded, it may happen that $\kappa_i \to 1$ and $\eta_i \not\to 1$. It is even possible that $\kappa_i \to 1$ and $\eta_i \to 0$, see [9, Example 3.3].

Proof. Assume that (m_i) is bounded, that $\kappa_i \to 1$ and, towards a contradiction, that $\liminf \eta_i < 1$. Then, one can find an increasing sequence of indices $(i_s)_{s\geq 1}$, an integer m, some $\varepsilon > 0$ and $a, b \in [\![0, m-1]\!]$ with $a < b$, such that $m_{i_s} = m$ and $\mu_{i_s}(a), \mu_{i_s}(b) > \varepsilon$ for all $s \geq 1$. Let $j := b - a$. If $s \geq 1$, then any set $D \subset [\![0, m-1]\!]$ such that $\mu_{i_s}(D) > 1 - \varepsilon$ must contain a and b, so it cannot satisfy $D \cap (D + j) = \varnothing$. So we have $\kappa_{i_s} \leq 1 - \varepsilon$ for all $s \geq 1$, a contradiction since $\kappa_i \to 1$ as $i \to \infty$. □

We can now improve Theorems 3.6 and 3.7 of [9] by exhibiting, for any sequence of integers (m_i), a measure μ on $\prod_{i\geq 1} \mathbb{Z}/m_i\mathbb{Z}$ for which $C_{\mathfrak{o}}$ is hypercyclic but not topologically mixing. In [9], this was done when (m_i) does not grow too quickly.

Example 5.10. Let (m_i) be any sequence of integers with $m_i \geq 2$ for all $i \geq 1$. There exists a sequence of probability measures (μ_i) such that $C_\mathfrak{o}$ is hypercyclic but not topologically mixing on $L_p(\Omega, \mu)$.

Proof. Let us define the measures μ_i. If $m_i = 2$, we set $\mu_i(0) := 2/3$ and $\mu_i(1) := 1/3$. Otherwise, if m_i is even, $m_i = 2\beta_i$, we set

$$\mu_i(j) := \frac{c_i}{2^{\beta_i - 1 - j}} \qquad \text{for } j = 0, \ldots, \beta_i - 1$$

and

$$\mu_i(j) := \frac{c_i}{2^{j - \beta_i}} \qquad \text{for } j = \beta_i, \ldots, 2\beta_i - 1,$$

where $c_i > 0$ is chosen so that μ_i is a probability measure. If m_i is odd, $m_i = 2\beta_i + 1$, we set

$$\mu_i(j) := \frac{c_i}{2^{\beta_i - j}} \qquad \text{for } j = 0, \ldots, \beta_i$$

and

$$\mu_i(j) := \frac{c_i}{2^{j - \beta_i}} \qquad \text{for } j = \beta_i + 1, \ldots, 2\beta_i,$$

where $c_i > 0$ is again chosen so that μ_i is a probability measure. Since $\eta_i \leq 2/3$ for all $i \in \mathbb{N}$, the product measure μ is non-atomic.

We first observe that $C_\mathfrak{o}$ is bounded. Indeed, for every $i \in \mathbb{N}$, we have

$$\frac{\mu_i(m_i - 1)}{\mu_i(0)} \leq 1$$

and

$$\frac{\mu_i(j-1)}{\mu_i(j)} \in \{1/2; 2\} \qquad \text{for all } 1 \leq j \leq m_i - 1.$$

So it is clear that the boundedness condition (1.2) is satisfied.

We now claim that $C_\mathfrak{o}$ is hypercyclic. Indeed, let $i \geq 1$ and assume that m_i is odd, $m_i = 2\beta_i + 1$. Then

$$\eta_i - \delta_i = \mu_i(\beta_i) - \mu_i(0) = c_i \left(1 - \frac{1}{2^{\beta_i}}\right) \geq \frac{c_i}{2}.$$

Now, by definition of μ_i, we have

$$2c_i \sum_{j=0}^{\infty} \frac{1}{2^j} \geq \sum_{k=0}^{2\beta_i + 1} \mu_i(k) = 1,$$

so that $c_i \geq 1/4$. Hence $\eta_i - \delta_i \geq 1/8$. We get a similar lower bound when m_i is even, and hence we may conclude that $C_\mathfrak{o}$ is hypercyclic by Corollary 5.2.

To show that $C_\mathfrak{o}$ is not mixing, we observe that for each $i \geq 1$, there exist at least two different integers $a < b \in [\![0, m_i - 1]\!]$ such that $\mu_i(a), \mu_i(b) \geq c_i/2 \geq 1/8$ (consider for instance $a = \beta_i$, $b = \beta_{i+1}$ if $m_i = 2\beta_i + 1$ is odd). If we set $j := b - a$, then $\mu(D) \leq 7/8$ for any $D \subset \Omega_i$ such that $D \cap (D + j) = \varnothing$, since D cannot contain both a and b. So $\kappa_i \leq 7/8$ for all i, and hence $C_\mathfrak{o}$ is not mixing by Theorem 5.8. $\square$

We also complete the picture by exhibiting, for any sequence of integers (m_i), a sequence of measures (μ_i) such that $C_\mathfrak{o}$ is topologically mixing.

Example 5.11. Let (m_i) be any sequence of integers with $m_i \geq 2$ for all $i \in \mathbb{N}$. There exists a sequence of probability measures (μ_i) such that $C_{\mathfrak{o}}$ is topologically mixing on $L_p(\Omega, \mu)$.

Proof. For $i > 1$ and $j \in [\![0, m_i - 1]\!]$, we set

$$\mu_i(j) := \left(1 - \frac{1}{i+1}\right) c_i^j = \frac{i}{i+1}\, c_i^j,$$

where c_i is chosen so that

$$\sum_{j=0}^{m_i - 1} c_i^j = \frac{i+1}{i}\cdot$$

Observe first that

$$\frac{1}{i+1} \leq c_i \leq \frac{1}{i},$$

the "limit" cases corresponding to $m_i = 2$ and $m_i = \infty$. Note also that $\eta_i = \mu_i(0) = \frac{i}{i+1}\cdot$ In particular $\prod_{i=1}^{\infty} \eta_i = 0$, so the measure μ is non-atomic. We check that $C_{\mathfrak{o}}$ is bounded on $L_p(\Omega, \mu)$. Let $l > 1$ and $j \in [\![0, m_i - 1]\!]$. Then

$$\begin{aligned}\frac{\mu_l(j-1)}{\mu_l(j)} \prod_{i=1}^{l-1} \frac{\mu_i(m_i - 1)}{\mu_i(0)} &\leq c_l^{-1} \prod_{i=1}^{l-1} c_i^{m_i - 1} \\ &\leq c_l^{-1} \prod_{i=1}^{l-1} c_i \\ &\leq (l+1) \prod_{i=1}^{l-1} \frac{1}{i},\end{aligned}$$

so the boundedness condition (1.2) is satisfied. Since $\eta_i \to 1$, $C_{\mathfrak{o}}$ is topologically mixing by Theorem 1.3. □

We point out that the previous known examples of topologically mixing odometers were obtained with measures having atoms (see the proof of [9, Theorem 3.2]).

5.4. **Frequent hypercyclity.** In this section, we show that some composition operators induced by odometers are frequently hypercyclic. We need to introduce two new quantities. For $i \in \mathbb{N}$ and $\kappa \in (0,1)$, we set

$$\begin{aligned}\gamma_i := \sup\big\{\gamma \in (0,1) :\ \exists D \subset \Omega_i\ \exists j \in [\![1, m_i - 1]\!]\ :& \\ \mu_i(D) > \gamma \text{ and } \mu_i(D+j) < 1 - \gamma&\big\}\end{aligned}$$

where addition is understood in $\mathbb{Z}/m_i\mathbb{Z}$; and

$$\omega_i(\kappa) := \mu_i\big([\![m_i - 1 - \kappa m_i m_{i+1}, m_i - 1]\!]\big).$$

To help digesting these definitions (and also for future reference), we quote the following remark.

Remark 5.12. Assume that $\Omega_i = \{0, 1\}$ for all $i \geq 1$. Then $\gamma_i = \max\big(\mu_i(0), \mu_i(1)\big)$ and $\omega_i(\kappa) = \mu_i(1) = 1 - \mu_i(0)$ for all i and any $\kappa < 1/4$.

Theorem 5.13. *Assume that there exists $\kappa \in (0,1)$ such that*

$$\limsup_{i\to\infty} \min(1-\omega_{i-1}(\kappa), \gamma_i) = 1.$$

Then $C_{\mathfrak{o}}$ is frequently hypercyclic on $L_p(\Omega,\mu)$, $1\le p<\infty$.

Proof. We apply Theorem 3.2, taking as $\mathcal{C}$ the family of all cylinder sets of Ω. Note that $\operatorname{span}(\mathbf{1}_B:\ B\in\mathcal{C})$ is dense in $L_p(\Omega,\mu)$.

In order to check that Condition (H_κ) is satisfied, let us fix $\varepsilon\in(0,1)$, $m\in\mathbb{N}$, and $B_1,\dots,B_r\in\mathcal{C}$. Let us also set $\delta:=\varepsilon/2$.

Let $N=i$ be a very large integer such that

$$\max(\omega_{N-1}(\kappa), 1-\gamma_N)<\delta,$$

and choose $D_N\subset\Omega_N$ and $j_N\in[\![1,m_N-1]\!]$ such that $\mu_N(D_N)>1-\delta$ and $\mu_N(D_N+j_N)<\delta$. We set

$$\begin{aligned} B &:= [\Omega_1,\dots,\Omega_{N-1}, D_N+j_N],\\ B' &:= [\Omega_1,\dots,\Omega_{N-1}, D_N],\\ n &:= j_N\prod_{i=1}^{N-1} m_i \quad \text{and}\\ d &:= \prod_{i=1}^{N} m_i. \end{aligned}$$

Observe that $\mathfrak{o}^{-d}(B)=B$, and that if N is large enough then $d\ge m$ and $\mathfrak{o}^{-d}(B_j)=B_j$ for $j=1,\dots,r$. Note also that $\mathfrak{o}^{-n}(B)=B'$.

We show that $\forall 0\le k\le\kappa d:\ \mu\big(\mathfrak{o}^{-k}(B)\big)\le\varepsilon$ and $\mu\big(\mathfrak{o}^{-n-k}(B)\big)\ge 1-\varepsilon$. In what follows, we fix $0\le k\le\kappa d$. Observe that $\kappa m_{N-1}m_N<m_{N-1}$ since $\omega_{N-1}(\kappa)<1$, so that $k\le\prod_{i=1}^{N-2}m_i\times(\kappa m_{N-1}m_N)<\prod_{i=1}^{N-1}m_i$.

Let us first show that $\mu\big(\mathfrak{o}^{-k}(B)\big)\le\varepsilon$. If $x\in\mathfrak{o}^{-k}(B)$, then either the computation of $\mathfrak{o}^k(x)$ has led to a carry at the N-th coordinate, or not. If no carry has occured, then $x_N=(\mathfrak{o}^k(x))_N\in D_N+j_N$. If a carry has occured, this can be so only if $x_{N-1}\ge m_{N-1}-1-\kappa m_{N-1}m_N$ since $k\le\prod_{i=1}^{N-2}m_i\times(\kappa m_{N-1}m_N)$. So we see that

$$\begin{aligned} \mu\big(\mathfrak{o}^{-k}(B)\big) &\le \mu_N(D_N+j_N)+\mu_{N-1}\big([\![m_{N-1}-1-\kappa m_{N-1}m_N, m_{N-1}-1]\!]\big)\\ &\le \delta+\omega_{N-1}(\kappa)\\ &\le\varepsilon. \end{aligned}$$

Now, let us show that $\mu\big(\mathfrak{o}^{-n-k}(B)\big)\ge 1-\varepsilon$, *i.e.* $\mu\big(\mathfrak{o}^{-k}(B')\big)\ge 1-\varepsilon$. Observe that if $x\in B'\setminus\mathfrak{o}^{-k}(B')$, then the computation of $\mathfrak{o}^k(x)$ has led to a carry at the N-th coordinate, so that (as above) $x_{N-1}\ge m_{N-1}-1-\kappa m_{N-1}m_N$. It follows that

$$\mu\big(B'\setminus\mathfrak{o}^{-k}(B')\big)\le\mu_{N-1}\big([\![m_{N-1}-1-\kappa m_{N-1}m_N, m_{N-1}-1]\!]\big)=\omega_{N-1}(\kappa),$$

and hence that

$$\mu\big(\mathfrak{o}^{-n-k}(B)\big)=\mu\big(\mathfrak{o}^{-k}(B')\big)\ge\mu(B')-\omega_{N-1}(\kappa)=\mu_N(D_N)-\delta\ge 1-\varepsilon.$$

So we have checked Condition (H_κ), which concludes the proof. □

Remark 5.14. Alternatively we could apply Corollary 3.8 with $d_i := \prod_{j=1}^{i} m_j$, but the heart of the proof remains the same.

The easiest way to estimate γ_i is to observe that $\gamma_i \geq \eta_i$ (choose $D = \{a\}$ where $\mu_i(a) = \eta_i$, and $j = 1$), which leads to the following corollary:

Corollary 5.15. *If $\limsup_{i\to\infty} \min(1 - \omega_{i-1}(\kappa), \eta_i) = 1$ for some $\kappa > 0$, then $C_{\mathfrak{o}}$ is frequently hypercyclic on $L_p(\Omega, \mu)$.*

When the sequence (m_i) is bounded, we can simplify the statement.

Corollary 5.16. *Suppose that the sequence (m_i) is bounded. Then $C_{\mathfrak{o}}$ is frequently hypercyclic on $L_p(\Omega, \mu)$ as soon as*

$$\limsup_{i\to\infty} \min(1 - \mu_{i-1}(m_{i-1} - 1), \eta_i) = 1.$$

Proof. Let $M := \sup_i m_i$, and take any $\kappa < 1/M^2$. We have $\omega_i(\kappa) = \mu_i(m_i - 1)$ for all $i \geq 1$, so Corollary 5.15 applies. ☐

Corollary 5.17. *If the sequence (m_i) is bounded and $\lim_{i\to\infty} \eta_i = 1$, then $C_{\mathfrak{o}}$ is frequently hypercyclic on $L_p(\Omega, \mu)$.*

Proof. By Corollary 5.17, we have just to prove that $\limsup\,(1 - \mu_i(m_i - 1)) = 1$. Otherwise, there exists $c > 0$ such that $\mu_i(m_i - 1) > c$ for all $i \geq 1$. Since $\eta_i \to 1$ as $i \to \infty$, it follows that $\lim_{i\to\infty} \mu_i(m_i - 1) = 1$, so that $\lim_{i\to\infty} \mu_i(0) = 0$; but this contradicts the boundedness condition (1.2). ☐

In view of Thorem 1.3, we immediately deduce the following corollary.

Corollary 5.18. *Assume that the sequence (m_i) is bounded. If $C_{\mathfrak{o}}$ is topologically mixing, then it is frequently hypercyclic.*

Example 5.19. Assume that $\Omega_i = \{0, 1\}$ for all $i \geq 1$, and let $\mu_i(0) := 1 - \frac{1}{i+1} = \frac{i}{i+1}$ and $\mu_i(1) := \frac{1}{i+1}\cdot$ Then $C_{\mathfrak{o}}$ is frequently hypercyclic on $L_p(\Omega, \mu)$.

Proof. The boundedness of $C_{\mathfrak{o}}$ follows from the identity

$$\prod_{i=1}^{l-1} \frac{\mu_i(1)}{\mu_i(0)} \times \max\left(\frac{\mu_l(0)}{\mu_l(1)}, \frac{\mu_l(1)}{\mu_l(0)}\right) = \frac{l}{(l-1)!}$$

which is valid for all $l \in \mathbb{N}$, whereas frequent hypercyclicity is a consequence of Corollary 5.17. ☐

Theorem 5.13 allows us to exhibit very simple examples of chaotic and frequently hypercyclic Hilbert space operators that are not topologically mixing. The first construction of such operators dates back to [3], and other examples can be found in [20]. Composition operators induced by odometers are arguably "easier" examples.

Example 5.20. Let $\Omega_i = \{0, 1\}$ for all $i \geq 1$, and define the measures μ_i as follows: $\mu_{3k+1}(0) = \mu_{3k+2}(0) := 1 - \frac{1}{k+1}$ and $\mu_{3k+3}(0) := 1/2$ for all $k \geq 0$. Then $C_{\mathfrak{o}}$ is frequently hypercyclic on $L_p(\Omega, \mu)$, but not topologically mixing.

Proof. The boundedness of $C_{\mathfrak{o}}$ is easily checked, as in Example 5.19. As for frequent hypercyclicity, observe that if we set $n_k := 3k+2$, then $\gamma_{n_k} = 1 - 1/(k+1)$ and $\omega_{n_k-1}(1/5) = 1/(k+1)$ for all $k \geq 0$ (see Remark 5.12), so Theorem 5.13 applies. Finally, $C_{\mathfrak{o}}$ is not topologically mixing by Theorem 1.3, since $\eta_{3k+3} = 1/2$ for all $k \geq 0$. □

Our examples leave open the following basic question.

Question 5.21. Does there exist a composition operator induced by an odometer which is hypercyclic but not frequently hypercyclic ?

If the answer to this question was "yes", this would give natural examples of chaotic operators which are not frequently hypercyclic. The existence of such operators was a longstanding problem in linear dynamics, which was solved by Q. Menet [26]. Since the construction of [26] is rather difficult, it would be nice to have a "simple" example. It seems that there is some flexibility in the odometer setting. Natural candidates are $\Omega_i = \{0,1\}$, $\mu_i(0) = 3/4$ for all $i \geq 1$, or an odometer for which $m_i \to \infty$. In both cases, the assumptions of Theorem 5.13 are clearly not satisfied: we have $\gamma_i = 3/4$ for all $i \geq 1$ in the first case (as well as $\omega_i(\kappa) \geq 1/4$ for any κ), whereas in the second case, $\omega_i(\kappa)$ is eventually equal to 1 for any $\kappa \in (0,1)$.

5.5. **$\mathcal{U}$-frequent hypercyclicity.** If we want to prove that the composition operator induced by an odometer is $\mathcal{U}$-frequently hypercyclic, we can slightly weaken the assumptions of Theorem 5.13.

Theorem 5.22. *Assume that there exists $\kappa > 0$ such that the following holds true: for every $\delta > 0$, there exist $i \geq 1$ arbitrarily large, $j \in [\![1, m_i - 1]\!]$ and $D \subset \Omega_i$ such that*

$$\mu_i(D) > 1-\delta\,, \quad \mu_i(D+j) < \delta \qquad \text{and}$$
$$\mu_{i-1}\big([\![m_{i-1} - \kappa j m_{i-1}, m_{i-1}-1]\!]\big) < \delta. \tag{5.1}$$

Then $C_{\mathfrak{o}}$ is $\mathcal{U}$-frequently hypercyclic on $L_p(\Omega,\mu)$.

Remark 5.23. In Theorem 5.13, the assumptions are exactly the same except that (5.1) is replaced by the stronger inequality

$$\mu_{i-1}\big([\![m_{i-1} - \kappa m_i m_{i-1}, m_{i-1}-1]\!]\big) < \delta.$$

Proof of Theorem 5.22. The proof is almost identical to that of Theorem 5.13. Given $\varepsilon > 0$, we set

$$B := [\Omega_1, \dots, \Omega_{N-1}, D_N + j_N], \qquad n := j_N \prod_{i=1}^{N-1} m_i$$

for a very large $N = i$, where $j_N \in [\![1, m_N - 1]\!]$ and $D_N \subset \Omega_N$ are such that $\mu(D_N + j_N) < \varepsilon/2$, $\mu(D_N) > 1-\varepsilon/2$ and $\mu_{N-1}\big([\![m_{N-1} - \kappa j_N m_{N-1}, m_{N-1}-1]\!]\big) < \varepsilon/2$. Then $\mu(B) < \varepsilon$; and for any $k \in [\![0, \kappa n]\!]$, the proof of Theorem 5.13 shows that $\mu(\mathfrak{o}^{-(n+k)}(B)) > 1 - \varepsilon$. Hence the assumptions of Theorem 4.3 are satisfied with $m := (1+\kappa n)$ and $\alpha := \kappa/(1+\kappa)$. □

Here is a corollary to be compared with Corollary 5.15.

Corollary 5.24. *If* $\limsup_{i\to\infty} \min\big(1-\mu_{i-1}\big([\![\kappa m_{i-1}, m_{i-1}-1]\!]\big), \eta_i\big) = 1$ *for some* $\kappa < 1$*, then* $C_{\mathfrak{o}}$ *is* $\mathcal{U}$*-frequently hypercyclic on* $L_p(\Omega,\mu)$*. This holds in particular if* $\mu_i(0)\to 1$ *as* $i\to\infty$ *or, more generally, if there exists an increasing sequence of integers* (i_s) *such that* $\mu_{i_s-1}(0)\to 1$*.*

Example 5.25. Let (m_i) be any sequence of integers with $m_i \geq 2$ for all $i\in\mathbb{N}$. There exists a sequence of probability measures (μ_i) such that $C_{\mathfrak{o}}$ is $\mathcal{U}$-frequently hypercyclic on $L_p(\Omega,\mu)$.

Proof. The odometer exhibited in Example 5.11 does the job since $\mu_i(0)\to 1$. □

6. Diagonal translation operators

6.1. **Notation.** In this section, we use the notation of Section 1.6. So we still have $\Omega = \prod_{i\geq 1}\mathbb{Z}/m_i\mathbb{Z}$, but this time addition is performed coordinatewise. We consider the composition operator $C_{\mathfrak{t}}$ defined by the translation $\mathfrak{t}(x) = x + a$, where $a = (1,1,\dots)$. We view $C_{\mathfrak{t}}$ as acting on $L_p(\Omega,\mu)$, $1\leq p<\infty$, so we assume that Condition (1.3) is satisfied. Recall that to get hypercyclic examples, the sequence (m_i) must be unbounded. Note also that the periodic vectors of $C_{\mathfrak{t}}$ are dense in $L_p(\Omega,\mu)$, so that $C_{\mathfrak{t}}$ is chaotic as soon as it is hypercyclic. This follows from Lemma 1.6, or just by observing that if $n\in\mathbb{N}$ and if we set $N=\prod_{i=1}^n m_i$, then $\mathfrak{t}^N(x)_i = x_i$ for all $x\in\Omega$ and $i=1,\dots,n$, from which it follows that $\mathbf{1}_B$ is a periodic vector of $C_{\mathfrak{t}}$ for any cylinder set $B\subset\Omega$.

6.2. **Hypercyclicity and mixing.** Let us introduce (again) some useful quantities. For $i,n\geq 1$, we set

$$\alpha_{i,n} := \sup\big\{\mu_i(D):\ D\subset\Omega_i,\ (D+n)\cap D = \varnothing\big\}$$

where addition is performed in $\mathbb{Z}/m_i\mathbb{Z}$, and we define

$$\beta_i := \sup_{n\geq 1}\alpha_{i,n} \qquad\text{and}\qquad \gamma_n := \sup_{i\geq 1}\alpha_{i,n}.$$

Lemma 6.1. *We have* $\limsup_{i\to\infty}\beta_i = 1$ *if and only if* $\limsup_{n\to\infty}\gamma_n = 1$*.*

Proof. It is clear that $\beta_i < 1$ for all $i\geq 1$. Hence,

$$\limsup \beta_i = 1 \iff \sup_i \beta_i = 1 \iff \sup_i\sup_n \alpha_{i,n} = 1.$$

Moreover, it is also true that $\gamma_n < 1$ for all $n\geq 1$. Indeed, since $C_{\mathfrak{t}}$ is bounded on $L_p(\Omega,\mu)$, there exists a constant K such that $\mu_i(D-1)\leq K\mu_i(D)$ for all $i\geq 1$ and every $D\subset\Omega_i$. Then $\mu_i(D)\leq K^n\mu_i(D+n)$ for all i and every $D\subset\Omega_i$, so that $\mu_i(D)\leq K^n(1-\mu_i(D))$ if $D\cap(D+n)=\varnothing$; and this shows that $\gamma_n\leq\frac{K^n}{1+K^n}$. Therefore,

$$\limsup\gamma_n = 1 \iff \sup_n\gamma_n = 1 \iff \sup_n\sup_i\alpha_{i,n} = 1.$$

□

Proposition 6.2. *If* $\limsup_{n\to\infty}\gamma_n = 1$*, then* $C_{\mathfrak{t}}$ *is hypercyclic on* $L_p(\Omega,\mu)$*; and if* $\gamma_n\to 1$*, then* $C_{\mathfrak{t}}$ *is topologically mixing.*

Proof. Assume that $\limsup \gamma_n = 1$, and let us show that $C_{\mathfrak{t}}$ is hypercyclic. By Corollary 2.5 (1), it is enough to show that for any $\varepsilon > 0$, one can find $B \subset \Omega$ and $n \in \mathbb{N}$ such that $\mu(B) \le \varepsilon$ and $\mu(B - na) \ge 1 - \varepsilon$. By assumption, there exist $i \ge 1$, $D \subset \Omega_i$ and $n \in \mathbb{N}$ such that $\mu_i(D) \ge 1 - \varepsilon$ and $(D + n) \cap D = \varnothing$. We take that n, and $B := \Omega_1 \times \cdots \times \Omega_{i-1} \times (D + n) \times \Omega_{i+1} \times \cdots$.

The proof of the mixing case is the same, using Corollary 2.5 (2). □

We now provide some examples.

Example 6.3. Assume that the sequence (m_i) is unbounded.

(a) There exists a sequence of measures (μ_i) such that $C_{\mathfrak{t}}$ is hypercyclic on $L_p(\Omega, \mu)$.

(b) If $\sum_{i=1}^{\infty} m_i^{-1} < \infty$ and $m_{i+1} = O(m_i)$, then there exists a sequence of measures (μ_i) such that $C_{\mathfrak{t}}$ is topologically mixing on $L_p(\Omega, \mu)$.

Proof. (a) Let $(i_s)_{s \ge 1}$ be an increasing sequence of integers such that $\sum_{s=1}^{\infty} m_{i_s}^{-1} < \infty$, and choose a sequence of positive numbers (δ_s) such that

$$\delta_s m_{i_s} \to \infty \qquad \text{and} \qquad \sum_{s=1}^{\infty} \delta_s < \infty.$$

We define (μ_i) as follows. If $i \ne i_s$ for all s, then μ_i is the uniform distribution on $[\![0, m_i - 1]\!]$. If $i = i_s$ for some s, we set $n_{i_s} := \lfloor m_{i_s}/2 \rfloor$ and we write $[\![0, m_{i_s} - 1]\!]$ as $J_{s,1} \cup J_{s,2}$, where $J_{s,1}$ and $J_{s,2}$ are two consecutive intervals with $\# J_{s,2} = n_{i_s}$ (so that $\# J_{s,1} = n_{i_s}$ or $n_{i_s} + 1$). We set $\rho_s := 1 + \delta_s$, and we choose ε_s such that

$$\# J_{s,1} \varepsilon_s + \frac{\rho_s^{n_{i_s}} - 1}{\rho_s - 1} \varepsilon_s = 1. \tag{6.1}$$

The measure μ_{i_s} is defined by

$$\begin{cases} \mu_{i_s}(k) := \varepsilon_s & \text{if } k \in J_{s,1}, \\ \mu_{i_s}(m_{i_s} - 1 - n_{i_s} + k) := \rho_s^{n_{i_s} - k} \varepsilon_s & \text{if } k = 1, \dots, n_{i_s}. \end{cases}$$

It is indeed a probability measure on $[\![0, m_{i_s} - 1]\!]$. Moreover, we have

$$\sup_{j \in \Omega_{i_s}} \frac{\mu_{i_s}(j-1)}{\mu_{i_s}(j)} = \rho_s \qquad \text{for all } s \ge 1,$$

whereas $\sup_{j \in \Omega_i} \frac{\mu_i(j-1)}{\mu_i(j)} = 1$ if i is not an i_s. So the boundedness condition (1.3) is satisfied since the infinite product $\prod_{s \ge 1} \rho_s$ is convergent. Note also that the measure μ is non-atomic since $\eta_{i_s} = \rho_s^{n_{i_s} - 1} \varepsilon_s \le \delta_s + \varepsilon_s$ by (6.1) and hence $\eta_{i_s} \to 0$ as $s \to \infty$.

Now, we observe that $\mu_{i_s}(J_{s,2}) \to 1$ as $s \to \infty$. Indeed, we have $(1 + \delta_s)^{n_{i_s}} \gg n_{i_s} \delta_s$ since $n_{i_s} \delta_s \to \infty$, so that

$$\frac{\mu_{i_s}(J_{s,2})}{\mu_{i_s}(J_{s,1})} = \frac{(1 + \delta_s)^{n_{i_s}} - 1}{n_{i_s} \delta_s} \to \infty.$$

Hence, if we set $D_{i_s} := J_{s,2}$, then $\mu_{i_s}(D_{i_s}) \to 1$ and $D_{i_s} \cap (D_{i_s} + n_{i_s}) = \varnothing$. It follows that $\limsup_i \beta_i \ge \limsup_s \mu_{i_s}(D_{i_s}) = 1$, so that $C_{\mathfrak{t}}$ is topologically transitive by Proposition 6.2 and Lemma 6.1.

(b) Let $\kappa > 0$ be such that $\kappa m_{i+1} + 1 \leq (1-\kappa) m_i$ for all $i \geq 1$. We set $n_i := \lfloor \kappa m_i \rfloor$ and we split $[\![0, m_i - 1]\!]$ into two consecutive intervals $J_{i,1} \cup J_{i,2}$ with $\# J_{i,2} = n_i$. We choose a sequence of positive numbers (δ_i) such that $m_i \delta_i \to \infty$ and $\sum_{i=1}^\infty \delta_i < \infty$, and we set $\rho_i := 1 + \delta_i$. We define ε_i by

$$\# J_{i,1} \varepsilon_i + \frac{\rho_i^{n_i} - 1}{\rho_i - 1} \varepsilon_i = 1,$$

and μ_i by

$$\begin{cases} \mu_i(k) := \varepsilon_i & \text{if } k \in J_{i,1}, \\ \mu_i(m_i - 1 - n_i + k) := \rho_i^{n_i - k} \varepsilon_i & \text{if } k = 1, \dots, n_i. \end{cases}$$

As before, the convergence of $\prod_{i \geq 1} \rho_i$ ensures the boundedness of $C_{\mathfrak{t}}$, and the measure μ is non-atomic. By Proposition 6.2, to show that $C_{\mathfrak{t}}$ is mixing it is enough to find a sequence $(D_i)_{i \geq 1}$ with $D_i \subset \Omega_i$ such that $\mu_i(D_i) \to 1$ as $i \to \infty$ and, for any $i_0 \geq 1$, every large enough integer n is such that $D_i \cap (D_i + n) = \varnothing$ for some $i \geq i_0$.

We set $D_i := J_{i,2}$ and observe that $\mu_i(D_i) \to 1$. Moreover, if $n \in [\![n_i, m_i - n_i - 1]\!]$, then $(D_i + n) \cap D_i = \varnothing$. Now,

$$m_i - n_i - 1 \geq (1-\kappa) m_i - 1 \geq \kappa m_{i+1} \geq n_{i+1};$$

so, for any $i_0 \in \mathbb{N}$, the set $\bigcup_{i \geq i_0} [\![n_i, m_i - n_i - 1]\!]$ has the form $[\![n_0, \infty)$ for some $n_0 \in \mathbb{Z}_+$. Hence, every large enough integer n is such that $D_i \cap (D_i + n) = \varnothing$ for some $i \geq i_0$. ☐

Question 6.4. Assuming only that the sequence (m_i) is unbounded, is it possible to find a sequence of measures (μ_i) such that $C_{\mathfrak{t}}$ is topologically mixing on $L_p(\Omega, \mu)$?

Let us now give another criterion for hypercyclicity or mixing, which is based on the behaviour of $\mathfrak{t}$ on several coordinates. For $i, n \geq 1$ we set

$$\theta_{i,n} := \sup\big\{ \mu_i(D) - \mu_i(D+n) : \ D \subset \Omega_i \big\}$$

where addition is understood in Ω_i, and

$$\widetilde{\gamma}_n := \sup \left\{ \frac{1}{\# I} \left(\sum_{i \in I} \theta_{i,n} \right)^2 \ : \ I \subset \mathbb{N} \text{ finite} \right\}.$$

Theorem 6.5. *If* $\limsup_{n \to \infty} \widetilde{\gamma}_n = \infty$, *then* $C_{\mathfrak{t}}$ *is hypercyclic on* $L_p(\Omega, \mu)$*; and if* $\widetilde{\gamma}_n \to \infty$, *then* $C_{\mathfrak{t}}$ *is topologically mixing.*

Proof. We only prove the hypercyclic case, the mixing case being completely similar. The proof follows that of Theorem 5.1, but it is easier since we do not have to worry about carries.

By Corollary 2.5, it is enough to show that given $\varepsilon > 0$, one can find a Borel set $B \subset \Omega$ and $n \in \mathbb{N}$ such that $\mu(B) \geq 1 - \varepsilon$ and $B \cap \mathfrak{t}^n(B) = \varnothing$.

Let $\varepsilon > 0$. By assumption, there exist $n, N \geq 1$ and $1 \leq i_1 < \cdots < i_N$ such that

$$\exp\left(-\frac{2}{9N} \left(\sum_{s=1}^N \theta_{i_s, n} \right)^2 \right) < \varepsilon.$$

For $s=1,\dots,N$, let $D_s\subset\Omega_{i_s}$ be such that $(D_s+n)\cap D_s=\varnothing$ and $\mu_{i_s}(D_s)-\mu_{i_s}(D_s+n)=\theta_{i_s,n}$. Define random variables $X_s,Y_s:(\Omega,\mathcal{B},\mu)\to\{0,1\}$ by

$$X_s:=\mathbf{1}_{\{x_{i_s}\in D_s\}}\qquad\text{and}\qquad Y_s:=\mathbf{1}_{\{x_{i_s}\in D_s+n\}}.$$

Now, consider

$$B_X:=\left\{\sum_{s=1}^N X_s\geq\sum_{s=1}^N\left(\mu_{i_s}(D_s)-\frac{\theta_{i_s,n}}{3}\right)\right\},$$
$$B_Y:=\left\{\sum_{s=1}^N Y_s\leq\sum_{k=1}^N\left(\mu_{i_s}(D_s+n)+\frac{\theta_{i_s,n}}{3}\right)\right\},$$

and

$$B=B_X\cap B_Y.$$

Observe that since $\mu_{i_s}(D_s)-\frac{\theta_{i_s,n}}{3}>\mu_{i_s}(D_s+n)+\frac{\theta_{i_s,n}}{3}$ (by the choice of D_s) and $Y_s(\mathfrak{t}^n x)=X_s(x)$ for all $x\in\Omega$, we have $\mathfrak{t}^n(B)\cap B=\varnothing$. Moreover,

$$\begin{aligned}\mu(B_X)&=1-\mu\left(\sum_{s=1}^N X_s<\sum_{s=1}^N\left(\mu_{i_s}(D_s)-\frac{\theta_{i_s,n}}{3}\right)\right)\\&=1-\mu\left(\sum_{s=1}^N\big(X_s-\mathbb{E}(X_s)\big)<-\frac{1}{3}\sum_{s=1}^N\theta_{i_s,n}\right).\end{aligned}$$

Since $X_1,\dots,X_s$ are independent Bernoulli variables, it follows (by Hoeffding's inequality) that

$$\mu(B_X)\geq1-\exp\left(-\frac{2}{9N}\left(\sum_{s=1}^N\theta_{i_s,n}\right)^2\right)\geq1-\varepsilon.$$

Similarly $\mu(B_Y)\geq1-\varepsilon$, and hence $\mu(B)\geq1-2\varepsilon$. ☐

We point out the following consequence of Theorem 6.5, to be compared with Theorem 5.1. Recall the notation

$$\theta_i=\sup\big\{\mu_i(D)-\mu_i(D+k):\ D\subset\Omega_i,\ k\in\mathbb{Z}\big\}.$$

Corollary 6.6. *Assume that the integers m_i are pairwise coprime, and that*

$$\sup\left\{\frac{1}{\#I}\left(\sum_{i\in I}\theta_i\right)^2\ :\ I\subset\mathbb{N}\ \text{finite}\right\}=\infty.$$

Then $C_\mathfrak{t}$ is hypercyclic on $L_p(\Omega,\mu)$, $1\leq p<\infty$.

Proof. It is enough to show that given $A>0$ and $n_0\in\mathbb{N}$, one can find $n\geq n_0$ such that $\widetilde{\gamma}_n>A$. By assumption, one can find a finite set $I\subset\mathbb{N}$ such that

$$\frac{1}{\#I}\left(\sum_{i\in I}\theta_i\right)^2>A.$$

For $i \in I$, choose a set $D_i \subset \Omega_i$ and an integer k_i such that $\mu_i(D_i) - \mu_i(D_i + k_i) = \theta_i$. By the Chinese Remainder Theorem, one can find an integer $n \geq n_0$ such that $n \equiv k_i\ [m_i]$ for all $i \in I$. Then $\theta_{i,n} = \theta_i$ for all $i \in I$; hence $\widetilde{\gamma}_n \geq \frac{1}{\#I}\left(\sum_{i\in I} \theta_i\right)^2 > A$. $\square$

Theorem 6.5 allows us to give examples of hypercyclic $C_{\mathfrak{t}}$ for which Proposition 6.2 cannot be applied.

Example 6.7. Let $m_i = 2^{l+1}$ if $l^2 \leq i < (l+1)^2$ for some $l \geq 1$. Then, one can find a sequence of measures (μ_i) such that $C_{\mathfrak{t}}$ is hypercyclic on $L_p(\Omega, \mu)$ and $\limsup \beta_i < 1$.

Proof. We write $m_i = 2n_i$, so $n_i = 2^l$ if $l^2 \leq i < (l+1)^2$ for some $l \geq 1$. Let $\delta_i := 1/n_i$ and $\rho_i := 1 + \delta_i$, let ε_i be such that

$$n_i\varepsilon_i + \frac{\rho_i^{n_i} - 1}{\rho_i - 1}\,\varepsilon_i = 1,$$

and define a probability measure μ_i on Ω_i as follows:

$$\begin{cases} \mu_i(k) := \varepsilon_i & \text{for } k = 0, \dots, n_i - 1, \\ \mu_i(n_i + k) := \rho_i^{n_i - 1 - k}\varepsilon_i & \text{for } k = 0, \dots, n_i - 1. \end{cases}$$

First, observe that

$$\sum_{i=1}^{\infty} \delta_i = \sum_{l=1}^{\infty} \sum_{l^2 \leq i < (l+1)^2} 2^{-l} < \infty,$$

so that $C_{\mathfrak{t}}$ is bounded on $L_p(\Omega, \mu)$.

Next, let $l \geq 1$ and consider $l^2 \leq i < (l+1)^2$. Taking $D := [\![2^l, 2^{l+1} - 1]\!] \subset \Omega_i$, we see that

$$\begin{aligned} \theta_{i,2^l} &\geq \left(\frac{\rho_i^{n_i} - 1}{\rho_i - 1} - n_i\right)\varepsilon_i \\ &\geq \left(\left(1 + 2^{-l}\right)^{2^l} - 2\right) 2^l \varepsilon_i. \end{aligned}$$

Since we also have

$$\left(\frac{\rho_i^{n_i} - 1}{\rho_i - 1} + n_i\right)\varepsilon_i = 1,$$

we get

$$\left(1 + 2^{-l}\right)^{2^l} 2^l \varepsilon_i = 1$$

which yields

$$\theta_{i,2^l} \geq \frac{\left(1 + 2^{-l}\right)^{2^l} - 2}{\left(1 + 2^{-l}\right)^{2^l}} \xrightarrow{l\to\infty} \frac{e - 2}{e}.$$

So, there exists $c > 0$ such that $\theta_{i,2^l} \geq c$ for any $l \geq 1$ and $l^2 \leq i < (l+1)^2$. Taking $n := 2^l$, $N := 2l + 1$ and $i_1 := l^2, \dots, i_N := l^2 + N - 1$ in the definition of $\widetilde{\gamma}_n$, it follows that $\widetilde{\gamma}_{2^l} \to \infty$ as $l \to \infty$. Hence, $C_{\mathfrak{t}}$ is hypercyclic by Theorem 6.5.

To see that Proposition 6.2 does not apply, observe that if $l^2 \leq i < (l+1)^2$ and if $D \subset \Omega_i = [\![0, 2^{l+1} - 1]\!]$ is such that $D \cap (D + n) = \varnothing$ for some $n \in \mathbb{Z}_+$, then $\#D \leq 2^l$, and that the set $D \subset [\![0, 2^{l+1}-1]\!]$ with cardinality 2^l and greatest μ_i-measure is $D = [\![2^l, 2^{l+1}-1]\!]$, for which

$$\mu_i([\![2^l, 2^{l+1}-1]\!]) = 1 - n_i\varepsilon_i = \frac{\left(1+2^{-l}\right)^{2^l}-1}{\left(1+2^{-l}\right)^{2^l}}\cdot$$

Hence, we have

$$\beta_i \leq \frac{\left(1+2^{-l}\right)^{2^l}-1}{\left(1+2^{-l}\right)^{2^l}} \qquad \text{if} \quad l^2 \leq i < (l+1)^2,$$

so that $\limsup_i \beta_i \leq (e-1)/e < 1$. □

6.3. **Frequent hypercyclicity.** The next theorem provides a simple sufficient condition for frequent hypercyclicity of $C_{\mathfrak{t}}$.

Theorem 6.8. *Let $(d_i)_{i\geq 1}$ be an increasing sequence of integers such that d_i is a multiple of $m_1, \dots, m_i$. Assume that there exists $\kappa > 0$ such that for any $\varepsilon > 0$, the following holds true: for all $i_0 \in \mathbb{N}$, there exist $i \geq i_0$, $n \in \mathbb{N}$ and $D \subset \Omega_i$ such that*

$$\forall 0 \leq k \leq \kappa d_i \ :\ \mu_i(D-k) \leq \varepsilon \quad \text{and} \quad \mu_i(D-(n+k)) \geq 1-\varepsilon.$$

Then $C_{\mathfrak{t}}$ is frequently hypercyclic.

Proof. We apply Corollary 3.8. The assumption on d_i implies that $d_i a \to 0$. For $i \geq 1$, let $B_i := [\Omega_1, \dots, \Omega_{i-1}, D_i]$, the cylinder defined by D_i. Then $B_i - d_i a = B_i$ because d_i is a multiple of m_i. So the assumptions of Corollary 3.8 are satisfied. □

Example 6.9. Assume m_{i+1} is a multiple of m_i for all $i \geq 1$.

(a) If $m_i \to \infty$, there exists a sequence of measures (μ_i) such that $C_{\mathfrak{t}}$ is frequently hypercyclic on $L_p(\Omega, \mu)$.

(b) If the sequence (m_{i+1}/m_i) is unbounded, there exists a sequence of measures (μ_i) such that $C_{\mathfrak{t}}$ is frequently hypercyclic on $L_p(\Omega, \mu)$ and *topologically rigid* along (m_i), *i.e.* $C_{\mathfrak{t}}^{m_i} f \to f$ for all $f \in L_p(\Omega, \mu)$. In particular, $C_{\mathfrak{t}}$ is not topologically mixing.

Proof. (a) Since $m_i \to \infty$, one can choose a sequence of positive numbers (δ_i) such that $\sum_{i=1}^{\infty} \delta_i < \infty$ and $\limsup \delta_i m_i = \infty$. We set $\rho_i := 1 + \delta_i$. For $i < 5$, we take as μ_i any probability measure on Ω_i. For $i \geq 5$, we define μ_i as follows. Let $n_i := \lfloor m_i/5 \rfloor$. We split Ω_i into three consecutive intervals, $\Omega_i = J_{i,1} \cup J_{i,2} \cup J_{i,3}$ with $\#J_{i,2} = \#J_{i,3} = n_i$. This implies that $3n_i \leq \#J_{i,1} \leq 3n_i + 4$. We now choose ε_i such that

$$(\#J_{i,1} + \#J_{i,3})\varepsilon_i + \frac{\rho_i^{n_i} - 1}{\rho_i - 1}\varepsilon_i = 1,$$

and we define μ_i by

$$\begin{cases} \mu_i(k) := \varepsilon_i & \text{if } k \in J_{i,1} \cup J_{i,3}, \\ \mu_i(\#J_{i,1} + k) := \rho_s^{n_i - k - 1}\varepsilon_s & \text{if } k = 0, \dots, n_i - 1. \end{cases}$$

As in the proof of Example 6.3, the convergence of the infinite product $\prod_{i\geq 1} \rho_i$ ensures that $C_{\mathfrak{t}}$ is bounded on $L_p(\Omega, \mu)$; and the measure μ is non-atomic because

$\eta_i \le \delta_i+\varepsilon_i \le \delta_i+1/2n_i$ for all $i \ge 5$ and hence $\eta_i \to 0$. Moreover, since $\limsup \delta_i n_i = \infty$, we have $\limsup \mu_i(J_{i,2}) = 1$, again as in the proof of Example 6.3.

Let $d_i := m_i$ and $D_i := J_{i,2} \cup J_{i,3}$. For $k = 0, \dots, n_i - 1$, we see that

$$D_i - k \supset J_{i,2} \qquad \text{and} \qquad D_i - (2n_i + k) \subset J_{i,1}.$$

Since $\limsup \mu_i(J_{i,2}) = 1$ and $n_i \ge m_i/6$, it follows that given $\varepsilon > 0$, one can find i arbitrarily large such

$$\forall 0 \le k \le d_i/6 \;:\; \mu_i(D_i - k) \ge 1 - \varepsilon \qquad \text{and} \qquad \mu_i\big(D_i - (2n_i + k)\big) \le \varepsilon.$$

Hence, $C_{\mathfrak{t}}$ is frequently hypercyclic by the "flipped" version of Theorem 6.8 (see Remark 3.3).

(b) Since the sequence (m_i/m_{i-1}) is unbounded, one can choose a sequence of positive numbers δ_i such that $\limsup \delta_i m_i = \infty$ and $\sum_{i=2}^{\infty} \delta_i m_{i-1} < \infty$. We take as (μ_i) the same sequence as in (a), so that $C_{\mathfrak{t}}$ is frequently hypercyclic on $L_p(\Omega, \mu)$. As for topological rigidity, note that since m_{i+1} is a multiple of m_i for all i, we already know that $C_{\mathfrak{t}}^{m_i} \mathbf{1}_B \to \mathbf{1}_B$ for every cylinder set $B \subset \Omega$. Hence, to show that $C_{\mathfrak{t}}$ is topologically rigid along (m_i), it is enough to show that $\sup_{i\ge 1} \|C_{\mathfrak{t}}^{m_i}\| < \infty$.

Let us first observe that

$$K := \sup_{i \ge 1} \prod_{j=i}^{\infty} \rho_j^{m_{i-1}} < \infty.$$

Indeed, since $\log(\rho_j) \le \delta_j$, we have for all $i \ge 2$:

$$\log\left(\prod_{j=i}^{\infty} \rho_j^{m_{i-1}}\right) \le m_{i-1} \sum_{j \ge i} \delta_j \le \sum_{j \ge 2} \delta_j m_{j-1}.$$

Now, let us show that $\|C_{\mathfrak{t}}^{m_{i-1}}\| \le K$ for all $i \ge 1$. It is enough to check that $\mu\big(\mathfrak{t}^{-m_{i-1}}(B)\big) \le K\mu(B)$ for every basic cylinder $B = [x_1, \dots, x_n] \subset \Omega$. Indeed, it then follows that $\mu(\mathfrak{t}^{-m_{i-1}}(B)) \le K\mu(B)$ for all Borel sets $B \subset \Omega$, which gives $\|C_{\mathfrak{t}}^{m_{i-1}}\| \le K$.

If $n \le i-1$, then $\mathfrak{t}^{-m_{i-1}}(B) = B$ and there is nothing to do. If $n \ge i$, then

$$\mathfrak{t}^{-m_{i-1}}(B) = [x_1, \dots, x_{i-1}, x_i - m_{i-1}, \dots, x_n - m_{i-1}],$$

so that

$$\begin{aligned}\mu\big(\mathfrak{t}^{-m_{i-1}}(B)\big) &\le \mu([x_1, \dots, x_n]) \times \prod_{j=i}^{\infty} \sup_{k \in \Omega_j} \frac{\mu_j(k - m_{i-1})}{\mu_j(k)} \\ &\le \prod_{j=i}^{\infty} \rho_j^{m_{i-1}} \times \mu(B) \le K\mu(B).\end{aligned}$$

This concludes the proof. □

Remark 6.10. In Example 6.9 (b), the translation $\mathfrak{t}$ is conservative, and hence $C_{\mathfrak{t}}$ cannot satisfy the Frequent Hypercyclicity Criterion if $p \ge 2$ by [16]. Indeed, the proof has shown that $\mu(B) \le K\mu(\mathfrak{t}^{m_{i-1}}(B))$ for all $i \ge 1$ and any Borel set $B \subset \Omega$. In particular, if $\mu(B) > 0$ then the series $\sum \mu(\mathfrak{t}^n(B))$ is divergent. Since μ is a probability measure, it follows that for any such B, one can find $n < n'$ such that $\mathfrak{t}^n(B) \cap \mathfrak{t}^{n'}(B) \neq \varnothing$; which implies that $\mathfrak{t}$ is conservative.

Example 6.9 (b) calls for some comments. It follows from the main results of [10] that if (m_i) is a sequence of integers such that m_{i+1} is a multiple of m_i for all $i \geq 1$ and the sequence (m_{i+1}/m_i) is unbounded, then there exists a Hilbert space operator T which is weakly mixing with respect to some nondegenerate invariant Gaussian measure (hence frequently hypercyclic) and uniformly rigid along (m_i), *i.e.* $\|T^{m_i} - \mathrm{Id}\| \to 0$. The operator T in [10] is specifically constructed in order to satisfy these requirements, and the construction is quite nontrivial. We find it interesting that the very simply defined operator $C_{\mathfrak{t}}$ from Example 6.9 (b) happens to have similar properties. Incidentally, it is plausible that in fact, $C_{\mathfrak{t}}$ is weakly mixing with respect to some nondegenerate invariant Gaussian measure. Since (for Hilbert spaces operators at least) this property is equivalent to having a perfectly spanning set of unimodular eigenvectors, this leads to the following question, that could be asked for odometers as well.

Question 6.11. What are the eigenvalues and the eigenvectors of $C_{\mathfrak{t}}$?

Concerning this question, the two extreme situations are *a priori* possible: it may be that all eigenvalues of $C_{\mathfrak{t}}$ are roots of unity (regardless of the measure μ), or that $C_{\mathfrak{t}}$ has a perfectly spanning set of unimodular eigenvectors as soon as it is hypercyclic. We are not ready to bet on any of the two alternatives. However, one can observe the following: if λ is an eigenvalue of $C_{\mathfrak{t}}$ and has an associated eigenvector which happens to be (μ-almost everywhere equal to) a continuous function, then $\lambda = \gamma(a)$ for some continuous character γ of $(\Omega, +)$; and hence λ has to be a root of unity due to the form of Ω. Indeed, if $f(x+a) = f(x)$ μ-almost everywhere for some non-zero continuous function $f : \Omega \to \mathbb{C}$, then in fact $f(x+a) = f(x)$ everywhere because the measure μ has full support (recall that $\mu_i(j) > 0$ for all i and every $j \in \Omega_i$). It follows that $\gamma(a)\widehat{f}(\gamma) = \lambda \widehat{f}(\gamma)$ for every character γ, which gives the result. But this seems far from telling the whole story.

6.4. **$\mathcal{U}$-frequent hypercyclicity.** In our context, Corollary 4.5 implies the following easy sufficient condition.

Theorem 6.12. *Assume as usual that $C_{\mathfrak{t}}$ is bounded on $L_p(\Omega, \mu)$.*

(1) *Suppose that there exists $\alpha > 0$ such that the following holds true: for any $\varepsilon > 0$, there exists an arbitrarily large $m \geq 1$, $i \geq 1$ and $D \subset \Omega_i$ such that $\mu_i(D) < \varepsilon$ and $\#\big\{k \in [\![1, m]\!] : \ \mu_i(D-k) > 1 - \varepsilon\big\} \geq \alpha m$. Then $C_{\mathfrak{t}}$ is $\mathcal{U}$-frequently hypercyclic.*
(2) *Suppose that for any $A \subset \mathbb{N}$ with $\overline{\mathrm{dens}}(A) > 0$, there exists $\alpha > 0$ such that: for any $\varepsilon > 0$, there exists an arbitrarily large $m \geq 1$, $i \geq 1$ and $D \subset \Omega_i$ such that $\mu_i(D) < \varepsilon$ and $\#\big\{k \in [\![1, m]\!] : \ k \in A$ and $\mu_i(D-k) > 1 - \varepsilon\big\} \geq \alpha m$. Then $C_{\mathfrak{t}}$ is hereditarily $\mathcal{U}$-frequently hypercyclic.*

Proof. Both proofs follow the same argument as the proof of Theorem 6.8. □

Example 6.13. Assume that the sequence (m_i) is unbounded. Then there exists a sequence of measures (μ_i) such that $C_{\mathfrak{t}}$ is $\mathcal{U}$-frequently hypercyclic on $L_p(\Omega, \mu)$.

Proof. We follow the proof of Example 6.3 but we now choose $n_{i_s} := \lfloor m_{i,s}/3 \rfloor$. Then $(D_{i_s} - k) \cap D_{i_s} = \varnothing$ for $n_{i_s} \leq k \leq 2n_{i_s}$ so that we can apply the "flipped" version of Theorem 6.12 with $m := n_{i_s}$ and $\alpha := 1/2$; see Remark 4.4. □

Theorem 6.12 also allows us to give new examples of hereditarily $\mathcal{U}$-frequently hypercyclic operators. We first need an elementary lemma.

Lemma 6.14. *Let $A \subset \mathbb{N}$ have positive upper density. There exists $\delta > 0$ such that, for any $N \geq 1$, there exists $n \geq N$ such that*
$$\frac{\#\left([\![2^n, 2^{n+1}[\![\cap A\right)}{2^n} \geq \delta.$$

Proof. Assume that this is not the case. Let $\delta > 0$. By assumption, there exists $N \geq 1$ such that
$$\forall n \geq N \;:\; \frac{\#\left([\![2^n, 2^{n+1}[\![\cap A\right)}{2^n} < \delta.$$
Let $m \geq 2^N$ and let $n \geq N$ be such that $2^n \leq m < 2^{n+1}$. Then
$$\begin{aligned}\frac{\#([\![1, m]\!] \cap A)}{m} &\leq \frac{\#([\![1, 2^{n+1}[\![\cap A)}{2^n}\\ &= \frac{\#([\![1, 2^N - 1]\!] \cap A)}{2^n} + \sum_{j=N}^{n} \frac{\#([\![2^j, 2^{j+1}[\![\cap A)}{2^j} \times \frac{1}{2^{n-j}}\\ &\leq \frac{2^{N+1}}{m} + \sum_{l=0}^{\infty} \frac{\delta}{2^l}.\end{aligned}$$
Since $m \geq 2^N$ is arbitrary, it follows that $\overline{\mathrm{dens}}(A) \leq 2\delta$ for any $\delta > 0$, a contradiction. □

Example 6.15. Let $n_i := 2^i$ and set $m_i := 3n_i$. Write $\Omega_i = [\![0, m_i - 1]\!]$ as $J_{1,i} \cup J_{2,i}$ where $J_{1,i}$ and $J_{2,i}$ are consecutive intervals with respective length $2n_i$ and n_i. As in Example 6.3, it is possible to define μ_i on Ω_i such that $C_{\mathfrak{t}}$ is bounded on $L_p(\Omega, \mu)$ and $\mu_i(J_{2,i}) \to 1$. Then, $C_{\mathfrak{t}}$ is hereditarily $\mathcal{U}$-frequently hypercyclic.

Proof. Let $A \subset \mathbb{N}$ have positive upper density, and choose $\delta > 0$ according to the above lemma, *i.e.* such that for arbitrarily large $n \in \mathbb{N}$, we have
$$\frac{\#([\![2^n, 2^{n+1}[\![\cap A)}{2^n} \geq \delta. \tag{6.2}$$
We show that the assumption of the "flipped" version of Theorem 6.12 is satisfied with $\alpha := \delta/2$; see Remark 4.4.

Let $\varepsilon > 0$, and consider n arbitrarily large so that (6.2) is satisfied. Set also $m := 2^{n+1}$, $i := n$ and $D := J_{2,i}$. Then $J_{2,i} - k \subset J_{1,i}$ for any $k \in [\![2^n, 2^{n+1}]\!]$, so that $\mu_i(D) \geq 1 - \varepsilon$ and $\mu_i(D - k) \leq \varepsilon$ provided n is large enough. Therefore,
$$\#\big\{k \in [\![1, m]\!] :\; k \in A \text{ and } \mu_i(D - k) \leq \varepsilon\big\} \geq \#([\![2^n, 2^{n+1}[\![\cap A) \geq \delta 2^n \geq \alpha m.$$
□

6.5. **A semigroup variant.** We can give a semigroup variant of all what we have done, in the following way. Assume now that
$$\Omega = \prod_{i=1}^{\infty} \mathbb{R}/m_i\mathbb{Z} =: \prod_{i=1}^{\infty} \Omega_i,$$

and let $\nu = \prod_{i=1}^{\infty} \nu_i$ be a product probability measure on Ω, where each ν_i is equivalent to $\mathcal{L}_i$, the normalized Lebesgue measure on $\Omega_i = \mathbb{R}/m_i\mathbb{Z}$. Define $T_t f(x) = f(x+ta)$ for $f \in L_p(\Omega, \mu)$ and $t > 0$. Then provided

$$\prod_{i=1}^{\infty} \sup_{A\subset\Omega_i,\ t\in[0,1]} \frac{\nu_i(A-t)}{\nu_i(A)} < \infty,$$

the semigroup (T_t) is (well-defined and) strongly continuous on $L_p(\Omega, \mu)$.

All of our previous statements admit semigroup versions. In the other direction, it is easy to adapt the examples given in this section so that they fit into this semigroup context. Starting from probability measures μ_i on $\mathbb{Z}/m_i\mathbb{Z}$, we simply define for any Borel set $A \subset [0, m_i)$,

$$\nu_i(A) = \sum_{j=0}^{m_i-1} \mu_i(j)\, \mathcal{L}_i\big(A \cap [j, j+1)\big).$$

In particular, if we start from Example 6.9, we get an example of a chaotic and frequently hypercyclic strongly continuous semigroup which fails to be topologically mixing. Previous examples have been obtained in [3], with a more difficult construction based on a renorming of some weighted ℓ_2 - space.

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