INFINITE GAMES, BANACH SPACE GEOMETRY
AND THE EIKONAL EQUATION

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Abstract

We study a class of infinite games which turns out to be closely related to Banach space geometry. Using one of these games, we construct a bounded, differentiable, almost everywhere solution of the Eikonal equation \( \|\nabla u\| = 1 \) on \( \mathbb{R}^d \), with \( d \geq 2 \).

1. Introduction

It is well known that derivative functions have many interesting properties. The following beautiful result goes back to A. Denjoy: if \( u : \mathbb{R} \to \mathbb{R} \) is everywhere differentiable, then, for any open set \( U \subset \mathbb{R} \), the set \( \{ x : u'(x) \in U \} \) is either empty or has positive Lebesgue measure; this property of derivatives is usually called the Denjoy–Clarkson property. The problem of extending Denjoy’s result to functions of several variables was raised in the 1960s by C. E. Weil [11], and since then it has been known as the Weil gradient problem. This problem was solved in 2002 by Z. Buczolich [3], who constructed an everywhere differentiable function \( u : \mathbb{R}^2 \to \mathbb{R} \) such that \( \nabla u(0) = 0 \) but \( \|\nabla u(x)\| \geq 1 \) for almost all points \( x \in \mathbb{R}^2 \); to see that \( \nabla u \) fails the Denjoy–Clarkson property, just consider the open unit ball \( U \subset \mathbb{R}^2 \).

The function \( u \) above is the limit of a sequence of smooth functions \( (s_n) \), and in order to ensure the differentiability of \( u \), one needs to ‘force’ the sequence \( (\nabla s_n) \) to converge at each point. Buczolich’s original arguments are very intricate, and they have been greatly simplified by J. Malý and M. Zelený [9]. The key new tool introduced in [9] is the following very interesting infinite game. There are two players, I and II. Player I starts the game by playing a point \( a_0 \) in the open unit ball of \( \mathbb{R}^2 \). Player II answers by playing a line \( L_0 \subset \mathbb{R}^2 \), which must pass through the point \( a_0 \). Then player I plays a point \( a_1 \) in the open unit ball, which must belong to the line \( L_0 \). Player II answers with a line \( L_1 \) passing through \( a_1 \), and so on. Player II wins if the infinite sequence \( (a_n) \) produced by the game is convergent in \( \mathbb{R}^2 \). This game is called the point–line game in [9]. One of the main results of [9] is that player II has a winning strategy in the point–line game, which is exactly what is needed to simplify Buczolich’s proof.

In the present paper, we study some more general versions of the point–line game. Instead of \( \mathbb{R}^2 \), one can consider a Banach space \( X \). In this setting, the obvious analogue of the point–line game should now be called the point–hyperplane game for the open unit ball \( B_X \). Rather unexpectedly, it turns out that this is not merely a formal generalization, and that the point–hyperplane game is in fact closely related to the geometry of the underlying Banach space \( X \). Indeed, we show that player II has a winning strategy in the point–hyperplane game for \( B_X \) if, and only if, \( X \) has the Radon–Nikodym property (RNP).

At first sight, the appearance of the Radon–Nikodym property may look a bit surprising, but it becomes quite natural if one decides to change the rule of the game by requiring player II to play slices of the unit ball rather than hyperplane sections. More explicitly, the new rule is the following: player I starts the game by playing a point \( a_0 \in B_X \), player II answers with a closed
half-space $S_0$ containing $a_0$ as a boundary point, player I then plays a point $a_1 \in B_X \cap S_0$, and so on. Since all the half-spaces played by II determine slices of the open unit ball, we call this modified game the point–slice game for $B_X$. Formally, the point–slice game is more difficult to win for player II than the point–hyperplane game, since it gives more freedom to player I. Yet, we show that if $X$ has the RNP, then II has a winning strategy in that game as well.

If the Banach space $X$ is very well behaved, one can get a stronger conclusion. Indeed, we show that if $X$ has a uniformly convex renorming, then player II even has a winning tactic in the point–slice game, that is, a winning strategy in which at each step, the answer to the points $a_0, \ldots, a_n$ already played by I depends only on the last move $a_n$. In other words, one can associate to each point $a \in B_X$ a closed half-space $S(a)$ containing $a$ as a boundary point in such a way that, whatever the moves of player I in the point–slice game may be, player II is sure to win if she answers $S(a_n)$ to each move $a_n$ of player I. Put in a slightly different way, this means that one can associate to each point $a \in B_X$ a linear functional $\Phi_a \in X^*$ in such a way that the following property holds true: each sequence $(a_n) \subset B_X$ satisfying $\langle \Phi_{a_n}, a_{n+1} \rangle \geq \langle \Phi_{a_n}, a_n \rangle$ for all $n \in \mathbb{N}$ is convergent. This can be viewed as a Banach space version of the fact that each bounded monotonic sequence of real numbers is convergent.

One may also consider other games of the same type, where player II is required to play members of some fixed family $A$ of affine subspaces of $X$. When $A$ is the family of all finite-codimensional affine subspaces of $X$, this leads to another well-known Banach space property, namely the point of continuity property (PCP). We show that player II has a winning strategy in the point–finite-codimensional subspace game for $B_X$ if and only the Banach space $X$ has the PCP.

Note that even in $\mathbb{R}^2$, the existence of a winning strategy for player II in the point–line game or the point–slice game for the unit ball is a non-trivial result. At first sight, one might think that player II should win by answering to each play $a_n \neq 0$ of player I the line $L(a_n)$ passing through $a_n$ and orthogonal to $\mathbb{R}a_n$. This is indeed natural because among all line sections of the unit ball passing through $a_n$, the one with smallest diameter is precisely $L(a_n)$. However, this ‘orthogonal strategy’ does not work. Indeed, let us define a sequence $(a_n)$ as follows: in polar coordinates, $a_n$ is given by $(r_n, \theta_n)$, where

$$r_n = \prod_{k=n+1}^{\infty} \cos\left(\frac{1}{k}\right) \quad \text{and} \quad \theta_n = \sum_{k=1}^{n} \frac{1}{k},$$

with the convention $\theta_0 = 0$. Since the line $L(a_n)$ is given in polar coordinates by

$$r = \frac{r_n}{\cos(\theta - \theta_n)},$$

we see that $a_{n+1} \in L(a_n)$ for all $n$. Thus, if player II follows the orthogonal strategy, then player I is allowed to play $a_0, a_1, \ldots$. But since $r_n$ tends to 1 and the sequence $(\theta_n)$ goes slowly to $+\infty$, the sequence $(a_n)$ is not convergent. Thus, player I has won the game.

Incidentally, this example shows that player II can lose the point–slice game even if she plays slices of the unit ball whose diameters tend to 0. Of course, II can sometimes win if the diameters of the slices do not tend to 0, for example if I decides to lose by always playing the same point. Notice also that a strategy for player II which would always produce a non-increasing sequence of slices cannot be winning for II. Indeed, assume that player II follows such a strategy $S$. Then I can force II to play always the same slice, simply by choosing at each step the point $a_{n+1}$ on the boundary of the half-space $S_n$ just played by II. Thus, choosing any point $a_0$ and then a point $a_1 \neq a_0$ on the boundary of $S(a_0)$, player I wins the game if she plays alternatively the two points $a_0, a_1$.

The paper is organized as follows. The first two parts deal with games. We first consider very abstract ‘point–set’ games and prove two general results concerning the existence of winning strategies or tactics for player II.
Then we apply these results in the Banach space setting. As explained above, this leads to characterizations of the Radon–Nikodym property and the point of continuity property, and to the existence of a winning tactic for player II in the point–slice game if the underlying Banach space X has a uniformly convex renorming.

In the final part, we elaborate a bit on Buczolich’s example and we prove the following slightly stronger result: if \( d \geq 2 \), then there exists a bounded differentiable function \( u : \mathbb{R}^d \to \mathbb{R} \) such that \( \nabla u(0) = 0 \) and \( u \) satisfies the Eikonal equation \( \| \nabla u(x) \| = 1 \) almost everywhere; here, \( \| \cdot \| \) is an arbitrary norm on \( \mathbb{R}^d \). More generally, we prove that if \( \Omega \) is an open subset of \( \mathbb{R}^d \), with \( d \geq 2 \), and if \( x_0 \in \Omega \) is given, then there exists a 1-Lipschitz function \( u : \overline{\Omega} \to \mathbb{R} \), which is bounded and differentiable at every point of \( \Omega \), such that \( \nabla u(x_0) = 0 \) and \( \| \nabla u(x) \| = 1 \) almost everywhere in \( \Omega \); moreover, \( u \) satisfies the boundary condition \( u_{|\partial \Omega} = 0 \). Thus, the Eikonal equation admits some rather exotic almost everywhere solutions, very different from the usual viscosity solution \( \text{dist}(\cdot, \partial \Omega) \), which is not everywhere differentiable.

2. Abstract games

Let \( (E,d) \) be a (non-empty) metric space, and for each point \( x \in E \), let \( \mathcal{A}(x) \) be a (non-empty) family of subsets of \( E \) containing \( x \). We denote by \( \mathcal{A} \) the disjoint union of all families \( \mathcal{A}(x) \), and we define a game \( G(E,\mathcal{A}) \) as follows. There are two players, I and II. Player I plays points \( a_0, a_1, \ldots , \in E \), and player II plays sets \( A_0, A_1, \ldots \in E \). Once player I has played a point \( a_n \), player II must choose a set \( A_n \in \mathcal{A}(a_n) \). Then, player I must choose the point \( a_{n+1} \) inside \( A_n \). Player II wins if the sequence \( (a_n) \) is \( d \)-Cauchy, otherwise player I wins.

**THEOREM 2.1.** Assume there exists a family \( \mathcal{C} \) of subsets of \( E \) such that the following properties hold:

1. \( \emptyset, E \in \mathcal{C} \);
2. \( \mathcal{C} \) is closed under intersections;
3. if \( C_1, C_2 \in \mathcal{C} \) satisfy \( C_1 \subset C_2 \) and \( C_2 \setminus C_1 \neq \emptyset \) then, for each \( \varepsilon > 0 \), one can find \( C \in \mathcal{C} \) such that \( C_1 \subset C \subset C_2 \), \( C_2 \setminus C \neq \emptyset \) and \( \text{diam}(C_2 \setminus C) < \varepsilon \);
4. for each \( C \in \mathcal{C} \) and each point \( x \in E \setminus C \), there exists \( A \in \mathcal{A}(x) \) such that \( A \cap C = \emptyset \).

Then player II has a winning strategy in the game \( G(E,\mathcal{A}) \).

The proof is based on the following lemma, which follows easily from (1) and (2). If \( I \) is a set, we denote by \( I^{<\omega} \) the set of all finite sequences of elements of \( I \). If \( s \in I^{<\omega} \) and \( i \in I \), the sequence ‘s followed by \( i \)’ is denoted by \( s \cdot i \). If \( I \) is a well-ordered set and \( s \in I^{<\omega} \) has the form \( t \cdot i \), where \( i \in I \) has a predecessor \( i^- \) in \( I \), we denote by \( s^- \) the sequence \( t \cdot i^- \), and we say that \( s^- \) is the predecessor of \( s \). This is indeed the predecessor of \( s \) in the ordering of \( I^{<\omega} \) defined by putting the lexicographic ordering on each set \( I^n \), with \( n \in \mathbb{N} \) and declaring that two sequences are comparable only if they have the same length. Finally, we denote by \( |s| \) the length of a sequence \( s \in I^{<\omega} \).

**LEMMA 2.2.** There exist some ordinal \( \eta \) and a family \( \{C_s\}_{s \in \eta^{<\omega}} \) of subsets of \( E \) such that:

1. \( C_s \in \mathcal{C} \) for each \( s \in \eta^{<\omega} \), with \( C_\emptyset = \emptyset \);
2. \( C_s \cdot 0 = E \) for each \( s \in \eta^{<\omega} \);
3. for each \( s \in \eta^{<\omega} \), the transfinite sequence \( \{C_s \cdot \xi\}_{\xi \in \eta} \) is non-increasing, with \( \bigcap_{\xi \in \lambda} C_{s \cdot \xi} = C_{s \cdot \lambda} \) for limit ordinals \( \lambda \);
4. \( \bigcap_{\xi < \eta} C_{s \cdot \xi} = C_s \) for each \( s \in \eta^{<\omega} \);
5. if \( s \in \eta^{<\omega} \) has a predecessor, then \( \text{diam}(C_{s^-} \setminus C_s) < 2^{-|s|} \).


Proof. Let \( \eta \) be a limit ordinal greater than the cardinal number of \( E \). Using (1) and (2), one can construct a non-increasing transfinite sequence \( (C_\xi)_{\xi < \eta} \subset C \) such that \( C_0 = E \), \( \bigcap_{\xi < \lambda} C_\xi = C_\lambda \) for limit ordinals \( \lambda \), and \( C_\xi \setminus C_{\xi+1} \neq \emptyset \). For cardinality reasons, we have \( \bigcap_{\xi < \eta} C_\xi = \emptyset \). Thus, we have found our sets \( C_s \) for all sequences \( s \in \eta < \omega \) of length 1. Clearly, this process can be repeated, so that one can construct \( C_s \) for all \( s \in \eta < \omega \) by induction on the length of a sequence \( s \).

Proof of Theorem 2.1. Let \( (C_s)_{s \in \eta < \omega} \) be the family of sets given by Lemma 2.2. Notice that if \( s \in \eta < \omega \) and \( a \in E \setminus C_s \), then, by (iii) there exists a smallest ordinal \( \xi < \eta \) such that \( a \notin C_{s \upharpoonright \xi} \), and by (i) and (ii) this ordinal is a successor ordinal. It follows that for each point \( a \in E \), there is a uniquely defined sequence of successor ordinals \( s(a) = (\xi_0(a), \xi_1(a), \ldots) \in \eta^\omega \) such that \( a \in C_{s(a)} \setminus C_{s(a)}^{\omega} \) for each non-zero \( n \in \omega \). Explicitly, \( \xi_0(a) = \min\{\xi < \eta : a \notin C_\xi\} \) and \( \xi_{n+1}(a) = \min\{\xi : a \notin C_{s(a)}^\xi\} \). The strategy of player II is defined as follows: once I has played \( a_n \), II chooses a set \( A_n \in A(a_n) \) such that \( A_n \cap C_{s(a_n)} = \emptyset \). This is possible by condition (3) above.

Let \( a_0, A_0, a_1, A_1, \ldots \) be a run of the game \( G(E, A) \), where II has played according to his strategy. Observe that if \( k \leq n \), then \( C_{s(a)}(a) \subset C_{s(a)}(a) \) for all \( a \in E \): this follows from (iii). Since \( a_{n+1} \notin A_n \), it follows that \( a_{n+1} \notin C_{s(a)}(a) \) whenever \( k \leq n \). Consequently, for each fixed \( k \in \omega \), the sequence \( (s(a))_{k \in \omega} \) is non-increasing in the well-ordered set \( \eta^k \), and hence stationary. Thus, we get an infinite sequence of successor ordinals \( s \in \eta^\omega \) such that, for each fixed \( k \in \omega \), we have \( a_n \in C_{s(a)}(a) \setminus C_{s(a)}^\omega \) for all large enough \( n \). By (iv), this implies that the sequence \( (a_n) \) is Cauchy. Thus, player II has won the game.

The proof of Theorem 2.1 shows that player II has a winning strategy of a very special type: at step \( n \) of the game, the set \( A_n \) depends only on \( n \) and on the \( n \)-th move of player I. In other words, the strategy of player II is given by a sequence of tactics: there is a sequence of maps \( t_n : E \to A \) such that II wins the game by answering \( t_n(a) \) when I has played \( a_n \). By strengthening condition (3) in Theorem 2.1, one can ensure that player II has in fact a single winning tactic in the game \( G(E, A) \). This is the content of the next theorem. We shall say that a sequence \( (D_n) \) of subsets of \( E \) accumulates to some point \( x \in E \) if every neighbourhood of \( x \) contains all but finitely many sets \( D_n \).

**Theorem 2.3.** Assume that the metric space \( (E, d) \) is bounded, and that there exist a point \( a \in E \) and a sequence \( (C_n)_{n \in \mathbb{N}} \) of families of non-empty subsets of \( E \) satisfying the following properties:

(0) \( C^n \subset C^{n+1} \) and \( \overline{B}(a, r) \subset C^n \) for all \( r > 0 \);
(1) if \( (C_i)_{i \in I} \) is a family of members of \( C^n \) such that \( \bigcap_i C_i \) has non-empty interior, then \( \bigcap_i C_i \in C^n \);
(2) if \( K_1 \in C^n \) and \( K_2 \subseteq \bigcup_{p \in \mathbb{N}} C^p \) satisfy \( K_1 \subset K_2 \) and \( K_2 \setminus K_1 \neq \emptyset \), then, for each \( \varepsilon > 0 \), one can find \( K \in C^{n+1} \) such that \( K_1 \subset K \subset K_2 \setminus K \neq \emptyset \) and \( \text{diam}(K_2 \setminus K) < \varepsilon \);
(3) if \( (C_n) \) is a non-decreasing sequence of subsets of \( E \) with \( C_n \in C^n \) for all \( n \) such that \( (C_{n+1} \setminus C_n) \) accumulates to some point \( x \in E \setminus \bigcup_n C_n \), then one can find \( A \in A(x) \) such that \( A \cap \bigcup_n C_n = \emptyset \).

Then player II has a winning tactic in the game \( G(E, A) \).

The proof is based on the following lemma, which is very similar to Lemma 2.2.

**Lemma 2.4.** There exist some ordinal \( \eta \) and a family \( (C_s)_{s \in \eta < \omega} \) of subsets of \( E \) such that:

(0) \( C_s \subset C^{s+1} \) for all \( s \in \eta < \omega \), with \( C_0 = \{a\} \);
(i) \( C_{s+0} = C_s \) if \( s \in \eta < \omega \) has a predecessor, and \( C_{s+0} = E \) otherwise;
(ii) for each \( s \in \eta^{<\omega} \), the transfinite sequence \( (C_{s,\xi})_{\xi<\eta} \) is non-increasing, with \( \bigcap_{\xi<\lambda} C_{s,\xi} = C_{s,\lambda} \) for limit ordinals \( \lambda \);

(iii) \( \bigcap_{\xi<\eta} C_{s,\xi} = C_s \) for each \( s \in \eta^{<\omega} \);

(iv) if \( s \in \eta^{<\omega} \) has a predecessor, then \( \text{diam}(C_{s^*} \setminus C_s) < 2^{-|s|} \).

Proof. Let \( \eta_0 \) be any limit ordinal greater than the cardinal number of \( E \), and put \( \eta = \eta_0 \cdot \omega \).

Let us also choose \( \tau > 0 \) such that \( E = \overline{B}(a,\tau) \). We first define the sets \( C_{\xi} \) for all ordinals \( \xi < \eta_0 \) in such a way that \( \bigcap_{\xi<\eta_0} C_{\xi} = \overline{B}(a,\tau/2^1) \). By (i), we must put \( C_0 = E \). Assume that \( C_{\xi} \) has been defined, with \( C_{\xi} \in C^1 \) and \( \overline{B}(a,\tau/2^1) \subset C_{\xi} \). If \( C_{\xi} \neq \overline{B}(a,\tau/2^1) \), we use (2) with \( K_2 = C_{\xi} \), \( K_1 = \overline{B}(0,\tau/2^1) \), and \( \varepsilon = 2^{-1} \). Since \( K_1 \in C^0 \), this gives a set \( K = C_{\xi+1} \subset C^1 \). Then (iv) is satisfied. If \( C_{\xi} \) is already equal to \( \overline{B}(a,\tau/2) \), we put \( C_{\xi+1} = C_{\xi} \). If \( \lambda \) is a limit ordinal, we have to put \( C_{\lambda} = \bigcap_{\xi<\lambda} C_{\xi} \); then \( C_{\lambda} \in C^1 \) by property (1), since all sets \( C_{\xi} \) already constructed contain \( \overline{B}(a,\tau/2) \). This defines the sets \( C_{\xi} \) for all \( \xi < \eta_0 \), and by definition of \( \eta_0 \), we have \( \bigcap_{\xi<\eta_0} C_{\xi} = \overline{B}(a,\tau/2) \). Now, we put \( C_{\eta_0} = \overline{B}(a,\tau/2) \) and we define the sets \( C_{\xi} \) for \( \eta_0 \leq \xi < \eta_0 \cdot 2 \) in exactly the same way, replacing \( \overline{B}(a,\tau/2^1) \) by \( \overline{B}(a,\tau/2^2) \in C^0 \). Continuing in that way, we construct the sets \( C_{\xi} \) for all \( \xi < \eta \). It should now be clear how to produce the whole family \( (C_s)_{s \in \eta^{<\omega}} \), by induction on the length of a sequence \( s \in \eta^{<\omega} \).

Proof of Theorem 2.3. Let \( (C_s)_{s \in \eta^{<\omega}} \) be the family of sets given by Lemma 2.4. By (i), (ii) and (iii), one can associate to each point \( x \in E \setminus \{a\} \) a uniquely defined sequence of successor ordinals \( s(x) = (\xi_0, \xi_1, \ldots) \) such that \( x \in C_{(s(x))_{\xi+n}^{-}} \setminus C_{(s(x))_{\xi+n}} \) for each \( n \geq 1 \). We put \( C_n(x) = C_{(s(x))_{\xi+n}} \). By (iii) the sequence \( (C_n(x)) \) is non-decreasing, and \( x \notin \bigcup_n C_n(x) \). Moreover, since \( x \in C_{(s(x))_{\xi+n}^{-}} \setminus C_{(s(x))_{\xi+n}} \) and \( C_{n+1}(x) \subset C_{(s(x))_{\xi+n}^{-}} \), it follows from (iv) that the sequence \( (C_{n+1}(x) \setminus C_n(x)) \) accumulates to \( x \). Hence, by (o) and property (3), one can find a set \( A(x) \in C(x) \) such that \( A(x) \subset \bigcup_n C_n(x) = \emptyset \). We put \( t(x) = A(x) \) for all \( x \in E \setminus \{a\} \). Finally, we choose \( t(a) \) to be any member of \( A(a) \). This defines the tactic of player II. If \( a_0, t(a_0), a_1, t(a_1), \ldots \) is a run of the game \( G(E,A) \) where II follows this tactic, then either \( a_n = a \) for all \( n \geq 0 \), in which case II has won, or \( a_n \neq a \) after some time because \( a \notin t(x) \) if \( x \neq a \). In that case, the same proof as in Theorem 2.1 shows that II has also won.

Remark 1. The reader may feel a bit unsatisfied when looking at Theorems 2.1 and 2.3 together, since the conclusion in Theorem 2.3 is stronger than that in Theorem 2.1 but some hypotheses are not. Indeed, in Theorem 2.3 the setting is formally more general than in Theorem 2.1 since one considers a sequence \( (C^n) \) rather than a single family \( C \), and condition (2) in Theorem 2.3 is weaker than the corresponding one in Theorem 2.1. However, looking at the proof of Theorem 2.3 and with the same notation, it is easy to convince oneself that player II has a winning strategy when conditions (0), (1), (2) of Theorem 2.3 are satisfied and (3) is replaced by the following weaker assumption: if \( C \in \bigcup_n C^n \) and \( x \in E \setminus C \), then one can find \( A \in \mathcal{A}(x) \) such that \( A \cap C = \emptyset \). Thus, one could formulate explicitly a variant of Theorem 2.1 which would be more closely related to Theorem 2.3. However, Theorem 2.1 as stated is just what we need for the applications we have in mind, while we have no interesting example to illustrate the modified version.

3. Banach space setting

For all background material concerning Banach space geometry, we refer to the books [1] and [4].

Let \( X \) be a real Banach space. If \( E \) is a subset of \( X \), a closed slice of \( E \) is the intersection of \( E \) with a closed half-space of \( X \); open slices of \( E \) are defined similarly. A hyperplane section of \( E \) is the intersection of \( E \) with a closed hyperplane of \( X \). For each point \( x \in E \), we denote
by \( \mathcal{H}(x) \) the family of all hyperplane sections of \( E \) containing \( x \), by \( S_n(x) \) the family of all closed slices of \( E \) of the form \( S \cap E \), where \( S \) is a closed half-space containing \( x \) as a boundary point, and by \( S^o_n(x) \) the family of all open slices of \( E \) containing \( x \). The corresponding games \( G(E, \mathcal{H}), G(E, S_n) \) and \( G(E, S^o_n) \) are called the point–hyperplane game for \( E \), the point–open slice game for \( E \), and the point–closed slice game for \( E \).

Notice that the point–closed slice game is clearly more difficult to win for player II than the point–hyperplane game. Moreover, a moment of thought reveals that if player II has a winning strategy in the point–open slice game, then she also has one in the point–closed slice game. Finally, it is clear that if \( E_1 \subset E_2 \), then the games in \( E_1 \) are easier to win for player II than the games in \( E_2 \).

3.1. The Radon–Nikodym property

Recall that a closed bounded convex set \( K \subset X \) is said to have the Radon–Nikodym property (RNP) if every bounded linear operator \( T : L^1(\Omega, \mu) \to X \) sending the positive unit sphere of \( L^1(\Omega, \mu) \) into \( K \) can be represented by an element of \( L^\infty(\Omega, \mu, X) \); here, \( (\Omega, \mu) \) is an arbitrary probability space. The Banach space \( X \) has the RNP if its closed unit ball has. Among the many beautiful characterizations of this property, we shall of course use the following one: the convex set \( K \) has the RNP if and only if each non-empty subset of \( K \) has non-empty open slices with arbitrarily small diameter.

**Theorem 3.1.** Let \( K \subset X \) be a (non-empty) bounded closed convex set. Then the point–open slice game for \( K \) is determined, and player II has a winning strategy if and only if \( K \) has the Radon–Nikodym property. More precisely:

(a) if \( K \) has the RNP, then II has a winning strategy;

(b) if \( K \) does not have the RNP, then there exists \( \varepsilon > 0 \), and a strategy for player I such that each run of the game where I plays according to this strategy produces a sequence \( \langle a_n \rangle \) such that \( \|a_{n+1} - a_n\| > \varepsilon \) for all \( n \in \mathbb{N} \).

**Proof.** Assume that \( K \) has the Radon–Nikodym property. Let \( C \) be the family of all closed convex subsets of \( K \). We check that \( C \) and \( \mathcal{A} = S_o \) satisfy the assumptions of Theorem 2.1. Condition (1) is obviously satisfied, and (3) follows from the Hahn–Banach theorem. To prove that (2) is also satisfied, let us fix \( C_1, C_2 \in \mathcal{C} \) with \( C_1 \subset C_2 \) and \( C_2 \setminus C_1 \neq \emptyset \). By the Hahn–Banach theorem, one can find \( x^* \in X^* \) and \( \alpha < \beta \) such that

\[
C_1 \subset \{ x : \langle x^*, x \rangle \leq \alpha \} \quad \text{and} \quad C_2 \cap \{ x : \langle x^*, x \rangle > \beta \} \neq \emptyset.
\]

Since the set \( C_2 \) has the RNP by assumption on \( K \), it follows from Stegall’s variational principle [10] that one can approximate \( x^* \) by \( y^* \in X^* \) strongly exposing some point of \( C_2 \). If \( y^* \) is close enough to \( x^* \), then

\[
C_1 \subset \{ x : \langle y^*, x \rangle \leq \beta \} \quad \text{and} \quad C_2 \cap \{ x : \langle y^*, x \rangle > \beta \} \neq \emptyset;
\]

and if \( \gamma \geq \beta \) is close enough to \( \sup C_2 y^* \), then the set \( \{ x \in C_2 : \langle y^*, x \rangle > \gamma \} \) has small diameter. Thus, putting \( C = \{ x \in C_2 : \langle y^*, x \rangle \leq \gamma \} \) for some suitable \( \gamma \), we see that condition (2) is satisfied. By Theorem 2.1, we conclude that player II has a winning strategy in the point–open slice game for \( K \).

Now, assume that \( K \) does not have the RNP. Then one can find \( \varepsilon > 0 \) and a non-empty set \( \tilde{K} \subset K \) such that each non-empty open slice of \( \tilde{K} \) has diameter greater than \( 2\varepsilon \). We define a strategy for player I as follows. First, I chooses some point \( a_0 \in \tilde{K} \). If II answers with some open slice \( A_0 \) containing \( a_0 \), then \( \text{diam} A_0 \cap \tilde{K} > 2\varepsilon \). By the triangle inequality, it follows that one can find \( a_1 \in A_0 \cap \tilde{K} \) such that \( \|a_1 - a_0\| > \varepsilon \); this point \( a_1 \) is the second move of player I. Repeating this procedure, we clearly get the announced strategy for player I. \( \square \)
Keeping in mind that the point–open slice game is harder to win for II than the point–closed game which is harder than the point–hyperplane game, and that the games in a larger set are harder to win for II, we get the following corollary.

**Corollary 3.2.** If the Banach space \( X \) has the RNP, then, for any (non-empty) bounded set \( \Omega \subset X \), player II has winning strategies in the point–slice games and in the point–hyperplane game for \( \Omega \).

The remark following Theorem 2.1 shows that if the convex set \( K \) has the RNP, then, in each of the above games, player II has a winning strategy given by a sequence of tactics: there is a sequence of maps \( t_n : E \to A \) such that II wins by answering \( t_n(a_n) \) when I has played \( a_n \). For the point–closed slice game, this result can be formulated in the following way, which is arguably very intuitive keeping in mind the fact that every bounded monotonic sequence of real numbers is convergent.

**Corollary 3.3.** If \( X \) has the Radon–Nikodym property, then there exists a sequence of maps \( \Phi_n : B_X \to S_X \), such that the following property holds true: if \( (a_n) \subset B_X \) is a sequence satisfying \( \langle \Phi_n(a_n), a_{n+1} \rangle \geq \langle \Phi_n(a_n), a_n \rangle \) for all \( n \in \mathbb{N} \), then \( (a_n) \) is convergent.

With a little more effort, one can show that, as far as the Radon–Nikodym property of the whole space \( X \) is concerned, the two point–slice games and the point–hyperplane game are essentially equivalent. This is the content of the next result.

**Theorem 3.4.** Let \( \Omega \) be a bounded subset of \( X \) with non-empty interior. Then the following are equivalent:

1. \( X \) has the Radon–Nikodym property;
2. player II has a winning strategy in the point–open slice game for \( \Omega \);
3. II has a winning strategy in the point–closed slice game for \( \Omega \);
4. II has a winning strategy in the point–hyperplane game for \( \Omega \).

**Proof.** One can find open balls \( B_1 \) and \( B_2 \) such that \( B_1 \subset \Omega \subset B_2 \). Since the games in a larger set are harder to win for II, it follows that II has a winning strategy in any of the above games for \( \Omega \) if and only if it has one in the same game for the open unit ball of \( X \). Thus, we may assume that \( \Omega \) is the open unit ball of \( X \).

That (1) implies (2) follows from Theorem 3.1: if \( X \) has the RNP, which means that \( \overline{\Omega} \) has the RNP, then II has a winning strategy in the game \( G(\overline{\Omega}, S_\eta) \), and hence also in \( G(\Omega, S_\eta) \). We have already observed that (2) implies (3) and (3) implies (4). To conclude the proof, we have to show that if \( X \) does not have the RNP, then player I also has a winning strategy in the point–hyperplane game \( G(\Omega, \mathcal{H}) \). So, assume that \( X \) does not have the RNP.

**Claim 1.** One can find a non-empty open convex set \( V \subset \Omega \) and \( \varepsilon > 0 \) such that all non-empty slices of \( V \) have diameter at least \( \varepsilon \).

**Proof of Claim 1.** Since \( X \) does not have the RNP, \( \Omega \) contains a non-empty closed convex set \( K \) such that all non-empty open slices of \( K \) have diameter at least \( 5\varepsilon \), for some fixed \( \varepsilon > 0 \). Moreover, we may assume that \( K \) is at positive distance from \( \partial \Omega \), that is, \( \eta_0 := \inf\{\text{dist}(x, \partial \Omega) : x \in K\} > 0 \). Then, for \( \eta < \eta_0 \), the convex open set \( V_\eta = \{x : \text{dist}(x, K) < \eta\} \) is contained in \( \Omega \). Let us check that one can take \( V = V_\eta \), if \( \eta \) is small enough. Since \( V_\eta \) is open, it is enough to consider only open slices of \( V_\eta \). Let \( S \) be a non-empty open slice of \( V_\eta \), that is \( S = U \cap V_\eta \), where \( U \) is an open half-space. Let \( x \) be any point of \( S \). Then one can find \( x' \in K \)
such that \( ||x' - x|| < \eta \). By translating the half-space \( U \) in the direction of \( x' - x \), one gets an open half-space \( U' \) containing \( x' \) such that \( \text{dist}(z, U) < \eta \) for all \( z \in U' \). Then \( S' = U' \cap K \) is a non-empty open slice of \( K \), so it has diameter at least \( 5\varepsilon \). By the triangle inequality, it follows that one can find \( y' \in U' \cap K \) such that \( ||y' - x'|| \geq 2\varepsilon \), and by the choice of \( U' \), one gets a point \( y \in U \) such that \( ||y - y'|| < \eta \). Then \( y \in V_\eta \) because \( y' \in K \), so that \( y \in S' \); and \( ||y - x|| \geq ||y' - x'|| - 2\eta \geq 2\varepsilon - 2\eta \). Thus, if \( \eta < \varepsilon/2 \), then the diameter of every non-empty slice of \( V_\eta \) is at least \( \varepsilon \). \( \square \)

Claim 2. Each non-empty hyperplane section of \( V \) has diameter at least \( \varepsilon/4 \).

Proof of Claim 2. Let \( \Phi \in X^* \) and, put \( m_\Phi := \inf_V \Phi \) and \( M_\Phi := \sup_V \Phi \). We have to show that for each \( r \in (m_\Phi, M_\Phi) \), the set \( A_r = \{ x \in V : \Phi(x) = r \} \) has diameter at least \( \varepsilon/4 \). Let \( x_0 \) be any point of \( V \), and assume \( r > \Phi(x_0) \). Let us denote by \( S_r \) the slice \( \{ x \in V : \Phi(x) > r \} \). For each point \( z \in S_r \), the line segment \([x_0, z]\) intersects \( A_r \) at a unique point \( \hat{z} \), and we have \( ||z - \hat{z}|| = \lambda_z ||z - x_0|| \), where

\[
\lambda_z = \frac{\Phi(z) - \Phi(\hat{z})}{\Phi(z) - \Phi(x_0)} \leq \frac{M_\Phi - r}{r - \Phi(x_0)}.
\]

It follows that \( S_r \subset A_r + B(0, \varepsilon_r) \), where \( \varepsilon_r \to 0 \) as \( r \to M_\Phi \). Since \( \text{diam}(S_r) \geq \varepsilon \), we have shown that if \( r \) is close to \( M_\Phi \), then \( \text{diam}(A_r) \geq \varepsilon/2 \); and likewise if \( r \) is close to \( m_\Phi \).

Now, for an arbitrary \( r \in (m_\Phi, M_\Phi) \), one can find \( r_1 < r < r_2 \) such that \( A_{r_1} \) and \( A_{r_2} \) have diameter at least \( \varepsilon/2 \). Moreover, we have \( \text{diam}(A_r) \geq \frac{1}{2} \min(\text{diam}(A_{r_1}), \text{diam}(A_{r_2})) \), and hence \( \text{diam}(A_r) \geq \varepsilon/4 \). Indeed, assume (as we may) that \( r - r_1 \leq \frac{1}{2}(r_2 - r_1) \). If \( x_1 \) and \( y_1 \) are any two points of \( A_{r_1} \), then, taking a point \( z \in A_{r_2} \) and drawing the triangle \( x_1 y_1 z \), we see that one can find \( x, y \in A_r \) such that \( ||x - y|| \geq \frac{1}{2} ||x_1 - y_1|| \). Since \( x_1 \) and \( y_1 \) are arbitrary, this gives \( \text{diam}(A_r) \geq \frac{1}{2} \text{diam}(A_{r_1}) \). \( \square \)

Exactly as in the proof of Theorem 3.1, it follows from Claim 2 that player I has an obvious winning strategy in the game \( G(V, \mathcal{H}) \): begin with any point \( a_0 \in V \), and then, thanks to Claim 2, play at each step a point \( a_{n+1} \in V \) such that \( ||a_{n+1} - a_n|| \geq \varepsilon/8 \). Since the game \( G(V, \mathcal{H}) \) is more difficult to win for I than the corresponding game in the larger set \( \Omega \), this concludes the proof. \( \square \)

3.2. The super-reflexive case

We observed above that if \( X \) has the RNP, then, in each of the three games we have considered, player II has a winning strategy which is given by a sequence of tactics. It is not clear for us whether II has in fact a single winning tactic. However, we can do more under a stronger assumption on \( X \). Recall that the Banach space \( X \) is said to be super-reflexive if every Banach space which is finitely representable in \( X \) is reflexive. By a deep result due to P. Enflo, the super-reflexive Banach spaces are exactly those which admit an equivalent uniformly convex norm.

Theorem 3.5. Assume that the Banach space \( X \) is super-reflexive. Then, for any bounded set \( \Omega \subset X \), player II has a winning tactic in the point–closed slice game for \( \Omega \).

As in Corollary 3.3 above, this result can be formulated in the following way.

Corollary 3.6. If \( X \) is super-reflexive, then one can associate to each point \( a \in B_X \) a linear functional \( \Phi_a \in S_X \), in such a way that the following property holds true: if a sequence \( (a_n) \subset B_X \) satisfies \( \langle \Phi_{a_n}, a_{n+1} \rangle \geq \langle \Phi_{a_n}, a_n \rangle \) for all \( n \in \mathbb{N} \), then \( (a_n) \) is convergent.
For the proof of Theorem 3.5, we need the following definition.

**Definition 1.** Let $K \subset X$ be a bounded closed convex set with non-empty interior. We say that $K$ is uniformly convex if the following property holds: there exists a function

$$\delta : (0, \infty) \to (0, \infty)$$

such that whenever $x, y \in K$ satisfy $\|x - y\| \geq \varepsilon$, it follows that $\text{dist}(\frac{1}{2}(x + y), \partial K) \geq \delta(\varepsilon)$. In such a case, we say that $K$ is uniformly convex with modulus $\delta$.

Thus, the balls of $X$ are uniformly convex if and only if the given norm is uniformly convex in the usual sense, and in that case, a ball of radius $r$ is uniformly convex with a modulus depending only on $r$.

**Proof of Theorem 3.5.** Since $X$ is super-reflexive, we may assume that the norm of $X$ is uniformly convex. For each $r > 0$, we choose a modulus of uniform convexity $\delta^r$ for balls of radius $r$. Clearly, we may assume that if $r \leq r'$, then $\delta^r(\varepsilon) \geq \delta^{r'}(\varepsilon)$ for small enough $\varepsilon$.

**Fact 1.** Let $(K_i)_{i \in I} \subset X$ be a family of closed convex bounded sets. Assume that $\bigcap_{i} K_i$ has non-empty interior, and that all sets $K_i$ are uniformly convex with some fixed modulus $\delta$. Then $K = \bigcap_{i \in I} K_i$ is uniformly convex with modulus $\delta$.

**Proof.** First, we observe that if $K_1$ and $K_2$ are uniformly convex subsets of $X$ with modulus $\delta_1$ and $\delta_2$, respectively, and if $K_1 \cap K_2$ has non-empty interior, then $K_1 \cap K_2$ is uniformly convex with modulus $\delta_3 \geq \inf(\delta_1, \delta_2)$. This is obvious since $\text{dist}(K_1 \cap K_2) \subset \partial K_1 \cup \partial K_2$.

Accordingly, we may assume that the family $(K_i)$ is stable under finite intersections, and of course that $K_i \neq K_j$ if $i \neq j$. Replacing each $K_i$ by $K_i \cap K_{i_0}$ for some fixed $i_0 \in I$, we may assume in addition that $\bigcup_{i} K_i$ is bounded. Finally, we may also assume that $0 \in \text{int}(K)$.

Let us fix $\varepsilon > 0$, and $x, y \in K$ with $\|x - y\| \geq \varepsilon$. Let $p \in \partial K$. Since $\bigcup_{i} K_i$ is bounded, one can find $\lambda \in (1, \infty)$ such that, for each $i \in I$, the segment $[p, \lambda p]$ intersects $\partial K_i$ at some point $p_i$. We order the index set $I$ in the obvious way: $i \leq j$ if $K_i \supset K_j$. Then $I$ is a directed set, and since $[p; \lambda p]$ is compact, the net $(p_i)_{i \in I}$ has a subnet converging to some $\bar{p} \in [p, \lambda p] \cap K$. Since $0 \in \text{int}(K)$, the half-open segment $[0, \bar{p})$ is contained in $\text{int}(K)$, and this implies that $\bar{p} = p$. Moreover, we have $\left\| \frac{1}{2}(x + y) - p_i \right\| \geq \delta(\varepsilon)$ for all $i \in I$; hence $\left\| \frac{1}{2}(x + y) - p \right\| \geq \delta(\varepsilon)$. This concludes the proof.

**Fact 2.** Let $R > 0$, and let $K_1$ and $K_2$ be non-empty closed convex sets of diameter less than $R$ such that $K_1 \subset K_2$ and $K_2 \setminus K_1 \neq \emptyset$. Assume that $K_2$ is uniformly convex, and that $K_1$ is the intersection of a family of balls of radii less than $R$. For each $\varepsilon > 0$, there exists a ball $B$ of radius less than $4R$ such that $K = \overline{B} \cap K_2$ satisfies $K_1 \subset K \subset K_2$, $K_2 \setminus K \neq \emptyset$ and $\text{diam}(K_2 \setminus K) < \varepsilon$.

**Proof.** Choose a point $x_0 \in K_2 \setminus K_1$. By assumption on $K_1$, one can find an open ball $B_0 = B(p_0, r_0)$ with $r_0 < R$ such that $K_1 \subset B_0$ and $x_0 \notin \overline{B}_0$. Since $X$ is reflexive, the set $K_2$ is weakly compact. By a classical result of K. S. Lau [6], the set of points $p \in X$ admitting a farthest point in $K_2$ is dense in $X$. Thus, one can find a point $p \in X$ very close to $p_0$ and $x \in \partial K_2$ such that $\|x - p\| \geq \|y - p\|$ for all $y \in K_2$. Notice that if $p$ is close enough to $p_0$, then $\|x - p\| \geq \|x_0 - p\| > \sup\{\|z - p\| : z \in K_1\}$, so that $x \notin K_1$. Notice also that if $p$ is close to $p_0$, then $\|x - p\| < 2R$. Now, let $\eta > 0$ and $\alpha > 0$. By the choice of $p$ and $x$, if a point $z \in K_2$ satisfies

$$\|z - p\| > r_\eta := \frac{1}{1 + \eta} \|x - p\|,$$
then $p + (1 + \eta)(z - p) \notin K_2$; in particular, the half-line $z + \mathbb{R}^+(z - p)$ must meet $\partial K_2$ at some point $z + t(z - p)$ with $t \leq \eta$, so
\[
\text{dist}(z, \partial K_2) \leq \eta\|z - p\| \leq 2R\eta.
\]
Thus, for each $\eta > 0$, we have found $r_\eta < 2R$ such that if $z \in K_2$ satisfies $\|z - p\| > r_\eta$, then $\text{dist}(z, \partial K_2) \leq 2R\eta$. Denoting by $\delta$ a modulus of uniform convexity for $K_2$, we see that $\|x - y\| < \alpha$ whenever $2R\eta < \delta(\alpha)$ and $y \in K_2$ satisfies $\|\frac{1}{2}(x + y) - p\| > r_\eta$. In other words, for small enough $\eta$, the set $K_2 \setminus B(2p - x, 2r_\eta)$ is contained in $B(x, \alpha)$. Thus, one can put $B = B(2p - x, 2r_\eta)$, for some suitably chosen $\eta > 0$.

**Fact 3.** Let $(K_n)$ be a non-decreasing sequence of uniformly convex subsets of $X$. Assume that the sequence $(K_{n+1} \setminus K_n)$ accumulates to some point $x \in X \setminus \bigcup_n K_n$. Then one can find a closed half-space containing $x$ as a boundary point and disjoint from $\bigcup_n K_n$.

**Proof.** Put $K = \bigcup_n K_n$. Since the sequence $(K_n)$ is non-decreasing, the set $K$ is convex, and $K$ has non-empty interior. If $x \notin \overline{K}$ (which may happen in the uninteresting case where the sequence $(K_n)$ is stationary), the Hahn–Banach theorem allows us to separate strictly $x$ from $K$ by some linear functional, and the result follows. Now, assume $x \in \overline{K}$. Then we have $\overline{K} = K \cup \{x\}$ because the sequence $(K_{n+1} \setminus K_n)$ accumulates to $x$. Since $K$ has non-empty interior, one can still separate $x$ from $K$ by some linear functional, but perhaps not strictly. In other words, one can find a non-zero linear functional $x^* \in X^*$ such that $\alpha := \langle x^*, x \rangle \geq \langle x^*, z \rangle$ for all $z \in K$. By contradiction, assume that equality occurs at some point $z \in K$. Then the segment $[z, x]$ is contained in $\overline{K} = K \cup \{x\}$, so that the half-open segment $[z, x]$ is contained in $K$. Since the sequence $(K_n)$ is non-decreasing and the sets $K_n$ are convex, it follows that one can find some integer $n$ such that $I := [z, \frac{1}{2}(x + z)] \subseteq K_n$; hence $I \subseteq \partial K_n$ because $I$ is contained in the hyperplane $\{x^* = \alpha\}$. This is a contradiction since, being uniformly convex, $K_n$ cannot contain non-trivial segments in its boundary. Thus, the half-space $M := \{x^* \geq \alpha\}$ satisfies $M \cap K = \emptyset$, as required.

We are now in position to apply Theorem 2.3. Let $\Omega$ be a bounded subset of $X$, and choose $R > 0$ such that $\overline{\Omega} \subset B(0, R/3)$. We apply Theorem 2.3 with $E = B(0, R/3)$; as observed above, it is enough to show that player $\Pi$ has a winning tactic in the point–closed slice game for $E$. We put $a = 0$, and for each $n \in \mathbb{N}$, we define $C^n$ to be the family of all uniformly convex sets $C \subseteq E$ which are intersections of balls of $X$ of radii less than $4^nR$. Then condition (0) in Theorem 2.3 is clearly satisfied. Conditions (1), (2) and (3) are also satisfied, thanks to the corresponding facts proved above. This concludes the proof of Theorem 3.5.

### 3.3. The point of continuity property

We conclude this section with a game characterization of another well-known Banach space property. Recall that the Banach space $X$ is said to have the point of continuity property (PCP) if each non-empty bounded set $A \subseteq X$ has non-empty relatively weakly open subsets with arbitrarily small diameter. More generally, let $(E, d)$ be a metric space, and let $\tau$ be a topology on $E$. The topological space $(E, \tau)$ is said to be fragmented by the metric $d$ if each non-empty subset of $E$ has non-empty relative $\tau$-open subsets with arbitrarily small diameter. Thus, a Banach space $X$ has the PCP if and only if its unit ball is norm-fragmented in the weak topology.

If $(E, d)$ is a metric space and $\tau$ is a topology on $E$, we define the $(\tau, d)$-game for $E$ to be the game $G(E, \mathcal{A})$, where for each $x \in E$, $\mathcal{A}(x)$ is the family of all $\tau$-open subsets of $E$ containing $x$. Thus, player $I$ starts the game by playing some point $a_0$, player $\Pi$ answers by some $\tau$-open set $U_0$ containing $a_0$, player $I$ then plays a point $a_1 \in U_0$ and so on.
**Theorem 3.7.** The topological space \((E, \tau)\) is fragmented by the metric \(d\) if and only if player II has a winning strategy in the \((\tau, d)\)-game for \(E\).

**Proof.** The ‘only if’ part follows from Theorem 2.1: just take for \(C\) the family of all \(\tau\)-closed subsets of \(E\). The only thing to be checked is property (2). Now, if \(C_1\) and \(C_2\) are \(\tau\)-closed with \(C_1 \subset C_2\) and \(C_2 \setminus C_1 \neq \emptyset\), then one can find a \(\tau\)-open set \(U\) such that \(U \cap (C_2 \setminus C_1) \neq \emptyset\) and \(\text{diam}(U \cap (C_1 \setminus C_1)) < \varepsilon\). Thus, one can put \(C = C_1 \cup (C_2 \setminus U)\) to get (2). The ‘if’ part is obvious, exactly as in Theorem 3.1. \(\square\)

**Corollary 3.8.** A Banach space \(X\) has the PCP if and only if player II has a winning strategy in the \((w, \| \cdot \|)\)-game for \(B_X\).

**Remark 2.** In \([7]\), P. S. Kenderov and W. B. Moors give a characterization of fragmentability by means of another natural topological game. See also \([8]\).

Using the same kind of arguments as in the proof of Theorem 3.4 above, we can also get a characterization of the point of continuity property by means of a game where the weak topology does not appear explicitly. Let us denote by \(A_{\text{cof}}\) the family of all finite-codimensional affine subspaces of \(X\).

**Theorem 3.9.** Let \(\Omega\) be a bounded subset of \(X\) with non-empty interior. The following are equivalent:

1. the Banach space \(X\) has the PCP;
2. player II has a winning strategy in the game \(G(\Omega, A_{\text{cof}})\).

**Proof.** As in the proof of Theorem 3.4, we may assume that \(\Omega\) is the open unit ball \(B_X\). From Corollary 3.8, we already know that (1) implies (2). Conversely, assume that \(X\) does not have the PCP. We show that player I has a winning strategy in the game \(G(B_X, A_{\text{cof}})\).

**Claim.** There exist \(\varepsilon > 0\) and an increasing sequence of open sets \((V_n)\), with \(V_n \subset B_X\) for all \(n\), such that the following property holds true: for each \(n \in \mathbb{N}\) and all finite-codimensional affine subspaces \(M \subset X\) such that \(M \cap V_n \neq \emptyset\), the diameter of \(M \cap V_{n+1}\) is at least \(\varepsilon\).

**Proof of the claim.** Since \(X\) does not have the PCP, one can find a non-empty set \(K \subset \frac{1}{2}B_X\) such that each non-empty relative weak open subset of \(K\) has diameter at least \(9\varepsilon\), for some fixed \(\varepsilon > 0\). As in the proof of Theorem 3.4, we find that for all sufficiently small \(\eta > 0\), the open set

\[
V_\eta = \{x : \text{dist}(x, K) < \eta\}
\]

has the same property, with \(9\varepsilon\) replaced by \(3\varepsilon\). Putting \(V_n = V_{\eta_n}\), for some suitable increasing sequence \((\eta_n)\), we get an increasing sequence of open sets \(V_n \subset B_X\) such that:

- (a) all non-empty relative weak open subsets of \(V_n\) have diameter at least \(3\varepsilon\);
- (b) \(\inf\{\text{dist}(x, \partial V_{n+1}) : x \in V_n\} > 0\) for all \(n \in \mathbb{N}\).

Let us fix \(n \in \mathbb{N}\), and let \(M\) be a finite-codimensional affine subspace of \(X\) such that \(M \cap V_n \neq \emptyset\). We write \(M = \bigcap_{i=1}^{n} H_i\), where \(H_i = \{\Phi_i = r_i\}\) is an affine hyperplane determined by some linear functional \(\Phi_i \in X^*\). Then \(\rho(x) = \sup_{1 \leq i \leq N} |\Phi_i(x)|\) induces a norm on the quotient space \(X/M\), where \(M = \bigcap_{i} \text{Ker}(\Phi_i)\); and since \(X/M\) is finite dimensional, this norm is equivalent to the one induced by \(\rho_0(x) = \text{dist}(x, M)\). Thus, we see that one can find some constant \(C = C(\Phi_1, \ldots, \Phi_N)\) such that the following property holds true:

\[
\text{for all } x \in X, \quad \text{dist}(x, M) \leq C \sup_{1 \leq i \leq N} |\Phi_i(x) - r_i|.
\] (3.1)
By the choice of $V_n$, for each $\eta > 0$, the set
\[ A_\eta := \left\{ x \in V_n : \sup_{i \in \mathbb{N}} |\Phi_i(x) - r_i| < \eta \right\} \]
has diameter at least $3\varepsilon$. So one can find $y_1, y_2 \in A_\eta$ such that $\|y_1 - y_2\| \geq 2\varepsilon$, and by (3.1) one can find $x_1, x_2 \in M$ such that $\|x_j - y_j\| \leq C\eta$, for $j = 1, 2$. Then $\|x_1 - x_2\| \geq \varepsilon$ if $\eta$ is small enough. Moreover, since $y_1$ and $y_2$ are in $V_n$, they are ‘far away’ from $\partial V_{n+1}$, so that we also have $x_1, x_2 \in V_{n+1}$ if $\eta$ is small enough. Thus we have proved that $\text{diam}(V_{n+1} \cap M) \geq \varepsilon$. \hfill \square

Now, we define a winning strategy for $\mathbf{I}$ in the game $G(B_X, \mathcal{A})$ as follows. The first move of player $\mathbf{I}$ is any point $a_0 \in V_0$. When $\mathbf{II}$ has played a finite-codimensional section $A_0 = M_0 \cap B_X$ containing $a_0$, then, by the claim, $\mathbf{I}$ can play a point $a_1 \in M_0 \cap V_1$ such that $\|a_1 - a_0\| \geq \varepsilon/2$. Then $\mathbf{II}$ plays $A_1 = M_1 \cap B_X$ containing $a_1$, $\mathbf{I}$ can play $a_2 \in M_1 \cap V_2$ such that $\|a_2 - a_1\| \geq \varepsilon/2$, and so on. \hfill \square

**Remark 3.** When the Banach space $X$ is separable and reflexive, it is very easy to describe a winning strategy for $\mathbf{II}$ in the game $G(B_X, \mathcal{A}_{\text{cof}})$, without appealing to Theorem 2.1. Since $X$ is separable, we may assume that the norm of $X$ has the Kadec–Klee property, which means that the weak and the norm topologies coincide on the unit sphere. Moreover, $X^*$ is separable since $(X^*)^* = X$; let $\{x_n^* : n \in \mathbb{N}\}$ be a countable dense subset of $X^*$. The strategy of player $\mathbf{II}$ is defined as follows. Once player $\mathbf{I}$ has played a point $a_n \in B_X$, player $\mathbf{II}$ chooses a linear functional $\Phi_n \in B_{X^*}$ such that $\Phi_n(a_n) = \|a_n\|$, and then she plays the affine subspace
\[ M_n := \{ x \in X : \langle \Phi_i(x) - \Phi_i(a_n) \rangle_{i=0}^n \text{ and } \langle x_i^*, x \rangle = \langle x_i^*, a_n \rangle, \ i = 0, \ldots, n \}. \]
If $\mathbf{II}$ plays according to this strategy, then, since $X$ is reflexive, the sequence $(a_n)$ produced by any run of the game is weakly convergent. Moreover, denoting by $a$ the weak limit of $(a_n)$, we have $\|a\| \geq \Phi_n(a) = \Phi_n(a_n) = \|a_n\|$ for all $n \in \mathbb{N}$; and since in any case $\|a\| \leq \liminf \|a_n\|$, we conclude that $\|a\| = \lim \|a_n\|$. By the Kadec–Klee property, it follows that the sequence $(a_n)$ is in fact $\| \cdot \|$-convergent, so that $\mathbf{II}$ has won the game. We thank G. Lancien for showing this strategy to us.

### 3.4. Concluding remarks

To conclude this section, let us mention some problems that we were not able to solve.

1. Does player $\mathbf{II}$ always have a winning tactic in the point–closed slice game for $B_X$ if the Banach space $X$ has the RNP? If not, what about the reflexive case?

2. In the super-reflexive case, does player $\mathbf{II}$ have a continuous tactic in the point–closed slice game? M. Zeleny has shown very recently [12] that this is indeed the case when $X = \mathbb{R}^d$.

3. Does player $\mathbf{II}$ have a winning tactic in the game $G(B_X, \mathcal{A}_{\text{cof}})$ when the Banach space $X$ has the PCP? If not, is there a natural class of Banach spaces for which player $\mathbf{II}$ does have a winning tactic? A plausible candidate might be the class of asymptotically uniformly convex spaces (see [5]).

4. What can be said about other games of the type $G(B_X, \mathcal{A})$ where $\mathcal{A}$ is a given family of affine subspaces of $X$? In particular, let $\mathcal{A}_\infty$ be the family of all infinite-dimensional subspaces of $X$. Does the game $G(B_X, \mathcal{A}_\infty)$ characterize some known Banach space property? It follows from Corollary 3.8 that player $\mathbf{II}$ has a winning strategy in $G(B_X, \mathcal{A}_\infty)$ if $X$ has an infinite-dimensional subspace with the PCP. Indeed, if $Y$ is such a subspace, then, for any point $a_0 \in X$, the subspace $Y(a_0) := Y \oplus \mathbb{R} a_0$ also has the PCP, so that $\mathbf{II}$ can win the game by playing inside $Y(a_0)$. We are not able to say more.

Of course, one could ask the same question as in (4) for the family of all finite-dimensional subspaces of $X$, but in that case the answer is easy. Actually, as soon as the family $\mathcal{A}$ contains
all \(d\)-dimensional subspaces of \(X\), for some non-negative integer \(d < \dim(X)\), then \(\mathbf{I}\) has a
winning strategy in \(G(B_X, A)\). Indeed, once player \(\mathbf{I}\) has played \(a_0\), player \(\mathbf{II}\) can answer
with any \(d\)-dimensional subspace \(F_0 = F(a_0)\). Then \(\mathbf{I}\) plays \(a_1\), and \(\mathbf{II}\)
can answer with a \(d\)-dimensional subspace \(F_1 = F(a_0, a_1)\) such that the affine subspace \(F\) generated by \(F_0\) and
\(F_1\) has dimension \(d + 1\). Then the next move \(a_2\) of player \(\mathbf{I}\) will belong to \(F\), so that
from this point on, \(\mathbf{II}\) can play according to some winning tactic in the point–hyperplane game inside \(F\).

4. Back to differentiable functions

In this section, we turn back to pathological differentiable functions. We fix once and for all
an integer \(d \geq 2\) and an arbitrary norm \(\| \cdot \|\) on \(\mathbb{R}^d\).

Let us recall that Buczolich has constructed an everywhere differentiable function \(u : \mathbb{R}^d \to \mathbb{R}\)
such that \(\nabla u(0) = 0\) and \(\|\nabla u(x)\| \geq 1\) for almost every \(x \in \mathbb{R}^d\), in the sense of Lebesgue
measure. Following the ideas of [9], our purpose here is to use the games introduced above
to obtain a bounded differentiable function \(u : \mathbb{R}^d \to \mathbb{R}\) which is a solution of the Eikonal
equation \(\|\nabla u\| = 1\) almost everywhere.

Of course, this equation admits unbounded smooth solutions (for example, any norm 1 linear
functional), as well as bounded almost everywhere solutions. The pathology comes from the
fact that the almost everywhere solution \(u\) is both bounded and everywhere differentiable.
Notice that, using Ekeland’s variational principle, it is easy to check that the gradient of any
bounded differentiable function on \(\mathbb{R}^d\) takes arbitrarily small values. This shows, in particular,
that the Eikonal equation does not have bounded differentiable solutions on \(\mathbb{R}^d\), and that
the gradient of any bounded, differentiable, almost everywhere solution fails the Denjoy–Clarkson
property.

**Theorem 4.1.** There exists an everywhere differentiable, bounded function \(u : \mathbb{R}^d \to \mathbb{R}\)
such that \(\nabla u(0) = 0\) and \(\|\nabla u(x)\| = 1\) for almost every \(x \in \mathbb{R}^d\).

We shall actually prove the following more general result. Here and below, all cubes of \(\mathbb{R}^d\)
will be half-open, that is, of the form \(\prod_{i=1}^d [a_i, b_i)\), where the \([a_i, b_i)\) are half-open intervals
of \(\mathbb{R}\).

**Theorem 4.2.** Let \(U\) be a bounded open subset of \(\mathbb{R}^d\) containing 0, and let \(Q_0 = [0, 1)^d\)
be the unit cube in \(\mathbb{R}^d\). Then, there exists an everywhere differentiable, \(\mathbb{Z}^d\)-periodic function
\(u : \mathbb{R}^d \to \mathbb{R}\) such that:

1. \(u\) and \(\nabla u\) vanish on \(\partial Q_0\);
2. \(\nabla u(x) \in \overline{U}\) for all \(x \in \mathbb{R}^d\);
3. \(\nabla u(x) \in \partial U\) for almost every \(x \in \mathbb{R}^d\).

Up to a constant, the function \(u\) in Theorem 4.2 will be the sum of a uniformly convergent
series of non-zero \(C^\infty\) smooth functions \(u_n : \mathbb{R}^d \to \mathbb{R}\). In order to prove that \(\sum_{n=1}^\infty u_n\) is
everywhere differentiable, we shall use the following differentiability criterion. Here and below,
if \(F\) is a function between two Banach spaces \(X\) and \(Y\) and if \(\varepsilon\) is a positive number, we put

\[
\text{osc}(F, \varepsilon) = \sup\{\|F(x) - F(y)\| : \|x - y\| < \varepsilon\}.
\]

**Lemma 4.3.** Let \((u_n)_{n \geq 1}\) be a sequence of \(C^1\)-functions between two Banach spaces \(X\) and
\(Y\). Assume that:

1. the series \(\sum u'_n(x)\) is pointwise convergent;
2. \((u'_n)\) converges uniformly to 0;
3. \(\|u_{n+1}\|_\infty = o(\|u_n\|_\infty)\).
(d) $\lim_{n \to -\infty} \text{osc} \left( \sum_{k=1}^{n} u_k', \|u_{n+1}\|_\infty \right) = 0$.

Then the series $\sum u_n$ is uniformly convergent, the function $f := \sum_1^\infty u_n$ is everywhere differentiable, and $f'(x) = \sum_1^\infty u'_n(x)$ for all $x \in X$.

Proof. By condition (c), one can find $n_0$ such that $\|u_{n+1}\|_\infty \leq \|u_n\|_\infty / 2$ for all $n \geq n_0$; therefore, the series $\sum u_n$ is uniformly convergent. If $u_n = 0$ for some $n \geq n_0$, then $u_k = 0$ for all $k \geq n$ and there is nothing to prove. So we assume that $\|u_n\|_\infty > 0$ for all $n$. Let us fix $x_0 \in X$.

Putting

$$S_n(x) = \sum_{k=1}^{n} u_k(x), \quad s_n(x) = S'_n(x) = \sum_{k=1}^{n} u'_k(x), \quad r_n(x) = \sum_{k=n+1}^{\infty} u_k(x),$$

we have

$$\left\| f(x) - f(x_0) - \sum_{k=1}^{\infty} u'_k(x_0) \cdot (x - x_0) \right\|$$

$$\leq \left\| S_{n-1}(x) - S_{n-1}(x_0) - s_{n-1}(x_0) \cdot (x - x_0) \right\|$$

$$+ \left\| u_n(x) - u_n(x_0) - u'_n(x_0) \cdot (x - x_0) \right\|$$

$$\leq \left\| u_n(x) - u_n(x_0) - u'_n(x_0) \cdot (x - x_0) \right\|$$

$$+ \left\| r_{n+1}(x) \right\| + \left\| r_{n+1}(x_0) \right\| + \left\| \sum_{k=n+2}^{\infty} u'_n(x_0) \right\| \times \|x - x_0\|$$

for all $x \in X$ and all $n \geq 2$.

By the mean value theorem, the first three terms in the right side can be estimated as follows:

$$\left\| S_{n-1}(x) - S_{n-1}(x_0) - s_{n-1}(x_0) \cdot (x - x_0) \right\| \leq \text{osc} (s_{n-1}, \|x - x_0\|) \times \|x - x_0\|;$$

$$\left\| u_n(x) - u_n(x_0) - u'_n(x_0) \cdot (x - x_0) \right\| \leq 2\|u_n\|_\infty \times \|x - x_0\|;$$

$$\left\| r_{n+1}(x) \right\| + \left\| r_{n+1}(x_0) \right\| + \left\| \sum_{k=n+2}^{\infty} u'_n(x_0) \right\| \times \|x - x_0\| \leq 2\|u_{n+1}\|_\infty \times \|x - x_0\|.$$

Since $\|r_{n+1}\|_\infty \leq \sum_{k=n+2}^{\infty} \|u_k\|_\infty$, it follows from condition (c) that

$$\|r_{n+1}\|_\infty = o(\|u_{n+1}\|_\infty) \quad \text{as } n \to \infty.$$

Finally, by condition (a),

$$\left\| \sum_{k=n+2}^{\infty} u'_k(x_0) \right\| \to 0 \quad \text{as } n \to \infty.$$

These estimates are valid for all $x \in X$ and all $n \geq 2$. Now, if $\|x - x_0\|$ is small enough, there is a uniquely defined integer $n = n(x) \geq n_0$ such that $\|u_{n+1}\|_\infty \leq \|x - x_0\| \leq \|u_n\|_\infty$, and clearly $n(x) \to \infty$ as $x \to x_0$. Using the above estimates and conditions (b) and (d), we see that

$$\left\| f(x) - f(x_0) - \sum_{k=1}^{\infty} u'_k(x_0) \cdot (x - x_0) \right\| = o(\|x - x_0\|)$$

as $x \to x_0$. In other words, $f$ is differentiable at $x_0$, with $f'(x_0) = \sum_1^\infty u'_k(x_0)$. This concludes the proof. \qed

Each function $u_n : \mathbb{R}^d \to \mathbb{R}$ will be constructed on small cubes, and will have the property that the image of each such cube by the gradient mapping $\nabla u_n$ is essentially equal to a segment. Precisely what is needed is stated in the next lemma. Here and afterwards, we denote by $\lambda_d$ the usual Lebesgue measure on $\mathbb{R}^d$. A function defined on a cube $Q$ will be said to be
piecewise constant on $Q$ if $Q$ can be partitioned into finitely many cubes on which the function is constant.

**Lemma 4.4.** Let $a$ be a non-zero vector in $\mathbb{R}^d$, let $Q$ be a cube in $\mathbb{R}^d$, and let $\varepsilon > 0$. Then, there exists a bounded, $C^\infty$-smooth function $u : \mathbb{R}^d \to \mathbb{R}$ satisfying the following properties:

(a) $u$ vanishes in a neighbourhood of $\partial Q$ and $\|u\|_\infty < \varepsilon$;
(b) $\lambda_d\left(\{x \in Q : |\nabla u(x)| = -a\} \cup \{x \in Q : |\nabla u(x)| = a\}\right) \geq (1 - \varepsilon)\lambda_d(Q)$;
(c) one can write $\nabla u = v + w$ with $\|w\|_\infty < \varepsilon$, the set $\{v(x) : x \in Q\}$ is included in the segment $[-a, a]$, and the function $v$ is piecewise constant on $Q$.

**Proof.** By translation and dilation, we may assume that $Q$ is the unit cube $[0, 1]^d$. Let $m$ be a positive number to be chosen later. Let $\varphi : [0, 1] \to \mathbb{R}$ be a $C^\infty$-smooth 1-periodic function such that $\|\varphi\|_\infty \leq 1$ and $\lambda_1(\{t \in [0, 1] : |\varphi'(t)| = 1\}) \geq 1 - \alpha(\varepsilon)$, where $\alpha(\varepsilon) > 0$ will be specified later. Finally, let $\psi : \mathbb{R}^d \to [0, 1]$ be a $C^\infty$-smooth ‘cut-off’ function vanishing on some neighbourhood of $\mathbb{R}^d \setminus \text{int}(Q)$ and such that

$$\lambda_d(\{x \in Q : \psi(x) = 1\}) \geq (1 - \varepsilon/2)\lambda_d(Q).$$

We define the function $u$ on $Q$ by setting

$$u(x) = \frac{\varphi(m\langle x, a \rangle)\psi(x)}{m},$$

where $\langle , \rangle$ is the usual scalar product on $\mathbb{R}^d$. Since $\psi$ vanishes in a neighbourhood of $\partial Q$, one can extend $u$ to a $\mathbb{Z}^d$-periodic, $C^\infty$-smooth function on $\mathbb{R}^d$, still denoted by $u$.

If $m$ is large enough, then condition (a) is satisfied.

To check condition (b), observe that we have

$$\lambda_d(\{x \in Q : |\varphi'(m\langle x, a \rangle)| = 1\}) \geq (1 - \varepsilon/2)\lambda_d(Q)$$

provided $\alpha(\varepsilon)$ is small enough. Together with (4.1), this implies that

$$\lambda_d(\{x \in Q : \psi(x) = 1 \text{ and } |\varphi'(m\langle x, a \rangle)| = 1\}) \geq (1 - \varepsilon)\lambda_d(Q).$$

Computing the derivative of $u$ and noting that $\nabla\psi(x) = 0$ when $\psi(x)$ has the maximal value 1, we see that (b) is satisfied.

We now turn to condition (c). If we set

$$v_1(x) := \varphi'(m\langle x, a \rangle)\psi(x)a \quad \text{and} \quad w_1(x) := \varphi(m\langle x, a \rangle)\nabla\psi(x)/m,$$

we see that $\nabla u = v_1 + w_1$ and that the set $\{v_1(x) : x \in \mathbb{R}^d\}$ is contained in $[-a, a]$. Moreover, we have $\|w_1\|_\infty < \varepsilon/2$ provided $m$ is large enough. We now fix $m$ large enough and $\alpha(\varepsilon)$ small enough, and we choose a positive integer $p$ such that $\text{osc}(v_1, 1/p) < \varepsilon/2$. We define the mapping $v : Q \to \mathbb{R}^d$ as follows: for each $g \in p^{-1}\mathbb{Z}^d \cap Q$ and all $x \in g + p^{-1}Q$, we put $v(x) = v_1(g)$. Then $v$ has values in the segment $[-a, a]$ and is piecewise constant on $Q$. Finally, we set $w = w_1 + v_1 - v$. We have $\nabla u = v + w$ and $\|w\|_\infty \leq \|w_1\|_\infty + \|v_1 - v\|_\infty < \varepsilon$. This concludes the proof of the lemma.

According to condition (a) in Lemma 4.3, the sequence of functions $u_n : \mathbb{R}^d \to \mathbb{R}$ should be constructed in such a way that for all $x \in \mathbb{R}^d$, the series $\sum_{k=1}^n u_k(x)$ is convergent. This will be guaranteed by the next lemma applied to $s_n(x) = \sum_{k=1}^n \nabla u_k(x)$. Here and below, we denote by $\langle , \rangle$ the usual scalar product on $\mathbb{R}^d$.

**Lemma 4.5.** Let $U$ be a bounded open subset of $\mathbb{R}^d$, and let $B$ be a closed ball containing $U$. Then, there exists a map $t : B \to \mathbb{R}^d$ such that the following property holds true: if $(s_n)$ is a sequence in $U$ and if there exists a sequence $(\sigma_n) \in B$ such that $s_n - \sigma_n$ converges and $\langle t(\sigma_n), \sigma_{n+1} - \sigma_n \rangle \geq 0$ for all $n$, then $(s_n)$ converges to some point $s \in \overline{U}$.
Proof. Since \( \mathbb{R}^d \) is super-reflexive (!), player II has a winning tactic in the point–closed slice game for \( B \). Identifying \( \mathbb{R}^d \) with \( (\mathbb{R}^d)^* \), we put \( t(a) = \Phi_a \), where the map \( a \mapsto \Phi_a \) is given by Corollary 3.6. By the definition of \( t \), each sequence \( (\sigma_n) \subset B \) such that \( (t(\sigma_n), \sigma_{n+1} - \sigma_n) \geq 0 \) for all \( n \) is convergent. If \( s_n - \sigma_n \) converges, it follows that the sequence \( (s_n) \) converges to some point \( s \in \mathbb{R}^d \), and of course we have \( s \in U \) if \( s_n \in U \) for all \( n \). \( \square \)

The next lemma follows from the fact that almost sure convergence implies convergence in probability.

**Lemma 4.6.** Let \( (s_n) \) be an almost everywhere convergent sequence of mappings from a probability space \( (\Omega, \mathcal{A}, \mathbb{P}) \) into a Banach space \( X \), and let \( \varepsilon > 0 \). Then \( \mathbb{P}(\|s_{n+1} - s_n\| \geq \varepsilon) \to 0. \)

After these preliminary lemmas, we are now ready to begin the proof of Theorem 4.2, but first, we introduce some terminology.

We shall consider **cube partitions** of \( \mathbb{R}^d \), that is, partitions of \( \mathbb{R}^d \) into half-open cubes. By a \( \mathbb{Z}^d \)-periodic cube partition of \( \mathbb{R}^d \), we mean a cube partition consisting of \( \mathbb{Z}^d \)-translates of some finite cube partition of the unit cube \( Q_0 = [0,1]^d \). We say that a partition \( \mathcal{Q}' \) is a **refinement** of a partition \( \mathcal{Q} \) if each cube \( Q \in \mathcal{Q} \) can be decomposed into finitely many cubes \( Q' \in \mathcal{Q}' \).

**Proof of Theorem 4.2.** The unit cube \( Q_0 = [0,1]^d \) and the bounded open set \( U \subset \mathbb{R}^d \) are given. For the proof, we fix once and for all a decreasing sequence of real numbers \( (\varepsilon_k) \) such that \( 0 < \varepsilon_k < 1 \) and \( \lim_{k \to \infty} \varepsilon_k = 0 \). Finally, for each \( \varepsilon > 0 \), we put

\[
\partial U_\varepsilon = \{ x \in U : \text{dist}(x, \partial U) < \varepsilon \}.
\]

Up to a constant, the function \( u \) will be the sum of a uniformly convergent series \( \sum_{n} u_n \) of non-zero, real-valued, \( \mathbb{Z}^d \)-periodic, \( C^\infty \)-smooth functions defined on \( \mathbb{R}^d \). We put \( s_n := \sum_{k=1}^{n} \nabla u_k \). Each function \( \nabla u_n \) will be of the form \( v_n + w_n \), and we put \( \sigma_n := \sum_{k=1}^{n} v_n \). Together with the sequence \( (u_n) \), we will construct a sequence \( (\sigma_n) \), where each \( \sigma_n \) is a \( \mathbb{Z}^d \)-periodic cube partition of \( \mathbb{R}^d \) and \( \mathcal{Q}_{n+1} \) is a refinement of \( \mathcal{Q}_n \). Finally, we will also define an increasing sequence of integers \( (N_k) \). The following conditions have to be fulfilled:

(o) \( N_0 = 0 \), \( u_0 = 0 \) is constant, \( v_0 = 0 = w_0 \), the partition \( \mathcal{Q}_0 \) is the family of all \( \mathbb{Z}^d \)-translates of \( Q_0 = [0,1]^d \), and for each \( n \geq 1 \), the function \( u_n \) is constant on a neighbourhood of \( \partial Q_0 \);

(i) \( v_n \) is constant on each cube \( Q \in \mathcal{Q}_n \);
(ii) \( \|u_n\|_\infty \leq 2^{-n} \);
(iii) \( s_n(x) \in U \) for all \( x \in \mathbb{R}^d \);
(iv) \( \|u_n(x)\| \leq R + 1 \) for all \( x \in \mathbb{R}^d \), where \( R = \sup \{ \|s\| : s \in U \} \);
(v) \( t(\sigma_n(x)) - \sigma_{n+1}(x) - \sigma_n(x) = 0 \) for all \( x \in \mathbb{R}^d \), where \( t \) is is the mapping given by Lemma 4.5 with \( B = \{ x \in \mathbb{R}^d : \| x \| \leq R + 1 \} \);
(vi) \( \|u_{n+1}\|_\infty \leq 2^{-n} \|u_n\|_\infty \) for all \( n \), and if \( N_{k-1} < n \leq N_k \), then \( \|v_n\|_\infty \leq \varepsilon_k/4 \) and \( \text{osc}(s_n, \|u_{n+1}\|_\infty) < \varepsilon_k/4 \);

(iii) for each \( k \geq 1 \), we have

\[
\lambda_d(\{ x \in [0,1]^d : s_{N_k}(x) \notin \partial U_{\varepsilon_k} \}) \leq 2^{-k}.
\]

**Inductive step.** Let us fix \( k \geq 1 \). Assume \( N_{k-1} \) has been defined, and that \( u_n \) has been constructed for some \( n \geq N_{k-1} \). By (i), the function \( \sigma_n \) is constant on each cube \( Q \in \mathcal{Q}_n \); we denote by \( \sigma_n(Q) \) the value taken by \( \sigma_n \) on such a cube \( Q \), and we choose some vector \( a = a(Q) \in \mathbb{R}^d \) such that \( \|a\| = \varepsilon_k/4 \) and \( \langle t(\sigma_n(Q)), a \rangle = 0 \); this can be done because \( d \geq 2 \). Finally, we choose a \( \mathbb{Z}^d \)-periodic cube partition \( \tilde{\mathcal{Q}}_n \) of \( \mathbb{R}^d \) refining \( \mathcal{Q}_n \), such that the oscillation of \( s_n \) on each cube \( Q \in \tilde{\mathcal{Q}}_n \) is less than \( \varepsilon_k/4 \). Recall that \( u_n \neq 0 \). Applying Lemma 4.4 to all
cubes $\tilde{Q} \in \tilde{\Omega}_n$, we see that one can construct a $\mathbb{Z}^d$-periodic, $C^\infty$-smooth function $\tilde{u} : \mathbb{R}^d \to \mathbb{R}$ and a $\mathbb{Z}^d$-periodic cube partition $\Omega_{n+1}$ refining $\Omega_n$ (hence a refinement of $\Omega_n$), such that

- (a) $\|\tilde{u}\|_\infty < 2^{-n}\|u\|_\infty$ and $\text{osc}(s_n, \|\tilde{u}\|_\infty) < \varepsilon_k/4$;
- (b) $\lambda_d(x \in Q : \nabla u(x) = \pm a) \geq (1 - 2^{-d})\lambda_d(Q)$ for each cube $Q \in \Omega_n$;
- (c) one can write $\nabla \tilde{u} = \tilde{v} + \tilde{w}$, where $\|\tilde{u}\| < \varepsilon_k/2^{n+2}$, the function $\tilde{u}$ is constant on each cube of the partition $\Omega_{n+1}$, and $\tilde{v}(Q) \subset [-\alpha(Q); \alpha(Q)]$ for each cube $\tilde{Q} \in \tilde{\Omega}_n$, where $Q$ is the unique cube of the partition $\Omega_n$ containing $Q$;
- (d) the function $\tilde{u}$ vanishes on a neighbourhood of $\partial \tilde{Q}$, for each cube $\tilde{Q} \in \tilde{\Omega}_n$.
Notice that $\tilde{u} \neq 0$. Let $c \in \mathbb{R}^d$ be a non-zero vector such that $\|c\| < \|\tilde{u}\|_\infty$ and $\|\tilde{u}\|_\infty + \|c\| \leq 2^{-n}\|u\|_\infty$.

Now, we define the function $u_{n+1}$ on each cube $\tilde{Q} \in \tilde{\Omega}_n$. Let us choose a point $g\tilde{Q} \in \tilde{Q}$ for each such cube $\tilde{Q}$, assuming (as we may) that this choice is compatible with the $\mathbb{Z}^d$-periodicity of $\Omega_n$.

If $s_n(gQ) \in \partial U_{3\varepsilon_k/4}$, we set $u_{n+1} = c$ on $\tilde{Q}$, and $v_{n+1} = w_{n+1} = 0$.

If $s_n(gQ) \notin \partial U_{3\varepsilon_k/4}$, we set $u_{n+1} = \tilde{u} + c$ on $\tilde{Q}$, and accordingly, $v_{n+1} = \tilde{v}$ and $w_{n+1} = \tilde{w}$ on $\tilde{Q}$. In this case, we have

$$\lambda_d\left(\{x \in Q : \|\nabla u_{n+1}(x)\| = \varepsilon_k/4\}\right) \geq (1 - 2^{-d})\lambda_d(Q). \tag{4.2}$$

The function $u_{n+1}$ is $\mathbb{Z}^d$-periodic, and it is $C^\infty$-smooth because the auxiliary function $\tilde{u}$ is smooth and vanishes on a neighbourhood of $\partial \tilde{Q}$, for each cube $\tilde{Q} \in \tilde{\Omega}_n$. Notice also that $u_{n+1} \neq 0$: this is clear if $s_n(gQ) \notin \partial U_{3\varepsilon_k/4}$ for at least one cube $Q$, and otherwise it is also clear because $\|c\|_\infty < \|\tilde{u}\|_\infty$.

Conditions (o), (i) and (ii) for $n+1$ are clearly satisfied, as well as (vi) (though the integer $N_k$ is not yet defined). Condition (iv) for $n+1$ will follow from (iii) and the inequality

$$\|s_{n+1} - \sigma_{n+1}\| \leq \sum_{k=1}^{n+1} \|w_k(x)\| \leq 1.$$

Let us check condition (iii) for $n+1$. Let $x \in \mathbb{R}^d$, and choose $\tilde{Q} \in \tilde{\Omega}_n$ such that $x \in \tilde{Q}$. If $s_n(gQ) \in \partial U_{3\varepsilon_k/4}$, then $s_{n+1}(x) = s_n(x) \in U$ by the induction hypothesis. If $s_n(gQ) \notin \partial U_{3\varepsilon_k/4}$, then, since the oscillation of $s_n$ on $\tilde{Q}$ is less than $\varepsilon_k/4$, we have

$$\text{dist}(s_n(x), \partial U) \geq 3\varepsilon_k/4 - \|s_n(x) - s_n(gQ)\| \geq \varepsilon_k/2,$$

and so $s_n(x) \notin \partial U_{\varepsilon_k/2}$. Observing that

$$s_{n+1}(x) = s_n(x) + \nabla u_{n+1}(x)$$

and

$$\|\nabla u_{n+1}(x)\| \leq \|v_{n+1}(x)\| + \|w_{n+1}(x)\| \leq \varepsilon_k/2,$$

we conclude that $s_{n+1}(x) \in U$.

Let us prove (v). If $Q \in \Omega_n$ and $x \in Q$, then $v_{n+1}(x)$ is proportional to $a(Q)$, and hence orthogonal to $t(\sigma_n(Q))$. Since $v_{n+1}(x) = \sigma_{n+1}(x) - \sigma_n(x)$ and $\sigma_n(x) = \sigma_n(x)$, this gives (v).

Now, we show that if we continue this construction, then we will find $N_k > N_{k-1}$ satisfying (vii). Assume by contradiction that for all $n > N_{k-1}$,

$$\lambda_d\left(\{x \in [0,1]^d : s_n(x) \notin \partial U_{\varepsilon_k}\right)} \geq 2^{-k}. \tag{4.3}$$

If $\tilde{Q} \in \tilde{\Omega}_n$ is a cube that meets $\{x \in [0,1]^d : s_n(x) \notin \partial U_{\varepsilon_k}\}$, then $s_n(g\tilde{Q}) \notin \partial U_{3\varepsilon_k/4}$ because the oscillation of $s_n$ on $\tilde{Q}$ is less than $\varepsilon_k/4$. By (4.2), it follows that for every such cube $\tilde{Q}$, we have

$$\lambda_d\left(\{y \in \tilde{Q} : \|s_{n+1}(y) - s_n(y)\| \geq \varepsilon_k/4\}\right) \geq (1 - 2^{-k})\lambda_d(\tilde{Q}).$$
On the other hand, condition (4.3) implies that the proportion of cubes $\tilde{Q} \in \tilde{\Omega}_n$ that meet \{ $x \in [0,1]^d$: $s_n(x) \notin \partial U_{\varepsilon_k}$ \} is at least $2^{-k}$. Therefore,

$$\lambda_d(\{ y \in [0,1]^d: \|s_{n+1}(y) - s_n(y)\| \geq \varepsilon/4 \}) \geq (1 - 2^{-k}) \times 2^{-k}.$$ (4.4)

This will contradict Lemma 4.6 if we can prove that the sequence $(s_n)$ is pointwise convergent. Now, it follows from (ii) that $s_n(x) - \sigma_n(x) = \sum_{k=1}^{n} w_k(x)$ converges at each point $x \in \mathbb{R}^d$, so that conditions (iii), (iv), (v) allow us to apply Lemma 4.5 to conclude that $(\nabla u)$ exists a 1-Lipschitz function

The function $u$. Let us denote by $c_n$ the constant value of $u_n$ on $\partial Q_0$. By (vi), we can put $c := \sum_{k=1}^{+\infty} c_n$ and define

$$u := -c + \sum_{n=1}^{+\infty} u_n.$$ 

The function $u$ is $\mathbb{Z}^d$-periodic. To show that it is also differentiable, we check the conditions of Lemma 4.3. For each $n \in \mathbb{N}$, let $k_n$ be the unique positive integer such that $N_{k_n-1} < n \leq N_{k_n}$. From (ii) and (vi), we get $\|u_n\|_\infty \leq \|u_n\|_\infty + \|u_n\|_\infty \leq \varepsilon_{k_n} + 2^{-n}$, so that $\|u_n\|_\infty$ tends to 0. By (vi), we have $\|u_n+1\|_\infty = o(\|u_n\|_\infty)$, and osc($s_n, \|u_n+1\|_\infty) < \varepsilon_{k_n}/4 \to 0$. Moreover, it follows as above from (ii)–(v) and Lemma 4.5 that the sequence $(s_n)$ is pointwise convergent; that is, the series $\sum u'_n(x)$ converges at every point $x \in \mathbb{R}^d$. Thus, one can apply Lemma 4.3 to conclude that $u$ is everywhere differentiable, and that $\nabla u$ is the pointwise limit of the sequence $s_n$.

The function $u$ vanishes on $\partial Q_0$, and by (o) we also have $\nabla u = 0$ on $\partial Q_0$. It follows from (ii) that $\nabla u(x) \in \overline{U}$ for all $x \in \mathbb{R}^d$. Finally, condition (vii) implies that if $k \leq \ell$, then

$$\lambda_d(\{ x \in [0,1]^d: s_{N_k}(x) \notin \partial U_{\varepsilon_k} \}) \leq 2^{-k}.$$ 

Sending $\ell$ to $\infty$, we get $\lambda_d(\{ x \in [0,1]^d: \nabla u(x) \notin \partial U_{\varepsilon_k} \}) \leq 2^{-k}$ for all $k \in \mathbb{N}$; and sending $k$ to $\infty$, we conclude that $\nabla u(x) \in \partial U$ for almost every $x \in \mathbb{R}^d$.

**Corollary 4.7.** Let $\Omega$ be an open subset of $\mathbb{R}^d$, and let $x_0 \in \Omega$. Also let $U$ be a bounded open subset of $\mathbb{R}^d$ containing 0, and put $K := \sup\{ \|y\|: y \in U \}$. Then, there exists a $K$-Lipschitz function $u: \overline{\Omega} \to \mathbb{R}$ with the following properties:

1. $u$ is bounded and everywhere differentiable in $\Omega$, with $\nabla u(\Omega) \subset U$;
2. $u(\xi) = 0$ for all $\xi \in \partial \Omega$;
3. $\nabla u(x_0) = 0$ and $\nabla u(x) \in \partial U$ for almost every $x \in \Omega$.

When $U$ is the open unit ball, this gives the result announced in the introduction: there exists a 1-Lipschitz function $u: \overline{\Omega} \to \mathbb{R}$, differentiable on $\Omega$, such that $u(x) = 0$ for all $x \in \partial \Omega$, $\nabla u(x_0) = 0$ and $\|\nabla u(x)\| = 1$ almost everywhere. More generally, Corollary 4.7 gives the existence of non-trivial, differentiable, almost everywhere solutions of the equation $F(\nabla u) = 0$, for any continuous function $F: \mathbb{R}^d \to \mathbb{R}$ such that $F(0) \neq 0$ and the connected component of the set $\{F \neq 0\}$ containing 0 is bounded.

Notice also that the condition $\nabla u(x_0) = 0$ is not really essential when the open set $\Omega$ is bounded. Indeed, by Rolle’s theorem, the boundary condition (2) forces $\nabla u$ to vanish somewhere in $\Omega$.

**Proof of Corollary 4.7.** Let $\Omega$ be a locally finite cube partition of the open set $\Omega$, with $\partial Q \subset \Omega$ for all $Q \in \Omega$ and $x_0 \in \partial Q_0$ for some cube $Q_0 \in \Omega$. By translation and dilation, it follows from Theorem 4.2 that for each cube $Q \in \Omega$, one can find an everywhere differentiable
function \( u_Q : \mathbb{R}^d \to \mathbb{R} \) such that \( \nabla u_Q(x) \in \mathcal{U} \) for all \( x \in \mathbb{R}^d \), \( \nabla u_Q(x) \in \partial U \) almost everywhere, and \( u_Q = 0, \nabla u_Q = 0 \) on \( \partial Q \). We define \( u \) on \( \Omega \) in the obvious way: \( u = u_Q \) on each cube \( Q \in \mathcal{Q} \). Then \( u \) is everywhere differentiable on \( \Omega \). Moreover, we have \( \nabla u(x) \in \mathcal{U} \) for all \( x \in \Omega \), so, by the mean value theorem, the restriction of \( u \) to the closure of each cube \( Q \in \mathcal{Q} \) is \( K \)-Lipschitz. Since \( u \equiv 0 \) on the boundary of each cube of the partition \( \mathcal{Q} \), it follows that \( u \) is in fact \( K \)-Lipschitz on \( \Omega \). Indeed, given \( x_1, x_2 \in \Omega \), one can find \( Q_1, Q_2 \in \mathcal{Q} \) such that \( x_i \in Q_i \), and the line segment \([x_1, x_2]\) intersects both \( \partial Q_1 \) and \( \partial Q_2 \), say at points \( q_1 \) and \( q_2 \). Since \( u(q_1) = 0 = u(q_2) \), we have
\[
\|u(x_2) - u(x_1)\| \leq \|u(x_2) - u(q_2)\| + \|u(q_1) - u(x_1)\| \\
\leq K (\|x_2 - q_2\| + \|q_1 - x_1\|) \\
\leq K \|x_2 - x_1\|.
\]
Therefore, \( u \) can be extended to a \( K \)-Lipschitz function on \( \Omega \) with the required properties. Notice that the boundary condition (2) is satisfied because \( u \) vanishes on the boundary of each cube of the partition \( \mathcal{Q} \) and the closure of \( \bigcup \{\partial Q : Q \in \mathcal{Q}\} \) contains \( \partial \Omega \).

**Remark 4.** In addition to the boundary condition \( u|_{\partial \Omega} = 0 \), one may also impose the condition ‘\( \nabla u = 0 \) on the boundary’, that is, \( u(x) = o(\|x - \xi\|) \) as \( x \to \xi \in \partial \Omega \). Actually, given any positive function \( \phi \) on \((0, \infty)\) such that \( \inf \{\phi(t) : t \geq \alpha\} > 0 \) for each \( \alpha > 0 \), one may require that \( |u(x)| \leq \phi(\text{dist}(x, \partial \Omega)) \) for all \( x \in \Omega \).

**Proof.** Let the function \( \phi \) be given. If we denote by \( Q_x \) the unique cube \( Q \in \mathcal{Q} \) containing \( x \in \Omega \), then \( |u(x)| \leq K \text{diam}(Q_x) \), because \( u \) is \( K \)-Lipschitz and vanishes on \( \partial Q_x \). Thus, it is enough to show that the partition \( \mathcal{Q} \) can be chosen in such a way that \( \text{diam}(Q_x) \leq \phi(\text{dist}(x, \partial \Omega)) \) for all \( x \to \partial \Omega \). To do this, start with a cube partition \( \mathcal{P} \), and let \( (\delta_k) \) be a sequence of positive numbers. For each \( k \in \mathbb{N} \), set
\[
\mathcal{P}_k = \left\{ P \in \mathcal{P} : \frac{1}{k+1} \leq \text{dist}(P, \partial \Omega) < \frac{1}{k} \right\}
\]
and subdivide each cube \( P \in \mathcal{P}_k \) into finitely many cubes \( Q \) of diameter less than \( \delta_k \). This gives a cube partition \( \mathcal{Q} \) with the required additional property if the sequence \( (\delta_k) \) is suitably chosen.

**Remark 5.** Very recently, M. Zelený was able to construct a differentiable function \( u \) on \( \mathbb{R}^d \) such that the set \( (\nabla u)^{-1}(B(0, 1)) \) is non-empty and has Hausdorff dimension 1 [12]. As far as the Denjoy–Clarkson property is concerned, this may be viewed as the ‘optimal’ improvement of Buczolich’s example. Indeed, Buczolich had shown earlier that for any differentiable function \( u \) on \( \mathbb{R}^d \), the set \( (\nabla u)^{-1}(B(0, 1)) \) is either empty or has positive 1-dimensional Hausdorff measure [2]. Zelený’s proof goes along the same lines as in [9], but several delicate arguments from geometric measure theory are additionally needed.

**Acknowledgements.** The authors would like to thank the anonymous referee for a careful reading of the paper.

**References**


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