A useful Lemma concerning subseries convergence

The purpose of this note is to show that several classical results in Functional Analysis can be deduced very easily from a simple lemma concerning subseries convergence.

Definition. A series $\sum g_n$ in an abelian topological group G is said to be *subseries convergent* if all series $\sum \alpha_n g_n$, $\alpha \in \{0,1\}^{\mathbb{N}}$, converge in G.

Main Lemma. *Let G be an abelian topological group. Let (A_n), (B_n) be two sequences of Borel subsets of G, and let (g_n) be a sequence in G. Assume that $A_n \cup (B_n + g_n) = G$ for each n, and that $\sum g_n$ is subseries convergent. Then one can find a subsequence (h_n) of (g_n) such that $x = \sum_0^\infty h_n$ belongs to infinitely many A_n's or to infinitely many B_n's.*

Proof.

It is easy to check that the convergence of the series $\sum \alpha_n g_n$ is uniform with respect to $\alpha \in \{0,1\}^{\mathbb{N}}$. Therefore, identifying $\{0,1\}^{\mathbb{N}}$ with the compact abelian group $\Delta = (\mathbb{Z}/2\mathbb{Z})^{\mathbb{N}}$, the formula $\varphi(\alpha) = \sum_0^\infty \alpha_n g_n$ defines a continuous map $\varphi : \Delta \to G$.

For each $n \in \mathbb{N}$, put $A'_n = \varphi^{-1}(A_n)$, $B'_n = \varphi^{-1}(B_n)$. The sets A'_n, B'_n are Borel subsets of Δ, and we have to show that $\limsup_n (A'_n \cup B'_n) = \bigcap_N \bigcup_{n>N} (A'_n \cup B'_n)$ is nonempty. To this end, it is enough to prove that $m(A'_n \cup B'_n) \geq 1/4$ for all n, where m is the normalized Haar measure on Δ.

Fix $n \in \mathbb{N}$ and put $W_0^{(n)} = \{\alpha \in \Delta : \ \alpha_n = 0\}$, $W_1^{(n)} = \{\alpha \in \Delta : \ \alpha_n = 1\}$. Observe that

$$\varphi^{-1}(B_n + g_n) \cap W_1^{(n)} = \delta^{(n)} + B'_n \cap W_0^{(n)},$$

where $\delta^{(n)} \in \Delta$ is defined by $\delta_n^{(n)} = 1$ and $\delta_i^{(n)} = 0$ if $i \neq n$.

Since $\Delta = A'_n \cup \varphi^{-1}(B_n + g_n)$, this implies that $W_1^{(n)} \subseteq A'_n \cup (\delta^{(n)} + B'_n)$, whence $m(A'_n \cup B'_n) \geq \frac{1}{2}\big(m(A'_n) + m(B'_n)\big) \geq m(W_1^{(n)})/2 = 1/4$.

We now turn to some applications of the Main Lemma. These include the Uniform Boundedness Theorem, a simple result on automatic continuity, Schur's l^1-theorem, the Nikodym boundedness theorem, the Vitali–Hahn–Sacks theorem, and the Orlicz–Pettis theorem. Of course, all these results are quite well known and very classical. Yet, we believe that the proofs given in this note may have some interest, mainly because they are almost identical. The Main Lemma is applied each time in exactly the same way, with sets $A_n = B_n$ arising immediately from the triangle inequality.

Uniform Boundedness Principle. *Let X be a Banach space. If Y is a normed space and if $\mathcal{F}$ is a pointwise bounded family of continuous linear operators from X into Y, then $\mathcal{F}$ is norm-bounded.*

Proof.

Assume that $\sup_{T\in\mathcal{F}} ||T|| = +\infty$. Then one can find two sequences $(x_n) \subseteq X$, $(T_n) \subseteq X$, such that $||x_n|| < 2^{-n}$ and $||T_n(x_n)|| > n$ for all n. For each $n \in \mathbb{N}$, put

$$A_n = \{x \in X;\ ||T_n(x)|| > n/2\}.$$

Each A_n is an open subset of X, and by the triangle inequality, one can write

$$X = \{x;\ ||T_n(x)|| > n/2\} \cup \{x;\ ||T_n(x - x_n)|| > n/2\} = A_n \cup (A_n + x_n)$$

for all $n \in \mathbb{N}$. Since the absolutely convergent series $\sum x_n$ is subseries convergent in the Banach space X, the Main Lemma allows us to find a point $x \in X$ such that $||T_n(x)|| > n/2$ for infinitely many n's. This shows that $\mathcal{F}$ is not pointwise bounded.

Automatic continuity. *Let X be Banach space and let Y be a normed space. If $T : X \to Y$ is a Borel linear mapping, then T is continuous*

Proof.

If T is not continuous, then one can find a sequence $(x_n) \subseteq X$ such that $||x_n|| < 2^{-n}$ and $||T(x_n)|| > n$ $(n \in \mathbb{N})$. Put $A_n = \{x \in X;\ ||T(x)|| > n/2\}$. The sets A_n are Borel subsets of X, and $X = A_n \cup (A_n + x_n)$ for all n. By the Main Lemma, we get a point $x \in X$ such that $||T(x)|| > n/2$ for infinitely many n's, which is a contradiction.

Schur's l_1-Theorem. *In the space l_1, weakly convergent and norm-convergent sequences are the same.*

Proof.

Let (x_n) be a weakly null sequence in l_1, and assume that $||x_n|| \geq \varepsilon$ for all n and for some $\varepsilon > 0$. Using the fact that (x_n) is weakly null, one can find a subsequence (y_n) of (x_n) and a normalized sequence (y_n^*) in l_∞ such that the y_n^* have finite, pairwise disjoint supports and $\langle y_n^*, y_n\rangle > \varepsilon/2$ for all n. Put $A_n = \{y^* \in l_\infty;\ |\langle y^*, y_n\rangle| > \varepsilon/4\}$, $n \in \mathbb{N}$. The sets A_n are w^*-open, and $X^* = A_n \cup (A_n + y_n^*)$ for all n. Moreover, since the y_n^* have norm 1 and are disjointely supported, the series $\sum y_n^*$ is subseries convergent in $G = (l_\infty, w^*)$. By the Main Lemma, one can find $y^* \in l_\infty$ such that $|\langle y^*, y_n\rangle| > \varepsilon/2$ for infinitely many n's. This is impossible because (y_n) is weakly null.

Nikodym Boundedness Theorem. *Let $(X, \mathcal{T})$ be a measurable space. If $\mathcal{M}$ is a family of countably additive set functions defined on $\mathcal{T}$ such that $\sup_{\mu\in\mathcal{M}} |\mu(E)| < +\infty$ for all $E \in \mathcal{T}$, then* $\sup\{|\mu(E)|;\ E \in \mathcal{T}, \mu \in \mathcal{M}\} < +\infty$.

Proof.

Assume that $M_E = \sup_{\mathcal{M}} |\mu(E)| < +\infty$ for all $E \in \mathcal{T}$, and that $\sup_{\mathcal{T}} M_E = +\infty$. Choose $E_1 \in \mathcal{T}$ and $\mu_1 \in \mathcal{M}$ such that $|\mu_1(E_1)| > 1 + M_X$; then $|\mu_1(E_1)| > 1$, and $|\mu_1(X \setminus E_1)| \geq |\mu_1(E_1)| - |\mu_1(X)| > 1$. Moreover, at least one of $\sup\{M_F;\ F \subseteq E_1\}$, $\sup\{M_F;\ F \subseteq X \setminus E_1\}$ is infinite; say $\sup_{F \subseteq X \setminus E_1} M_F = +\infty$. Repeating this argument, one can construct a sequence $(\mu_n) \subseteq \mathcal{M}$ and a sequence (E_n) of pairwise disjoint sets in $\mathcal{T}$ such that $|\mu_n(E_n)| > n$ for all n.

Denote by $ca(\mathcal{T})$ the family of all countably additive set-functions defined on $\mathcal{T}$, and let G be the group of all bounded, scalar-valued, $\mathcal{T}$-measurable functions on X, endowed with the topology of pointwise convergence on $ca(\mathcal{T})$. Since the E_n's are pairwise disjoint, the series $\sum \mathbf{1}_{E_n}$ is subseries convergent in G; moreover, the limit of any subseries of $\sum \mathbf{1}_{E_n}$ is the characteristic function of some set $E \in \mathcal{T}$. Now, put $A_n = \{f \in G;\ |\int f d\mu_n| > n/2\}$. Each set A_n is open in G, and $G = A_n \cup (A_n + \mathbf{1}_{E_n})$ for all n. Applying the Main Lemma, we get that $\mathcal{M}$ is not pointwise bounded.

Vitali–Hahn–Sacks Theorem. *Let $(X, \mathcal{T}, \mu)$ be a measure space ($\mu \geq 0$), and let (μ_n) be a sequence of countably additive measures defined on $\mathcal{T}$. Assume that each μ_n is μ-continuous, and that $\lim_n \mu_n(E)$ exists for all sets $E \in \mathcal{T}$. Then the sequence (μ_n) is uniformly μ-continuous.*

Proof.

If (μ_n) is not uniformly μ-continuous, then one can find a subsequence (ν_n) of (μ_n) and a sequence $(E_n) \subseteq \mathcal{T}$ such that $\mu(E_n) \to 0$ and $|\nu_n(E_n)| > \varepsilon > 0$ for all n. Using the μ-continuity of the ν_n's and extracting further subsequences if necessary, we may assume that $|\nu_n(E)| < \varepsilon/3$ for all n and for every set $E \in \mathcal{T}$ contained in $\cup_{k>n} E_k$. Thus, putting $F_n = E_n \setminus \big(\cup_{k>n} E_k\big)$, we get a sequence of pairwise disjoint sets in $\mathcal{T}$ such that $|\nu_n(F_n)| > 2\varepsilon/3$ and $|\nu_n(F_{n+1})| < \varepsilon/3$, $n \in \mathbb{N}$; in particular, $|\nu_{n+1}(F_{n+1}) - \nu_n(F_{n+1})| > \varepsilon/3$ for all n. The proof now proceeds exactly in the same way as for the Nikodym Boundedness Theorem: applying the Main Lemma, we get a set $F \in \mathcal{T}$ such that $|\nu_{n+1}(F) - \nu_n(F)| > \varepsilon/6$ for infinitely many n's, which contradicts the convergence of the sequence $(\nu_n(F))$.

Orlicz–Pettis Theorem. *If X is a normed space, then a series $\sum x_n$ in X is subseries convergent for the weak topology if and only if it is subseries convergent for the norm topology.*

Proof.

It is enough to prove that if $\sum x_n$ is subseries convergent in X for the weak topology, then $||x_n|| \to 0$. Indeed, when applied to series of the form $\sum \Big(\sum_{k \in F_n} \alpha_k x_k\Big)$, where (F_k) is a sequence of pairwise disjoint intervals of $\mathbb{N}$, this result yields that for each $\alpha \in \{0;1\}^{\mathbb{N}}$, the partial sums of $\sum \alpha_n x_n$ form a norm-Cauchy sequence, and since they converge weakly, it follows that $\sum x_n$ is norm subseries convergent. Moreover, it follows from Mazur's theorem that if $\sum x_n$ is weakly subseries convergent in X, then it is weakly subseries convergent in the norm-closed linear span of the x_n's, hence it is enough to consider the case of a *separable* normed space. So, assume that X is separable, that $\sum x_n$ is subseries convergent for the weak topology, and that $||x_n|| > \varepsilon > 0$ for all n.

First, we observe that for any point $a \in X$, the set $\{n;\ ||x_n - a|| \leq \varepsilon/2\}$ is finite. Indeed, it is even empty if $||a|| \leq \varepsilon/2$ (by the triangle inequality), and if $||a|| > \varepsilon/2$, the result follows because the set $\{x;\ ||x|| > \varepsilon/2\}$ is weakly open and $a - x_n \to a$ weakly.

Now, let (F_n) be an increasing sequence of finite subsets of X such that $\bigcup_n F_n$ is norm-dense in X. Using the preceding remark, one can construct by induction a subsequence (z_n) of (x_n) such that $dist(z_n, F_n - F_n) > \varepsilon/2$ for all n. The sets $A_n = \{x;\ dist(x, F_n) > \varepsilon/4\}$ are weakly Borel (actually weakly open), and $X = A_n \cup (z_n + A_n)$ for all n. By the Main Lemma, one can find a point $x \in X$ such that $dist(x, F_n) > \varepsilon/4$ for infinitely many n, hence for all n because (F_n) is increasing. Since $\bigcup_n F_n$ is dense in X, this is the required contradiction.

Remark. Obvious modifications of the above proof give the following Orlicz-Pettis type result due to N. Kalton: *Let G be an abelian group endowed with two (Hausdorff) group topologies $\tau_1 \subseteq \tau_2$. Assume that τ_2 is separable and admits a neighbourhood basis consisting of τ_1-closed sets. Then subseries convergence in (G, τ_1) is equivalent to subseries convergence in (G, τ_2).*