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INFINITE GAMES, BANACH SPACE GEOMETRY AND THE EIKONAL EQUATION

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Abstract

We study a class of infinite games which turns out to be closely related to Banach space geometry. Using one of these games, we construct a bounded, differentiable, almost everywhere solution of the Eikonal equation $\|\nabla u\| = 1$ on $\mathbb{R}^d$, with $d \geqslant 2$.

1. *Introduction*

It is well known that derivative functions have many interesting properties. The following beautiful result goes back to A. Denjoy: *if $u : \mathbb{R} \to \mathbb{R}$ is everywhere differentiable, then, for any open set $U \subset \mathbb{R}$, the set $\{x\colon u'(x) \in U\}$ is either empty or has positive Lebesgue measure*; this property of derivatives is usually called the Denjoy–Clarkson property. The problem of extending Denjoy's result to functions of several variables was raised in the 1960s by C. E. Weil [**11**], and since then it has been known as the Weil gradient problem. This problem was solved in 2002 by Z. Buczolich [**3**], who constructed an everywhere differentiable function $u : \mathbb{R}^2 \to \mathbb{R}$ such that $\nabla u(0) = 0$ but $\|\nabla u(x)\| \geqslant 1$ for almost all points $x \in \mathbb{R}^2$; to see that ∇u fails the Denjoy–Clarkson property, just consider the open unit ball $U \subset \mathbb{R}^2$.

The function u above is the limit of a sequence of smooth functions (s_n), and in order to ensure the differentiability of u, one needs to 'force' the sequence (∇s_n) to converge at each point. Buczolich's original arguments are very intricate, and they have been greatly simplified by J. Malý and M. Zelený [**9**]. The key new tool introduced in [**9**] is the following very interesting infinite game. There are two players, **I** and **II**. Player **I** starts the game by playing a point a_0 in the open unit ball of $\mathbb{R}^2$. Player **II** answers by playing a line $L_0 \subset \mathbb{R}^2$, which must pass through the point a_0. Then player **I** plays a point a_1 in the open unit ball, which must belong to the line L_0. Player **II** answers with a line L_1 passing through a_1, and so on. Player **II** wins if the infinite sequence (a_n) produced by the game is convergent in $\mathbb{R}^2$. This game is called the *point–line game* in [**9**]. One of the main results of [**9**] is that player **II** has a winning strategy in the point–line game, which is exactly what is needed to simplify Buczolich's proof.

In the present paper, we study some more general versions of the point–line game. Instead of $\mathbb{R}^2$, one can consider a Banach space X. In this setting, the obvious analogue of the point–line game should now be called the *point–hyperplane game* for the open unit ball B_X. Rather unexpectedly, it turns out that this is not merely a formal generalization, and that the point–hyperplane game is in fact closely related to the geometry of the underlying Banach space X. Indeed, we show that player **II** has a winning strategy in the point–hyperplane game for B_X if, and only if, X has the Radon–Nikodym property (RNP).

At first sight, the appearance of the Radon–Nikodym property may look a bit surprising, but it becomes quite natural if one decides to change the rule of the game by requiring player **II** to play *slices* of the unit ball rather than hyperplane sections. More explicitly, the new rule is the following: player **I** starts the game by playing a point $a_0 \in B_X$, player **II** answers with a closed

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half-space S_0 containing a_0 as a boundary point, player **I** then plays a point $a_1 \in B_X \cap S_0$, and so on. Since all the half-spaces played by **II** determine slices of the open unit ball, we call this modified game the *point–slice game* for B_X. Formally, the point–slice game is more difficult to win for player **II** than the point–hyperplane game, since it gives more freedom to player **I**. Yet, we show that if X has the RNP, then **II** has a winning strategy in that game as well.

If the Banach space X is very well behaved, one can get a stronger conclusion. Indeed, we show that if X has a uniformly convex renorming, then player **II** even has a winning *tactic* in the point–slice game, that is, a winning strategy in which at each step, the answer to the points $a_0, \ldots, a_n$ already played by **I** depends only on the last move a_n. In other words, one can associate to each point $a \in B_X$ a closed half-space $S(a)$ containing a as a boundary point in such a way that, whatever the moves of player **I** in the point–slice game may be, player **II** is sure to win if she answers $S(a_n)$ to each move a_n of player **I**. Put in a slightly different way, this means that one can associate to each point $a \in B_X$ a linear functional $\Phi_a \in X^*$ in such a way that the following property holds true: *each sequence* $(a_n) \subset B_X$ *satisfying* $\langle \Phi_{a_n}, a_{n+1}\rangle \geqslant \langle \Phi_{a_n}, a_n\rangle$ *for all* $n \in \mathbb{N}$ *is convergent.* This can be viewed as a Banach space version of the fact that each bounded monotonic sequence of real numbers is convergent.

One may also consider other games of the same type, where player **II** is required to play members of some fixed family $\mathcal{A}$ of affine subspaces of X. When $\mathcal{A}$ is the family of all *finite-codimensional* affine subspaces of X, this leads to another well-known Banach space property, namely the point of continuity property (PCP). We show that player **II** has a winning strategy in the point–finite-codimensional subspace game for B_X if and only the Banach space X has the PCP.

Note that even in $\mathbb{R}^2$, the existence of a winning strategy for player **II** in the point–line game or the point–slice game for the unit ball is a non-trivial result. At first sight, one might think that player **II** should win by answering to each play $a_n \neq 0$ of player **I** the line $L(a_n)$ passing through a_n and orthogonal to $\mathbb{R}a_n$. This is indeed natural because among all line sections of the unit ball passing through a_n, the one with smallest diameter is precisely $L(a_n)$. However, this 'orthogonal strategy' does not work. Indeed, let us define a sequence (a_n) as follows: in polar coordinates, a_n is given by (r_n, θ_n), where

$$r_n = \prod_{k=n+1}^{\infty} \cos\left(\frac{1}{k}\right) \qquad \text{and} \qquad \theta_n = \sum_{k=1}^{n} \frac{1}{k},$$

with the convention $\theta_0 = 0$. Since the line $L(a_n)$ is given in polar coordinates by

$$r = \frac{r_n}{\cos(\theta - \theta_n)},$$

we see that $a_{n+1} \in L(a_n)$ for all n. Thus, if player **II** follows the orthogonal strategy, then player **I** is allowed to play $a_0, a_1, \ldots$. But since r_n tends to 1 and the sequence (θ_n) goes slowly to $+\infty$, the sequence (a_n) is not convergent. Thus, player **I** has won the game.

Incidentally, this example shows that player **II** can lose the point–slice game even if she plays slices of the unit ball whose diameters tend to 0. Of course, **II** can sometimes win if the diameters of the slices do *not* tend to 0, for example if **I** decides to lose by always playing the same point. Notice also that a strategy for player **II** which would always produce a *non-increasing* sequence of slices cannot be winning for **II**. Indeed, assume that player **II** follows such a strategy S. Then **I** can force **II** to play always the same slice, simply by choosing at each step the point a_{n+1} on the boundary of the half-space S_n just played by **II**. Thus, choosing any point a_0 and then a point $a_1 \neq a_0$ on the boundary of $S(a_0)$, player **I** wins the game if she plays alternatively the two points a_0, a_1.

The paper is organized as follows.

The first two parts deal with games. We first consider very abstract 'point–set' games and prove two general results concerning the existence of winning strategies or tactics for player **II**.

Then we apply these results in the Banach space setting. As explained above, this leads to characterizations of the Radon–Nikodym property and the point of continuity property, and to the existence of a winning tactic for player **II** in the point–slice game if the underlying Banach space X has a uniformly convex renorming.

In the final part, we elaborate a bit on Buczolich's example and we prove the following slightly stronger result: *if $d \geqslant 2$, then there exists a bounded differentiable function $u : \mathbb{R}^d \to \mathbb{R}$ such that $\nabla u(0) = 0$ and u satisfies the Eikonal equation $\|\nabla u(x)\| = 1$ almost everywhere;* here, $\|\cdot\|$ is an arbitrary norm on $\mathbb{R}^d$. More generally, we prove that if Ω is an open subset of $\mathbb{R}^d$, with $d \geqslant 2$, and if $x_0 \in \Omega$ is given, then there exists a 1-Lipschitz function $u : \overline{\Omega} \to \mathbb{R}$, which is bounded and differentiable at every point of Ω, such that $\nabla u(x_0) = 0$ and $\|\nabla u(x)\| = 1$ almost everywhere in Ω; moreover, u satisfies the boundary condition $u_{|\partial\Omega} = 0$. Thus, the Eikonal equation admits some rather exotic almost everywhere solutions, very different from the usual viscosity solution $\operatorname{dist}(\,\cdot\,, \partial\Omega)$, which is not everywhere differentiable.

2. *Abstract games*

Let (E, d) be a (non-empty) metric space, and for each point $x \in E$, let $\mathcal{A}(x)$ be a (non-empty) family of subsets of E containing x. We denote by $\mathcal{A}$ the disjoint union of all families $\mathcal{A}(x)$, and we define a game $\mathbf{G}(E, \mathcal{A})$ as follows. There are two players, **I** and **II**. Player **I** plays points $a_0, a_1, \ldots \in E$, and player **II** plays sets $A_0, A_1, \ldots \subset E$. Once player **I** has played a point a_n, player **II** must choose a set $A_n \in \mathcal{A}(a_n)$. Then, player **I** must choose the point a_{n+1} inside A_n. Player **II** wins if the sequence (a_n) is d-Cauchy, otherwise player **I** wins.

THEOREM 2.1. *Assume there exists a family $\mathcal{C}$ of subsets of E such that the following properties hold:*

(0) *$\emptyset, E \in \mathcal{C}$;*
(1) *$\mathcal{C}$ is closed under intersections;*
(2) *if $C_1, C_2 \in \mathcal{C}$ satisfy $C_1 \subset C_2$ and $C_2 \setminus C_1 \neq \emptyset$ then, for each $\varepsilon > 0$, one can find $C \in \mathcal{C}$ such that $C_1 \subset C \subset C_2$, $C_2 \setminus C \neq \emptyset$ and $\operatorname{diam}(C_2 \setminus C) < \varepsilon$;*
(3) *for each $C \in \mathcal{C}$ and each point $x \in E \setminus C$, there exists $A \in \mathcal{A}(x)$ such that $A \cap C = \emptyset$.*

*Then player **II** has a winning strategy in the game $\mathbf{G}(E, \mathcal{A})$.*

The proof is based on the following lemma, which follows easily from (1) and (2). If I is a set, we denote by $I^{<\omega}$ the set of all finite sequences of elements of I. If $s \in I^{<\omega}$ and $i \in I$, the sequence 's followed by i' is denoted by $s * i$. If I is a well-ordered set and $s \in I^{<\omega}$ has the form $t * i$, where $i \in I$ has a predecessor i^- in I, we denote by s^- the sequence $t * i^-$, and we say that s^- is the *predecessor* of s. This is indeed the predecessor of s in the ordering of $I^{<\omega}$ defined by putting the lexicographic ordering on each set I^n, with $n \in \mathbb{N}$ and declaring that two sequences are comparable only if they have the same length. Finally, we denote by $|s|$ the length of a sequence $s \in I^{<\omega}$.

LEMMA 2.2. *There exist some ordinal η and a family $(C_s)_{s\in\eta^{<\omega}}$ of subsets of E such that:*

(o) *$C_s \in \mathcal{C}$ for each $s \in \eta^{<\omega}$, with $C_\emptyset = \emptyset$;*
(i) *$C_{s*0} = E$ for each $s \in \eta^{<\omega}$;*
(ii) *for each $s \in \eta^{<\omega}$, the transfinite sequence $(C_{s*\xi})_{\xi<\eta}$ is non-increasing, with $\bigcap_{\xi<\lambda} C_{s*\xi} = C_{s*\lambda}$ for limit ordinals λ;*
(iii) *$\bigcap_{\xi<\eta} C_{s*\xi} = C_s$ for each $s \in \eta^{<\omega}$;*
(iv) *if $s \in \eta^{<\omega}$ has a predecessor, then $\operatorname{diam}(C_{s^-} \setminus C_s) < 2^{-|s|}$.*

Proof. Let η be a limit ordinal greater than the cardinal number of E. Using (1) and (2), one can construct a non-increasing transfinite sequence $(C_\xi)_{\xi<\eta} \subset \mathcal{C}$ such that $C_0 = E$, $\bigcap_{\xi<\lambda} C_\xi = C_\lambda$ for limit ordinals λ, and $C_\xi \setminus C_{\xi+1} \neq \emptyset$, $\operatorname{diam}(C_\xi \setminus C_{\xi+1}) < 2^{-1}$ if $C_\xi \neq \emptyset$. For cardinality reasons, we have $\bigcap_{\xi<\eta} C_\xi = \emptyset$. Thus, we have found our sets C_s for all sequences $s \in \eta^{<\omega}$ of length 1. Clearly, this process can be repeated, so that one can construct C_s for all $s \in \eta^{<\omega}$ by induction on the length of a sequence s. □

Proof of Theorem 2.1. Let $(C_s)_{s\in\eta^{<\omega}}$ be the family of sets given by Lemma 2.2. Notice that if $s \in \eta^{<\omega}$ and $a \in E \setminus C_s$, then, by (iii) there exists a smallest ordinal $\xi < \eta$ such that $a \notin C_{s*\xi}$, and by (i) and (ii) this ordinal is a successor ordinal. It follows that for each point $a \in E$, there is a uniquely defined sequence of successor ordinals $\mathbf{s}(a) = (\xi_0(a), \xi_1(a), \dots) \in \eta^\omega$ such that $a \in C_{(\mathbf{s}(a)_{|n})^-} \setminus C_{\mathbf{s}(a)_{|n}}$ for each non-zero $n \in \omega$. Explicitly, $\xi_0(a) = \min\{\xi < \eta\colon a \notin C_\xi\}$ and $\xi_{n+1}(a) = \min\{\xi\colon a \notin C_{\mathbf{s}(a)_{|n}*\xi}\}$. The strategy of player **II** is defined as follows: once **I** has played a_n, **II** chooses a set $A_n \in \mathcal{A}(a_n)$ such that $A_n \cap C_{\mathbf{s}(a_n)_{|n}} = \emptyset$. This is possible by condition (3) above.

Let $a_0, A_0, a_1, A_1, \dots$ be a run of the game $\mathbf{G}(E, \mathcal{A})$, where **II** has played according to his strategy. Observe that if $k \leqslant n$, then $C_{\mathbf{s}(a)_{|k}} \subset C_{\mathbf{s}(a)_{|n}}$ for all $a \in E$: this follows from (iii). Since $a_{n+1} \in A_n$, it follows that $a_{n+1} \notin C_{\mathbf{s}(a_n)_{|k}}$ whenever $k \leqslant n$. Consequently, for each fixed $k \in \omega$, the sequence $(\mathbf{s}(a_n)_{|k})_{n\geqslant k}$ is non-increasing in the well-ordered set η^k, and hence stationary. Thus, we get an infinite sequence of successor ordinals $\mathbf{s} \in \eta^\omega$ such that, for each fixed $k \in \omega$, we have $a_n \in C_{(\mathbf{s}_{|k})^-} \setminus C_{\mathbf{s}_{|k}}$ for all large enough n. By (iv), this implies that the sequence (a_n) is Cauchy. Thus, player **II** has won the game. □

The proof of Theorem 2.1 shows that player **II** has a winning strategy of a very special type: at step n of the game, the set A_n depends only on n and on the nth move of player **I**. In other words, the strategy of player **II** is given by a sequence of *tactics*: there is a sequence of maps $t_n : E \to \mathcal{A}$ such that **II** wins the game by answering $t_n(a_n)$ when **I** has played a_n. By strengthening condition (3) in Theorem 2.1, one can ensure that player **II** has in fact a single winning tactic in the game $\mathbf{G}(E, \mathcal{A})$. This is the content of the next theorem. We shall say that a sequence (D_n) of subsets of E *accumulates* to some point $x \in E$ if every neighbourhood of x contains all but finitely many sets D_n.

THEOREM 2.3. *Assume that the metric space (E, d) is bounded, and that there exist a point $a \in E$ and a sequence $(\mathcal{C}^n)_{n\in\mathbb{N}}$ of families of non-empty subsets of E satisfying the following properties:*

(0) *$\mathcal{C}^n \subset \mathcal{C}^{n+1}$ and $\overline{B}(a, r) \in \mathcal{C}^n$ for all $r \geqslant 0$;*
(1) *if $(C_i)_{i\in I}$ is a family of members of $\mathcal{C}^n$ such that $\bigcap_i C_i$ has non-empty interior, then $\bigcap_i C_i \in \mathcal{C}^n$;*
(2) *if $K_1 \in \mathcal{C}^n$ and $K_2 \in \bigcup_{p\in\mathbb{N}} \mathcal{C}^p$ satisfy $K_1 \subset K_2$ and $K_2 \setminus K_1 \neq \emptyset$, then, for each $\varepsilon > 0$, one can find $K \in \mathcal{C}^{n+1}$ such that $K_1 \subset K \subset K_2$, $K_2 \setminus K \neq \emptyset$ and $\operatorname{diam}(K_2 \setminus K) < \varepsilon$;*
(3) *if (C_n) is a non-decreasing sequence of subsets of E with $C_n \in \mathcal{C}^n$ for all n such that $(C_{n+1} \setminus C_n)$ accumulates to some point $x \in E \setminus \bigcup_n C_n$, then one can find $A \in \mathcal{A}(x)$ such that $A \cap \bigcup_n C_n = \emptyset$.*

*Then player **II** has a winning tactic in the game $\mathbf{G}(E, \mathcal{A})$.*

The proof is based on the following lemma, which is very similar to Lemma 2.2.

LEMMA 2.4. *There exist some ordinal η and a family $(C_s)_{s\in\eta^{<\omega}}$ of subsets of E such that:*

(o) *$C_s \in \mathcal{C}^{|s|}$ for all $s \in \eta^{<\omega}$, with $C_\emptyset = \{a\}$;*
(i) *$C_{s*0} = C_{s^-}$ if $s \in \eta^{<\omega}$ has a predecessor, and $C_{s*0} = E$ otherwise;*

(ii) *for each* $s \in \eta^{<\omega}$, *the transfinite sequence* $(C_{s*\xi})_{\xi<\eta}$ *is non-increasing, with* $\bigcap_{\xi<\lambda} C_{s*\xi} = C_{s*\lambda}$ *for limit ordinals* λ*;*

(iii) $\bigcap_{\xi<\eta} C_{s*\xi} = C_s$ *for each* $s \in \eta^{<\omega}$*;*

(iv) *if* $s \in \eta^{<\omega}$ *has a predecessor, then* $\operatorname{diam}(C_{s^-} \setminus C_s) < 2^{-|s|}$.

Proof. Let η_0 be any limit ordinal greater than the cardinal number of E, and put $\eta = \eta_0 \cdot \omega$. Let us also choose $\tau > 0$ such that $E = \overline{B}(a,\tau)$. We first define the sets C_ξ for all ordinals $\xi < \eta_0$ in such a way that $\bigcap_{\xi<\eta_0} C_\xi = \overline{B}(a,\tau/2^1)$. By (i), we must put $C_0 = E$. Assume that C_ξ has been defined, with $C_\xi \in \mathcal{C}^1$ and $\overline{B}(a,\tau/2^1) \subset C_\xi$. If $C_\xi \neq \overline{B}(a,\tau/2^1)$, we use (2) with $K_2 = C_\xi$, $K_1 = \overline{B}(0,\tau/2^1)$, and $\varepsilon = 2^{-1}$. Since $K_1 \in \mathcal{C}^0$, this gives a set $K = C_{\xi+1} \in \mathcal{C}^1$. Then (iv) is satisfied. If C_ξ is already equal to $\overline{B}(a,\tau/2)$, we put $C_{\xi+1} = C_\xi$. If λ is a limit ordinal, we have to put $C_\lambda = \bigcap_{\xi<\lambda} C_\xi$; then $C_\lambda \in \mathcal{C}^1$ by property (1), since all sets C_ξ already constructed contain $\overline{B}(a,\tau/2)$. This defines the sets C_ξ for all $\xi < \eta_0$, and by definition of η_0, we have $\bigcap_{\xi<\eta_0} C_\xi = \overline{B}(a,\tau/2)$. Now, we put $C_{\eta_0} = \overline{B}(a,\tau/2)$ and we define the sets C_ξ for $\eta_0 \leqslant \xi < \eta_0 \cdot 2$ in exactly the same way, replacing $\overline{B}(a,\tau/2^1)$ by $\overline{B}(a,\tau/2^2) \in \mathcal{C}^0$. Continuing in that way, we construct the sets C_ξ for all $\xi < \eta$. It should now be clear how to produce the whole family $(C_s)_{s\in\eta^{<\omega}}$, by induction on the length of a sequence $s \in \eta^{<\omega}$. □

Proof of Theorem 2.3. Let $(C_s)_{s\in\eta^{<\omega}}$ be the family of sets given by Lemma 2.4. By (i), (ii) and (iii), one can associate to each point $x \in E \setminus \{a\}$ a uniquely defined sequence of successor ordinals $\mathbf{s}(x) = (\xi_0, \xi_1, \ldots)$ such that $x \in C_{(\mathbf{s}(x)_{|n})^-} \setminus C_{\mathbf{s}(x)_{|n}}$ for each $n \geqslant 1$. We put $C_n(x) = C_{\mathbf{s}(x)_{|n}}$, with $n \geqslant 1$. By (iii) the sequence $(C_n(x))$ is non-decreasing, and $x \notin \bigcup_n C_n(x)$. Moreover, since $x \in C_{(\mathbf{s}(x)_{|n})^-} \setminus C_n(x)$ and $C_{n+1}(x) \subset C_{(\mathbf{s}(x)_{|n})^-}$ for all n by (i) and (ii), it follows from (iv) that the sequence $(C_{n+1}(x) \setminus C_n(x))$ accumulates to x. Hence, by (o) and property (3), one can find a set $A(x) \in \mathcal{A}(x)$ such that $A(x) \cap \bigcup_n C_n(x) = \emptyset$. We put $t(x) = A(x)$ for all $x \in E \setminus \{a\}$. Finally, we choose $t(a)$ to be any member of $\mathcal{A}(a)$. This defines the tactic of player **II**. If $a_0, t(a_0), a_1, t(a_1), \ldots$ is a run of the game $\mathbf{G}(E,\mathcal{A})$ where **II** follows this tactic, then either $a_n = a$ for all $n \geqslant 0$, in which case **II** has won, or $a_n \neq a$ after some time because $a \notin t(x)$ if $x \neq a$. In that case, the same proof as in Theorem 2.1 shows that **II** has also won. □

Remark 1. The reader may feel a bit unsatisfied when looking at Theorems 2.1 and 2.3 together, since the conclusion in Theorem 2.3 is stronger than that in Theorem 2.1 but some hypotheses are not. Indeed, in Theorem 2.3 the setting is formally more general than in Theorem 2.1 since one considers a sequence $(\mathcal{C}^n)$ rather than a single family $\mathcal{C}$, and condition (2) in Theorem 2.3 is weaker than the corresponding one in Theorem 2.1. However, looking at the proof of Theorem 2.3 and with the same notation, it is easy to convince oneself that player **II** has a winning strategy when conditions (0), (1), (2) of Theorem 2.3 are satisfied and (3) is replaced by the following weaker assumption: *if* $C \in \bigcup_n \mathcal{C}^n$ *and* $x \in E \setminus C$*, then one can find* $A \in \mathcal{A}(x)$ *such that* $A \cap C = \emptyset$. Thus, one could formulate explicitly a variant of Theorem 2.1 which would be more closely related to Theorem 2.3. However, Theorem 2.1 as stated is just what we need for the applications we have in mind, while we have no interesting example to illustrate the modified version.

3. *Banach space setting*

For all background material concerning Banach space geometry, we refer to the books [**1**] and [**4**].

Let X be a real Banach space. If E is a subset of X, a *closed slice* of E is the intersection of E with a closed half-space of X; *open* slices of E are defined similarly. A *hyperplane section* of E is the intersection of E with a closed hyperplane of X. For each point $x \in E$, we denote by $\mathcal{H}(x)$ the family of all hyperplane sections of E containing x, by $\mathcal{S}_c(x)$ the family of all closed slices of E of the form $S \cap E$, where S is a closed half-space containing x as a boundary point, and by $\mathcal{S}_o(x)$ the family of all open slices of E containing x. The corresponding games $\mathbf{G}(E, \mathcal{H})$, $\mathbf{G}(E, \mathcal{S}_o)$ and $\mathbf{G}(E, \mathcal{S}_c)$ are called the *point–hyperplane game* for E, the *point–open slice game* for E, and the *point–closed slice game* for E.

Notice that the point–closed slice game is clearly more difficult to win for player **II** than the point–hyperplane game. Moreover, a moment of thought reveals that if player **II** has a winning strategy in the point–open slice game, then she also has one in the point–closed slice game. Finally, it is clear that if $E_1 \subset E_2$, then the games in E_1 are easier to win for player **II** than the games in E_2.

3.1. *The Radon–Nikodym property*

Recall that a closed bounded convex set $K \subset X$ is said to have the *Radon–Nikodym property* (RNP) if every bounded linear operator $T : L^1(\Omega, \mu) \to X$ sending the positive unit sphere of $L^1(\Omega, \mu)$ into K can be represented by an element of $L^\infty(\Omega, \mu, X)$; here, (Ω, μ) is an arbitrary probability space. The Banach space X has the RNP if its closed unit ball has. Among the many beautiful characterizations of this property, we shall of course use the following one: the convex set K has the RNP if and only if each non-empty subset of K has non-empty open slices with arbitrarily small diameter.

THEOREM 3.1. *Let $K \subset X$ be a (non-empty) bounded closed convex set. Then the point–open slice game for K is determined, and player* **II** *has a winning strategy if and only if K has the Radon–Nikodym property. More precisely:*
(a) *if K has the RNP, then* **II** *has a winning strategy;*
(b) *if K does not have the RNP, then there exists $\varepsilon > 0$, and a strategy for player* **I** *such that each run of the game where* **I** *plays according to this strategy produces a sequence (a_n) such that $\|a_{n+1} - a_n\| > \varepsilon$ for all $n \in \mathbb{N}$.*

Proof. Assume that K has the Radon–Nikodym property. Let $\mathcal{C}$ be the family of all closed convex subsets of K. We check that $\mathcal{C}$ and $\mathcal{A} = \mathcal{S}_o$ satisfy the assumptions of Theorem 2.1. Condition (1) is obviously satisfied, and (3) follows from the Hahn–Banach theorem. To prove that (2) is also satisfied, let us fix $C_1, C_2 \in \mathcal{C}$ with $C_1 \subset C_2$ and $C_2 \setminus C_1 \neq \emptyset$. By the Hahn–Banach theorem, one can find $x^* \in X^*$ and $\alpha < \beta$ such that

$$C_1 \subset \{x \colon \langle x^*, x\rangle \leqslant \alpha\} \qquad \text{and} \qquad C_2 \cap \{x \colon \langle x^*, x\rangle > \beta\} \neq \emptyset.$$

Since the set C_2 has the RNP by assumption on K, it follows from Stegall's variational principle [**10**] that one can approximate x^* by $y^* \in X^*$ strongly exposing some point of C_2. If y^* is close enough to x^*, then

$$C_1 \subset \{x \colon \langle y^*, x\rangle \leqslant \beta\} \qquad \text{and} \qquad C_2 \cap \{x \colon \langle y^*, x\rangle > \beta\} \neq \emptyset;$$

and if $\gamma \geqslant \beta$ is close enough to $\sup_{C_2} y^*$, then the set $\{x \in C_2 \colon \langle y^*, x\rangle > \gamma\}$ has small diameter. Thus, putting $C = \{x \in C_2 \colon \langle y^*, x\rangle \leqslant \gamma\}$ for some suitable γ, we see that condition (2) is satisfied. By Theorem 2.1, we conclude that player **II** has a winning strategy in the point–open slice game for K.

Now, assume that K does not have the RNP. Then one can find $\varepsilon > 0$ and a non-empty set $\tilde{K} \subset K$ such that each non-empty open slice of $\tilde{K}$ has diameter greater than 2ε. We define a strategy for player **I** as follows. First, **I** chooses some point $a_0 \in \tilde{K}$. If **II** answers with some open slice A_0 containing a_0, then $\operatorname{diam} A_0 \cap \tilde{K} > 2\varepsilon$. By the triangle inequality, it follows that one can find $a_1 \in A_0 \cap \tilde{K}$ such that $\|a_1 - a_0\| > \varepsilon$; this point a_1 is the second move of player **I**. Repeating this procedure, we clearly get the announced strategy for player **I**. □

Keeping in mind that the point–open slice game is harder to win for **II** than the point–closed game which is harder than the point–hyperplane game, and that the games in a larger set are harder to win for **II**, we get the following corollary.

COROLLARY 3.2. *If the Banach space X has the RNP, then, for any (non-empty) bounded set $\Omega \subset X$, player* **II** *has winning strategies in the point–slice games and in the point–hyperplane game for Ω.*

The remark following Theorem 2.1 shows that if the convex set K has the RNP, then, in each of the above games, player **II** has a winning strategy given by a sequence of tactics: there is a sequence of maps $t_n : E \to \mathcal{A}$ such that **II** wins by answering $t_n(a_n)$ when **I** has played a_n. For the point–closed slice game, this result can be formulated in the following way, which is arguably very intuitive keeping in mind the fact that every bounded monotonic sequence of real numbers is convergent.

COROLLARY 3.3. *If X has the Radon–Nikodym property, then there exists a sequence of maps $\Phi_n : B_X \to S_{X^*}$ such that the following property holds true: if $(a_n) \subset B_X$ is a sequence satisfying $\langle \Phi_n(a_n), a_{n+1}\rangle \geqslant \langle \Phi_n(a_n), a_n\rangle$ for all $n \in \mathbb{N}$, then (a_n) is convergent.*

With a little more effort, one can show that, as far as the Radon–Nikodym property of the whole space X is concerned, the two point–slice games and the point–hyperplane game are essentially equivalent. This is the content of the next result.

THEOREM 3.4. *Let Ω be a bounded subset of X with non-empty interior. Then the following are equivalent:*

1. *X has the Radon–Nikodym property;*
2. *player* **II** *has a winning strategy in the point–open slice game for Ω;*
3. **II** *has a winning strategy in the point–closed slice game for Ω;*
4. **II** *has a winning strategy in the point–hyperplane game for Ω.*

Proof. One can find open balls B_1 and B_2 such that $B_1 \subset \Omega \subset B_2$. Since the games in a larger set are harder to win for player **II**, it follows that **II** has a winning strategy in any of the above games for Ω if and only if it has one in the same game for the open unit ball of X. Thus, we may assume that Ω is the open unit ball of X.

That (1) implies (2) follows from Theorem 3.1: if X has the RNP, which means that $\overline{\Omega}$ has the RNP, then **II** has a winning strategy in the game $\mathbf{G}(\overline{\Omega}, \mathcal{S}_o)$, and hence also in $\mathbf{G}(\Omega, \mathcal{S}_o)$. We have already observed that (2) implies (3) and (3) implies (4). To conclude the proof, we have to show that if X does not have the RNP, then player **I** also has a winning strategy in the point–hyperplane game $\mathbf{G}(\Omega, \mathcal{H})$. So, assume that X does not have the RNP.

CLAIM 1. *One can find a non-empty open convex set $V \subset \Omega$ and $\varepsilon > 0$ such that all non-empty slices of V have diameter at least ε.*

Proof of Claim 1. Since X does not have the RNP, Ω contains a non-empty closed convex set K such that all non-empty open slices of K have diameter at least 5ε, for some fixed $\varepsilon > 0$. Moreover, we may assume that K is at positive distance from $\partial\Omega$, that is, $\eta_0 := \inf\{\operatorname{dist}(x, \partial\Omega) \colon x \in K\} > 0$. Then, for $\eta < \eta_0$, the convex open set $V_\eta = \{x \colon \operatorname{dist}(x, K) < \eta\}$ is contained in Ω. Let us check that one can take $V = V_\eta$, if η is small enough. Since V_η is open, it is enough to consider only open slices of V_η. Let S be a non-empty open slice of V_η, that is $S = U \cap V_\eta$, where U is an open half-space. Let x be any point of S. Then one can find $x' \in K$

such that $\|x' - x\| < \eta$. By translating the half-space U in the direction of $x' - x$, one gets an open half-space U' containing x' such that $\mathrm{dist}(z, U) < \eta$ for all $z \in U'$. Then $S' = U' \cap K$ is a non-empty open slice of K, so it has diameter at least 5ε. By the triangle inequality, it follows that one can find $y' \in U' \cap K$ such that $\|y' - x'\| \geqslant 2\varepsilon$, and by the choice of U', one gets a point $y \in U$ such that $\|y - y'\| < \eta$. Then $y \in V_\eta$ because $y' \in K$, so that $y \in S$; and $\|y - x\| \geqslant \|y' - x'\| - 2\eta \geqslant 2\varepsilon - 2\eta$. Thus, if $\eta < \varepsilon/2$, then the diameter of every non-empty slice of V_η is at least ε. □

CLAIM 2. *Each non-empty hyperplane section of V has diameter at least $\varepsilon/4$.*

Proof of Claim 2. Let $\Phi \in X^*$, and put $m_\Phi := \inf_V \Phi$ and $M_\Phi := \sup_V \Phi$. We have to show that for each $r \in (m_\Phi, M_\Phi)$, the set $A_r = \{x \in V : \Phi(x) = r\}$ has diameter at least $\varepsilon/4$. Let x_0 be any point of V, and assume $r > \Phi(x_0)$. Let us denote by S_r the slice $\{x \in V : \Phi(x) > r\}$. For each point $z \in S_r$, the line segment $[x_0, z]$ intersects A_r at a unique point $\tilde{z}$, and we have $\|z - \tilde{z}\| = \lambda_z \|z - x_0\|$, where

$$\lambda_z = \frac{\Phi(z) - \Phi(\tilde{z})}{\Phi(z) - \Phi(x_0)} \leqslant \frac{M_\Phi - r}{r - \Phi(x_0)}.$$

It follows that $S_r \subset A_r + B(0, \varepsilon_r)$, where $\varepsilon_r \to 0$ as $r \to M_\Phi$. Since $\mathrm{diam}(S_r) \geqslant \varepsilon$, we have shown that if r is close to M_Φ, then $\mathrm{diam}(A_r) \geqslant \varepsilon/2$; and likewise if r is close to m_Φ. Now, for an arbitrary $r \in (m_\Phi, M_\Phi)$, one can find $r_1 < r < r_2$ such that A_{r_1} and A_{r_2} have diameter at least $\varepsilon/2$. Moreover, we have $\mathrm{diam}(A_r) \geqslant \frac{1}{2}\min(\mathrm{diam}(A_{r_1}), \mathrm{diam}(A_{r_2}))$, and hence $\mathrm{diam}(A_r) \geqslant \varepsilon/4$. Indeed, assume (as we may) that $r - r_1 \leqslant \frac{1}{2}(r_2 - r_1)$. If x_1 and y_1 are any two points of A_{r_1}, then, taking a point $z \in A_{r_2}$ and drawing the triangle $x_1 y_1 z$, we see that one can find $x, y \in A_r$ such that $\|x - y\| \geqslant \frac{1}{2}\|x_1 - y_1\|$. Since x_1 and y_1 are arbitrary, this gives $\mathrm{diam}(A_r) \geqslant \frac{1}{2}\,\mathrm{diam}(A_{r_1})$. □

Exactly as in the proof of Theorem 3.1, it follows from Claim 2 that player **I** has an obvious winning strategy in the game $\mathbf{G}(V, \mathcal{H})$: begin with any point $a_0 \in V$, and then, thanks to Claim 2, play at each step a point $a_{n+1} \in V$ such that $\|a_{n+1} - a_n\| \geqslant \varepsilon/8$. Since the game $\mathbf{G}(V, \mathcal{H})$ is more difficult to win for **I** than the corresponding game in the larger set Ω, this concludes the proof. □

3.2. *The super-reflexive case*

We observed above that if X has the RNP, then, in each of the three games we have considered, player **II** has a winning strategy which is given by a sequence of tactics. It is not clear for us whether **II** has in fact a single winning tactic. However, we can do more under a stronger assumption on X. Recall that the Banach space X is said to be *super-reflexive* if every Banach space which is finitely representable in X is reflexive. By a deep result due to P. Enflo, the super-reflexive Banach spaces are exactly those which admit an equivalent uniformly convex norm.

THEOREM 3.5. *Assume that the Banach space X is super-reflexive. Then, for any bounded set $\Omega \subset X$, player* **II** *has a winning tactic in the point–closed slice game for Ω.*

As in Corollary 3.3 above, this result can be formulated in the following way.

COROLLARY 3.6. *If X is super-reflexive, then one can associate to each point $a \in B_X$ a linear functional $\Phi_a \in S_{X^*}$ in such a way that the following property holds true: if a sequence $(a_n) \subset B_X$ satisfies $\langle \Phi_{a_n}, a_{n+1} \rangle \geqslant \langle \Phi_{a_n}, a_n \rangle$ for all $n \in \mathbb{N}$, then (a_n) is convergent.*

For the proof of Theorem 3.5, we need the following definition.

Definition 1. Let $K \subset X$ be a bounded closed convex set with non-empty interior. We say that K is *uniformly convex* if the following property holds: there exists a function

$$\delta : (0,\infty) \to (0,\infty)$$

such that whenever $x, y \in K$ satisfy $\|x-y\| \geqslant \varepsilon$, it follows that $\operatorname{dist}\big(\frac{1}{2}(x+y), \partial K\big) \geqslant \delta(\varepsilon)$. In such a case, we say that K is uniformly convex *with modulus* δ.

Thus, the balls of X are uniformly convex if and only if the given norm is uniformly convex in the usual sense, and in that case, a ball of radius r is uniformly convex with a modulus depending only on r.

Proof of Theorem 3.5. Since X is super-reflexive, we may assume that the norm of X is uniformly convex. For each $r > 0$, we choose a modulus of uniform convexity δ^r for balls of radius r. Clearly, we may assume that if $r \leqslant r'$, then $\delta^r(\varepsilon) \geqslant \delta^{r'}(\varepsilon)$ for small enough ε .

Fact 1. *Let $(K_i)_{i\in I} \subset X$ be a family of closed convex bounded sets. Assume that $\bigcap_i K_i$ has non-empty interior, and that all sets K_i are uniformly convex with some fixed modulus δ. Then $K = \bigcap_{i\in I} K_i$ is uniformly convex with modulus δ.*

Proof. First, we observe that if K_1 and K_2 are uniformly convex subsets of X with modulus δ_1 and δ_2, respectively, and if $K_1 \cap K_2$ has non-empty interior, then $K_1 \cap K_2$ is uniformly convex with modulus $\delta_3 \geqslant \inf(\delta_1, \delta_2)$. This is obvious since $\partial(K_1 \cap K_2) \subset \partial K_1 \cup \partial K_2$. Accordingly, we may assume that the family (K_i) is stable under finite intersections, and of course that $K_i \neq K_j$ if $i \neq j$. Replacing each K_i by $K_i \cap K_{i_0}$ for some fixed $i_0 \in I$, we may assume in addition that $\bigcup_i K_i$ is bounded. Finally, we may also assume that $0 \in \operatorname{int}(K)$.

Let us fix $\varepsilon > 0$, and $x, y \in K$ with $\|x-y\| \geqslant \varepsilon$. Let $p \in \partial K$. Since $\bigcup_i K_i$ is bounded, one can find $\lambda \in (1,\infty)$ such that, for each $i \in I$, the segment $[p, \lambda p]$ intersects ∂K_i at some point p_i. We order the index set I in the obvious way: $i \preccurlyeq j$ if $K_i \supset K_j$. Then I is a directed set, and since $[p; \lambda p]$ is compact, the net $(p_i)_{i\in I}$ has a subnet converging to some $\tilde{p} \in [p, \lambda p] \cap K$. Since $0 \in \operatorname{int}(K)$, the half-open segment $[0, \tilde{p})$ is contained in $\operatorname{int}(K)$, and this implies that $\tilde{p} = p$. Moreover, we have $\big\|\frac{1}{2}(x+y) - p_i\big\| \geqslant \delta(\varepsilon)$ for all $i \in I$; hence $\big\|\frac{1}{2}(x+y) - p\big\| \geqslant \delta(\varepsilon)$. This concludes the proof. □

Fact 2. *Let $R > 0$, and let K_1 and K_2 be non-empty closed convex sets of diameter less than R such that $K_1 \subset K_2$ and $K_2 \setminus K_1 \neq \emptyset$. Assume that K_2 is uniformly convex, and that K_1 is the intersection of a family of balls of radii less than R. For each $\varepsilon > 0$, there exists a ball B of radius less than $4R$ such that $K = \overline{B} \cap K_2$ satisfies $K_1 \subset K \subset K_2$, $K_2 \setminus K \neq \emptyset$ and* $\operatorname{diam}(K_2 \setminus K) < \varepsilon$.

Proof. Choose a point $x_0 \in K_2 \setminus K_1$. By assumption on K_1, one can find an open ball $B_0 = B(p_0, r_0)$ with $r_0 < R$ such that $K_1 \subset B_0$ and $x_0 \notin \overline{B}_0$. Since X is reflexive, the set K_2 is weakly compact. By a classical result of K. S. Lau [**6**], the set of points $p \in X$ admitting a farthest point in K_2 is dense in X. Thus, one can find a point $p \in X$ very close to p_0 and $x \in \partial K_2$ such that $\|x-p\| \geqslant \|y-p\|$ for all $y \in K_2$. Notice that if p is close enough to p_0, then $\|x-p\| \geqslant \|x_0-p\| > \sup\{\|z-p\| \colon z \in K_1\}$, so that $x \notin K_1$. Notice also that if p is close to p_0, then $\|x-p\| < 2R$. Now, let $\eta > 0$ and $\alpha > 0$. By the choice of p and x, if a point $z \in K_2$ satisfies

$$\|z-p\| > r_\eta := \frac{1}{1+\eta}\,\|x-p\|,$$

then $p+(1+\eta)(z-p)\notin K_2$; in particular, the half-line $z+\mathbb{R}^+(z-p)$ must meet ∂K_2 at some point $z+t(z-p)$ with $t\leqslant\eta$, so

$$\operatorname{dist}(z,\partial K_2)\leqslant \eta\|z-p\|\leqslant 2R\eta.$$

Thus, for each $\eta>0$, we have found $r_\eta<2R$ such that if $z\in K_2$ satisfies $\|z-p\|>r_\eta$, then $\operatorname{dist}(z,\partial K_2)\leqslant 2R\eta$. Denoting by δ a modulus of uniform convexity for K_2, we see that $\|x-y\|<\alpha$ whenever $2R\eta<\delta(\alpha)$ and $y\in K_2$ satisfies $\left\|\frac{1}{2}(x+y)-p\right\|>r_\eta$. In other words, for small enough η, the set $K_2\setminus\overline{B}(2p-x,2r_\eta)$ is contained in $B(x,\alpha)$. Thus, one can put $B=B(2p-x,2r_\eta)$, for some suitably chosen $\eta>0$. □

FACT 3. *Let (K_n) be a non-decreasing sequence of uniformly convex subsets of X. Assume that the sequence $(K_{n+1}\setminus K_n)$ accumulates to some point $x\in X\setminus\bigcup_n K_n$. Then one can find a closed half-space containing x as a boundary point and disjoint from $\bigcup_n K_n$.*

Proof. Put $K=\bigcup_n K_n$. Since the sequence (K_n) is non-decreasing, the set K is convex, and K has non-empty interior. If $x\notin\overline{K}$ (which may happen in the uninteresting case where the sequence (K_n) is stationary), the Hahn–Banach theorem allows us to separate strictly x from K by some linear functional, and the result follows. Now, assume $x\in\overline{K}$. Then we have $\overline{K}=K\cup\{x\}$ because the sequence $(K_{n+1}\setminus K_n)$ accumulates to x. Since K has non-empty interior, one can still separate x from K by some linear functional, but perhaps not strictly. In other words, one can find a non-zero linear functional $x^*\in X^*$ such that $\alpha:=\langle x^*,x\rangle\geqslant\langle x^*,z\rangle$ for all $z\in K$. By contradiction, assume that equality occurs at some point $z\in K$. Then the segment $[z,x]$ is contained in $\overline{K}=K\cup\{x\}$, so that the half-open segment $[z,x)$ is contained in K. Since the sequence (K_n) is non-decreasing and the sets K_n are convex, it follows that one can find some integer n such that $I:=\left[z,\frac{1}{2}(x+z)\right]\subset K_n$; hence $I\subset\partial K_n$ because I is contained in the hyperplane $\{x^*=\alpha\}$. This is a contradiction since, being uniformly convex, K_n cannot contain non-trivial segments in its boundary. Thus, the half-space $M:=\{x^*\geqslant\alpha\}$ satisfies $M\cap K=\emptyset$, as required. □

We are now in position to apply Theorem 2.3. Let Ω be a bounded subset of X, and choose $R>0$ such that $\overline{\Omega}\subset B(0,R/3)$. We apply Theorem 2.3 with $E=B(0,R/3)$; as observed above, it is enough to show that player **II** has a winning tactic in the point–closed slice game for E. We put $a=0$, and for each $n\in\mathbb{N}$, we define $\mathcal{C}^n$ to be the family of all uniformly convex sets $C\subset E$ which are intersections of balls of X of radii less than 4^nR. Then condition (0) in Theorem 2.3 is clearly satisfied. Conditions (1), (2) and (3) are also satisfied, thanks to the corresponding facts proved above. This concludes the proof of Theorem 3.5. □

3.3. *The point of continuity property*

We conclude this section with a game characterization of another well-known Banach space property. Recall that the Banach space X is said to have the *point of continuity property* (PCP) if each non-empty bounded set $A\subset X$ has non-empty relatively weakly open subsets with arbitrarily small diameter. More generally, let (E,d) be a metric space, and let τ be a topology on E. The topological space (E,τ) is said to be *fragmented* by the metric d if each non-empty subset of E has non-empty relative τ-open subsets with arbitrarily small diameter. Thus, a Banach space X has the PCP if and only if its unit ball is norm-fragmented in the weak topology.

If (E,d) is a metric space and τ is a topology on E, we define the *(τ,d)-game* for E to be the game $\mathbf{G}(E,\mathcal{A})$, where for each $x\in E$, $\mathcal{A}(x)$ is the family of all τ-open subsets of E containing x. Thus, player **I** starts the game by playing some point a_0, player **II** answers by some τ-open set U_0 containing a_0, player **I** then plays a point $a_1\in U_0$ and so on.

THEOREM 3.7. *The topological space (E,τ) is fragmented by the metric d if and only if player* **II** *has a winning strategy in the (τ,d)-game for E.*

Proof. The 'only if' part follows from Theorem 2.1: just take for $\mathcal{C}$ the family of all τ-closed subsets of E. The only thing to be checked is property (2). Now, if C_1 and C_2 are τ-closed with $C_1 \subset C_2$ and $C_2 \setminus C_1 \neq \emptyset$, then one can find a τ-open set U such that $U \cap (C_2 \setminus C_1) \neq \emptyset$ and $\operatorname{diam}(U \cap (C_1 \setminus C_1)) < \varepsilon$. Thus, one can put $C = C_1 \cup (C_2 \setminus U)$ to get (2). The 'if' part is obvious, exactly as in Theorem 3.1. □

COROLLARY 3.8. *A Banach space X has the PCP if and only if player* **II** *has a winning strategy in the $(w, \|\cdot\|)$-game for B_X.*

REMARK 2. In [**7**], P. S. Kenderov and W. B. Moors give a characterization of fragmentability by means of another natural topological game. See also [**8**].

Using the same kind of arguments as in the proof of Theorem 3.4 above, we can also get a characterization of the point of continuity property by means of a game where the weak topology does not appear explicitly. Let us denote by $\mathcal{A}_{\mathrm{cof}}$ the family of all finite-codimensional affine subspaces of X.

THEOREM 3.9. *Let Ω be a bounded subset of X with non-empty interior. The following are equivalent:*
(1) *the Banach space X has the PCP;*
(2) *player* **II** *has a winning strategy in the game $\mathbf{G}(\Omega, \mathcal{A}_{\mathrm{cof}})$.*

Proof. As in the proof of Theorem 3.4, we may assume that Ω is the open unit ball B_X. From Corollary 3.8, we already know that (1) implies (2). Conversely, assume that X does not have the PCP. We show that player **I** has a winning strategy in the game $\mathbf{G}(B_X, \mathcal{A}_{\mathrm{cof}})$.

CLAIM. *There exist $\varepsilon > 0$ and an increasing sequence of open sets (V_n), with $V_n \subset B_X$ for all n, such that the following property holds true: for each $n \in \mathbb{N}$ and all finite-codimensional affine subspaces $M \subset X$ such that $M \cap V_n \neq \emptyset$, the diameter of $M \cap V_{n+1}$ is at least ε.*

Proof of the claim. Since X does not have the PCP, one can find a non-empty set $K \subset \frac{1}{2}B_X$ such that each non-empty relative weak open subset of K has diameter at least 9ε, for some fixed $\varepsilon > 0$. As in the proof of Theorem 3.4, we find that for all sufficiently small $\eta > 0$, the open set

$$V_\eta = \{x\colon\ \operatorname{dist}(x,K) < \eta\}$$

has the same property, with 9ε replaced by 3ε. Putting $V_n = V_{\eta_n}$, for some suitable increasing sequence (η_n), we get an increasing sequence of open sets $V_n \subset B_X$ such that:
(a) all non-empty relative weak open subsets of V_n have diameter at least 3ε;
(b) $\inf\{\operatorname{dist}(x, \partial V_{n+1})\colon x \in V_n\} > 0$ for all $n \in \mathbb{N}$.

Let us fix $n \in \mathbb{N}$, and let M be a finite-codimensional affine subspace of X such that $M \cap V_n \neq \emptyset$. We write $M = \bigcap_{i=1}^N H_i$, where $H_i = \{\Phi_i = r_i\}$ is an affine hyperplane determined by some linear functional $\Phi_i \in X^*$. Then $\rho(x) = \sup_{1\leqslant i\leqslant n} |\Phi_i(x)|$ induces a norm on the quotient space $X/\mathbf{M}$, where $\mathbf{M} = \bigcap_i \operatorname{Ker}(\Phi_i)$; and since $X/\mathbf{M}$ is finite dimensional, this norm is equivalent to the one induced by $\rho_0(x) = \operatorname{dist}(x, \mathbf{M})$. Thus, we see that one can find some constant $C = C(\Phi_1, \ldots, \Phi_N)$ such that the following property holds true:

$$\text{for all } x \in X, \qquad \operatorname{dist}(x, M) \leqslant C \sup_{1\leqslant i\leqslant N} |\Phi_i(x) - r_i|. \tag{3.1}$$

By the choice of V_n, for each $\eta > 0$, the set

$$A_\eta := \left\{ x \in V_n \colon \sup_{1 \leqslant i \leqslant N} |\Phi_i(x) - r_i| < \eta \right\}$$

has diameter at least 3ε. So one can find $y_1, y_2 \in A_\eta$ such that $\|y_1 - y_2\| \geqslant 2\varepsilon$, and by (3.1) one can find $x_1, x_2 \in M$ such that $\|x_j - y_j\| \leqslant C\eta$, for $j = 1, 2$. Then $\|x_1 - x_2\| \geqslant \varepsilon$ if η is small enough. Moreover, since y_1 and y_2 are in V_n, they are 'far away' from ∂V_{n+1}, so that we also have $x_1, x_2 \in V_{n+1}$ if η is small enough. Thus we have proved that $\mathrm{diam}(V_{n+1} \cap M) \geqslant \varepsilon$. □

Now, we define a winning strategy for **I** in the game $\mathbf{G}(B_X, \mathcal{A})$ as follows. The first move of player **I** is any point $a_0 \in V_0$. When **II** has played a finite-codimensional section $A_0 = M_0 \cap B_X$ containing a_0, then, by the claim, **I** can play a point $a_1 \in M_0 \cap V_1$ such that $\|a_1 - a_0\| \geqslant \varepsilon/2$. Then **II** plays $A_1 = M_1 \cap B_X$ containing a_1, **I** can play $a_2 \in M_1 \cap V_2$ such that $\|a_2 - a_1\| \geqslant \varepsilon/2$, and so on. □

REMARK 3. When the Banach space X is separable and reflexive, it is very easy to describe a winning strategy for **II** in the game $\mathbf{G}(B_X, \mathcal{A}_{\mathrm{cof}})$, without appealing to Theorem 2.1. Since X is separable, we may assume that the norm of X has the Kadec–Klee property, which means that the weak and the norm topologies coincide on the unit sphere. Moreover, X^* is separable since $(X^*)^* = X$ is; let $\{x_n^* \colon n \in \mathbb{N}\}$ be a countable dense subset of X^*. The strategy of player **II** is defined as follows. Once player **I** has played a point $a_n \in B_X$, player **II** chooses a linear functional $\Phi_n \in B_{X^*}$ such that $\Phi_n(a_n) = \|a_n\|$, and then she plays the affine subspace

$$M_n := \{x \in X \colon \Phi_i(x) = \Phi_i(a_n) \text{ and } \langle x_i^*, x \rangle = \langle x_i^*, a_n \rangle,\ i = 0, \dots, n\}.$$

If **II** plays according to this strategy, then, since X is reflexive, the sequence (a_n) produced by any run of the game is weakly convergent. Moreover, denoting by a the weak limit of (a_n), we have $\|a\| \geqslant \Phi_n(a) = \Phi_n(a_n) = \|a_n\|$ for all $n \in \mathbb{N}$; and since in any case $\|a\| \leqslant \liminf \|a_n\|$, we conclude that $\|a\| = \lim \|a_n\|$. By the Kadec–Klee property, it follows that the sequence (a_n) is in fact $\|\cdot\|$-convergent, so that **II** has won the game. We thank G. Lancien for showing this strategy to us.

3.4. *Concluding remarks*

To conclude this section, let us mention some problems that we were not able to solve.

(1) Does player **II** always have a winning tactic in the point–closed slice game for B_X if the Banach space X has the RNP? If not, what about the reflexive case?

(2) In the super-reflexive case, does player **II** have a *continuous* tactic in the point–closed slice game? M. Zelený has shown very recently [**12**] that this is indeed the case when $X = \mathbb{R}^d$.

(3) Does player **II** have a winning tactic in the game $\mathbf{G}(B_X, \mathcal{A}_{\mathrm{cof}})$ when the Banach space X has the PCP? If not, is there a natural class of Banach spaces for which player **II** does have a winning tactic? A plausible candidate might be the class of *asymptotically uniformly convex* spaces (see [**5**]).

(4) What can be said about other games of the type $\mathbf{G}(B_X, \mathcal{A})$ where $\mathcal{A}$ is a given family of affine subspaces of X? In particular, let $\mathcal{A}_\infty$ be the family of all infinite-dimensional subspaces of X. Does the game $\mathbf{G}(B_X, \mathcal{A}_\infty)$ characterize some known Banach space property? It follows from Corollary 3.8 that player **II** has a winning strategy in $\mathbf{G}(B_X, \mathcal{A}_\infty)$ if X has an infinite-dimensional subspace with the PCP. Indeed, if Y is such a subspace, then, for any point $a_0 \in X$, the subspace $Y(a_0) := Y \oplus \mathbb{R}a_0$ also has the PCP, so that **II** can win the game by playing inside $Y(a_0)$. We are not able to say more.

Of course, one could ask the same question as in (4) for the family of all *finite*-dimensional subspaces of X, but in that case the answer is easy. Actually, as soon as the family $\mathcal{A}$ contains all d-dimensional subspaces of X, for some non-negative integer $d < \dim(X)$, then **II** has a winning strategy in $\mathbf{G}(B_X, \mathcal{A})$. Indeed, once player **I** has played a_0, player **II** can answer with any d-dimensional subspace $F_0 = F(a_0)$. Then **I** plays a_1, and **II** can answer with a d-dimensional subspace $F_1 = F(a_0, a_1)$ such that the affine subspace F generated by F_0 and F_1 has dimension $d+1$. Then the next move a_2 of player **I** will belong to F, so that from this point on, **II** can play according to some winning tactic in the point–hyperplane game inside F.

4. *Back to differentiable functions*

In this section, we turn back to pathological differentiable functions. We fix once and for all an integer $d \geqslant 2$ and an arbitrary norm $\|\cdot\|$ on $\mathbb{R}^d$.

Let us recall that Buczolich has constructed an everywhere differentiable function $u : \mathbb{R}^d \to \mathbb{R}$ such that $\nabla u(0) = 0$ and $\|\nabla u(x)\| \geqslant 1$ for almost every $x \in \mathbb{R}^d$, in the sense of Lebesgue measure. Following the ideas of [**9**], our purpose here is to use the games introduced above to obtain a bounded differentiable function $u : \mathbb{R}^d \to \mathbb{R}$ which is a solution of the Eikonal equation $\|\nabla u\| = 1$ almost everywhere.

Of course, this equation admits unbounded smooth solutions (for example, any norm 1 linear functional), as well as bounded almost everywhere solutions. The pathology comes from the fact that the almost everywhere solution u is both bounded and *everywhere* differentiable. Notice that, using Ekeland's variational principle, it is easy to check that the gradient of any bounded differentiable function on $\mathbb{R}^d$ takes arbitrarily small values. This shows, in particular, that the Eikonal equation does not have bounded differentiable solutions on $\mathbb{R}^d$, and that the gradient of any bounded, differentiable, almost everywhere solution fails the Denjoy–Clarkson property.

THEOREM 4.1. *There exists an everywhere differentiable, bounded function $u : \mathbb{R}^d \to \mathbb{R}$ such that $\nabla u(0) = 0$ and $\|\nabla u(x)\| = 1$ for almost every $x \in \mathbb{R}^d$.*

We shall actually prove the following more general result. Here and below, all cubes of $\mathbb{R}^d$ will be *half-open*, that is, of the form $\prod_{i=1}^d [a_i, b_i)$, where the $[a_i, b_i)$ are half-open intervals of $\mathbb{R}$.

THEOREM 4.2. *Let U be a bounded open subset of $\mathbb{R}^d$ containing 0, and let $Q_0 = [0,1)^d$ be the unit cube in $\mathbb{R}^d$. Then, there exists an everywhere differentiable, $\mathbb{Z}^d$-periodic function $u : \mathbb{R}^d \to \mathbb{R}$ such that:*
(1) *u and ∇u vanish on ∂Q_0;*
(2) *$\nabla u(x) \in \overline{U}$ for all $x \in \mathbb{R}^d$;*
(3) *$\nabla u(x) \in \partial U$ for almost every $x \in \mathbb{R}^d$.*

Up to a constant, the function u in Theorem 4.2 will be the sum of a uniformly convergent series of non-zero $\mathcal{C}^\infty$-smooth functions $u_n : \mathbb{R}^d \to \mathbb{R}$. In order to prove that $\sum_1^\infty u_n$ is everywhere differentiable, we shall use the following differentiability criterion. Here and below, if F is a function between two Banach spaces X and Y and if ε is a positive number, we put

$$\operatorname{osc}(F, \varepsilon) = \sup\{\|F(x) - F(y)\| \colon \|x - y\| < \varepsilon\}.$$

LEMMA 4.3. *Let $(u_n)_{n \geqslant 1}$ be a sequence of $\mathcal{C}^1$-functions between two Banach spaces X and Y. Assume that:*
(a) *the series $\sum u_n'(x)$ is pointwise convergent;*
(b) *(u_n') converges uniformly to 0;*
(c) *$\|u_{n+1}\|_\infty = o(\|u_n\|_\infty)$;*

(d) $\lim_{n\to\infty} \operatorname{osc}\,(\sum_{k=1}^n u_k', \|u_{n+1}\|_\infty) = 0$.

Then the series $\sum u_n$ *is uniformly convergent, the function* $f := \sum_1^\infty u_n$ *is everywhere differentiable, and* $f'(x) = \sum_1^\infty u_n'(x)$ *for all* $x \in X$.

Proof. By condition (c), one can find n_0 such that $\|u_{n+1}\|_\infty \leqslant \|u_n\|_\infty/2$ for all $n \geqslant n_0$; therefore, the series $\sum u_n$ is uniformly convergent. If $u_n = 0$ for some $n \geqslant n_0$, then $u_k = 0$ for all $k \geqslant n$ and there is nothing to prove. So we assume that $\|u_n\|_\infty > 0$ for all n. Let us fix $x_0 \in X$.

Putting

$$S_n(x) = \sum_{k=1}^n u_k(x), \qquad s_n(x) = S_n'(x) = \sum_{k=1}^n u_k'(x), \qquad r_n(x) = \sum_{k=n+1}^\infty u_k(x),$$

we have

$$\begin{aligned}\Big\| f(x) - f(x_0) - \sum_{k=1}^\infty u_k'(x_0)\cdot(x-x_0)\Big\| \\ \leqslant \|S_{n-1}(x) - S_{n-1}(x_0) - s_{n-1}(x_0)\cdot(x-x_0)\| \\ + \|u_n(x) - u_n(x_0) - u_n'(x_0)\cdot(x-x_0)\| \\ + \|u_{n+1}(x) - u_{n+1}(x_0) - u_{n+1}'(x_0)\cdot(x-x_0)\| \\ + \|r_{n+1}(x)\| + \|r_{n+1}(x_0)\| + \Big\|\sum_{k=n+2}^\infty u_n'(x_0)\Big\| \times \|x-x_0\|\end{aligned}$$

for all $x \in X$ and all $n \geqslant 2$.

By the mean value theorem, the first three terms in the right side can be estimated as follows:

$$\begin{aligned}\|S_{n-1}(x) - S_{n-1}(x_0) - s_{n-1}(x_0)\cdot(x-x_0)\| &\leqslant \operatorname{osc}\,(s_{n-1}, \|x-x_0\|) \times \|x-x_0\|;\\ \|u_n(x) - u_n(x_0) - u_n'(x_0)\cdot(x-x_0)\| &\leqslant 2\|u_n'\|_\infty \times \|x-x_0\|;\\ \|u_{n+1}(x) - u_{n+1}(x_0) - u_{n+1}'(x_0)\cdot(x-x_0)\| &\leqslant 2\|u_{n+1}'\|_\infty \times \|x-x_0\|.\end{aligned}$$

Since $\|r_{n+1}\|_\infty \leqslant \sum_{k=n+2}^\infty \|u_k\|_\infty$, it follows from condition (c) that

$$\|r_{n+1}\|_\infty = o\big(\|u_{n+1}\|_\infty\big) \qquad \text{as } n\to\infty.$$

Finally, by condition (a),

$$\Big\|\sum_{k=n+2}^\infty u_k'(x_0)\Big\| \to 0 \qquad \text{as } n\to\infty.$$

These estimates are valid for all $x \in X$ and all $n \geqslant 2$. Now, if $\|x-x_0\|$ is small enough, there is a uniquely defined integer $n = n(x) \geqslant n_0$ such that $\|u_{n+1}\|_\infty \leqslant \|x-x_0\| \leqslant \|u_n\|_\infty$, and clearly $n(x) \to \infty$ as $x \to x_0$. Using the above estimates and conditions (b) and (d), we see that

$$\Big\| f(x) - f(x_0) - \sum_{k=1}^\infty u_k'(x_0)\cdot(x-x_0)\Big\| = o(\|x-x_0\|)$$

as $x \to x_0$. In other words, f is differentiable at x_0, with $f'(x_0) = \sum_1^\infty u_k'(x_0)$. This concludes the proof. □

Each function $u_n : \mathbb{R}^d \to \mathbb{R}$ will be constructed on small cubes, and will have the property that the image of each such cube by the gradient mapping ∇u_n is essentially equal to a segment. Precisely what is needed is stated in the next lemma. Here and afterwards, we denote by λ_d the usual Lebesgue measure on $\mathbb{R}^d$. A function defined on a cube Q will be said to be *piecewise constant* on Q if Q can be partitioned into finitely many cubes on which the function is constant.

LEMMA 4.4. *Let a be a non-zero vector in $\mathbb{R}^d$, let Q be a cube in $\mathbb{R}^d$, and let $\varepsilon > 0$. Then, there exists a bounded, $\mathcal{C}^\infty$-smooth function $u : \mathbb{R}^d \to \mathbb{R}$ satisfying the following properties:*

(a) *u vanishes in a neighbourhood of ∂Q and $\|u\|_\infty < \varepsilon$;*
(b) *$\lambda_d\big(\{x \in Q \colon \nabla u(x) = -a \text{ or } \nabla u(x) = a\}\big) \geqslant (1-\varepsilon)\lambda_d(Q)$;*
(c) *one can write $\nabla u = v + w$ with $\|w\|_\infty < \varepsilon$, the set $\{v(x) \colon x \in Q\}$ is included in the segment $[-a, a]$, and the function v is piecewise constant on Q.*

Proof. By translation and dilation, we may assume that Q is the unit cube $[0,1)^d$. Let m be a positive number to be chosen later. Let $\varphi : \mathbb{R} \to \mathbb{R}$ be a $\mathcal{C}^\infty$-smooth 1-periodic function such that $\|\varphi'\|_\infty \leqslant 1$ and $\lambda_1\big(\{t \in [0,1) \colon |\varphi'(t)| = 1\}\big) \geqslant 1 - \alpha(\varepsilon)$, where $\alpha(\varepsilon) > 0$ will be specified later. Finally, let $\psi : \mathbb{R}^d \to [0,1]$ be a $\mathcal{C}^\infty$-smooth 'cut-off' function vanishing on some neighbourhood of $\mathbb{R}^d \setminus \mathrm{int}(Q)$ and such that

$$\lambda_d\big(\{x \in Q \colon \psi(x) = 1\}\big) \geqslant (1 - \varepsilon/2)\lambda_d(Q). \tag{4.1}$$

We define the function u on Q by setting

$$u(x) = \frac{\varphi(m\langle x, a\rangle)\psi(x)}{m},$$

where $\langle\ ,\ \rangle$ is the usual scalar product on $\mathbb{R}^d$. Since ψ vanishes in a neighbourhood of ∂Q, one can extend u to a $\mathbb{Z}^d$-periodic, $\mathcal{C}^\infty$-smooth function on $\mathbb{R}^d$, still denoted by u.

If m is large enough, then condition (a) is satisfied.

To check condition (b), observe that we have

$$\lambda_d\big(\{x \in Q \colon |\varphi'(m\langle x, a\rangle)| = 1\}\big) \geqslant (1 - \varepsilon/2)\lambda_d(Q)$$

provided $\alpha(\varepsilon)$ is small enough. Together with (4.1), this implies that

$$\lambda_d\big(\{x \in Q \colon \psi(x) = 1 \text{ and } |\varphi'(m\langle x, a\rangle)| = 1\}\big) \geqslant (1 - \varepsilon)\lambda_d(Q).$$

Computing the derivative of u and noting that $\nabla\psi(x) = 0$ when $\psi(x)$ has the maximal value 1, we see that (b) is satisfied.

We now turn to condition (c). If we set

$$v_1(x) := \varphi'(m\langle x, a\rangle)\psi(x)a \qquad \text{and} \qquad w_1(x) := \varphi(m\langle x, a\rangle)\nabla\psi(x)/m,$$

we see that $\nabla u = v_1 + w_1$ and that the set $\{v_1(x) \colon x \in \mathbb{R}^d\}$ is contained in $[-a, a]$. Moreover, we have $\|w_1\|_\infty < \varepsilon/2$ provided m is large enough. We now fix m large enough and $\alpha(\varepsilon)$ small enough, and we choose a positive integer p such that $\mathrm{osc}(v_1, 1/p) < \varepsilon/2$. We define the mapping $v : Q \to \mathbb{R}^d$ as follows: for each $g \in p^{-1}\mathbb{Z}^d \cap Q$ and all $x \in g + p^{-1}Q$, we put $v(x) = v_1(g)$. Then v has values in the segment $[-a, a]$ and is piecewise constant on Q. Finally, we set $w = w_1 + v_1 - v$. We have $\nabla u = v + w$ and $\|w\|_\infty \leqslant \|w_1\|_\infty + \|v_1 - v\|_\infty < \varepsilon$. This concludes the proof of the lemma. □

According to condition (a) in Lemma 4.3, the sequence of functions $u_n : \mathbb{R}^d \to \mathbb{R}$ should be constructed in such a way that for all $x \in \mathbb{R}^d$, the series $\sum u_n'(x)$ is convergent. This will be guaranteed by the next lemma applied to $s_n(x) = \sum_{k=1}^n \nabla u_k(x)$. Here and below, we denote by $\langle\ ,\ \rangle$ the usual scalar product on $\mathbb{R}^d$.

LEMMA 4.5. *Let U be a bounded open subset of $\mathbb{R}^d$, and let B be a closed ball containing U. Then, there exists a map $t : B \to \mathbb{R}^d$ such that the following property holds true: if (s_n) is a sequence in U and if there exists a sequence $(\sigma_n) \in B$ such that $s_n - \sigma_n$ converges and $\langle t(\sigma_n), \sigma_{n+1} - \sigma_n\rangle \geqslant 0$ for all n, then (s_n) converges to some point $s \in \overline{U}$.*

Proof. Since $\mathbb{R}^d$ is super-reflexive (!), player **II** has a winning tactic in the point–closed slice game for B. Identifying $\mathbb{R}^d$ with $(\mathbb{R}^d)^*$, we put $t(a) = \Phi_a$, where the map $a \mapsto \Phi_a$ is given by Corollary 3.6. By the definition of t, each sequence $(\sigma_n) \subset B$ such that $\langle t(\sigma_n), \sigma_{n+1} - \sigma_n\rangle \geqslant 0$ for all n is convergent. If $s_n - \sigma_n$ converges, it follows that the sequence (s_n) converges to some point $s \in \mathbb{R}^d$, and of course we have $s \in \overline{U}$ if $s_n \in U$ for all n. □

The next lemma follows from the fact that almost sure convergence implies convergence in probability.

LEMMA 4.6. *Let (s_n) be an almost everywhere convergent sequence of mappings from a probability space $(\Omega, \mathcal{A}, \mathbb{P})$ into a Banach space X, and let $\varepsilon > 0$. Then $\mathbb{P}\big(\|s_{n+1} - s_n\| \geqslant \varepsilon\big) \to 0$.*

After these preliminary lemmas, we are now ready to begin the proof of Theorem 4.2, but first, we introduce some terminology.

We shall consider *cube partitions* of $\mathbb{R}^d$, that is, partitions of $\mathbb{R}^d$ into half-open cubes. By a *$\mathbb{Z}^d$-periodic cube partition of* $\mathbb{R}^d$, we mean a cube partition consisting of $\mathbb{Z}^d$-translates of some finite cube partition of the unit cube $Q_0 = [0, 1)^d$. We say that a partition $\mathfrak{Q}'$ is a *refinement* of a partition $\mathfrak{Q}$ if each cube $Q \in \mathfrak{Q}$ can be decomposed into finitely many cubes $Q' \in \mathfrak{Q}'$.

Proof of Theorem 4.2. The unit cube $Q_0 = [0, 1)^d$ and the bounded open set $U \subset \mathbb{R}^d$ are given. For the proof, we fix once and for all a decreasing sequence of real numbers (ε_k) such that $0 < \varepsilon_k < 1$ and $\lim_{k\to\infty} \varepsilon_k = 0$. Finally, for each $\varepsilon > 0$, we put

$$\partial U_\varepsilon = \{x \in U \colon \operatorname{dist}(x, \partial U) < \varepsilon\}.$$

Up to a constant, the function u will be the sum of a uniformly convergent series $\sum u_n$ of non-zero, real-valued, $\mathbb{Z}^d$-periodic, $\mathcal{C}^\infty$-smooth functions defined on $\mathbb{R}^d$. We put $s_n := \sum_{k=1}^n \nabla u_k$. Each function ∇u_n will be of the form $v_n + w_n$, and we put $\sigma_n := \sum_{k=1}^n v_k$. Together with the sequence (u_n), we will construct a sequence $(\mathfrak{Q}_n)$, where each $\mathfrak{Q}_n$ is a $\mathbb{Z}^d$-periodic cube partition of $\mathbb{R}^d$ and $\mathfrak{Q}_{n+1}$ is a refinement of $\mathfrak{Q}_n$. Finally, we will also define an increasing sequence of integers (N_k). The following conditions have to be fulfilled:

- (o) $N_0 = 0$, u_0 is constant, $v_0 = 0 = w_0$, the partition $\mathfrak{Q}_0$ is the family of all $\mathbb{Z}^d$-translates of $Q_0 = [0, 1)^d$, and for each $n \geqslant 1$, the function u_n is constant on a neighbourhood of ∂Q_0;
- (i) v_n is constant on each cube $Q \in \mathfrak{Q}_n$;
- (ii) $\|w_n\|_\infty \leqslant 2^{-n}$;
- (iii) $s_n(x) \in U$ for all $x \in \mathbb{R}^d$;
- (iv) $\|\sigma_n(x)\| \leqslant R + 1$ for all $x \in \mathbb{R}^d$, where $R = \sup\{\|s\| \colon s \in U\}$;
- (v) $\langle t\left(\sigma_n(x)\right), \sigma_{n+1}(x) - \sigma_n(x)\rangle = 0$ for all $x \in \mathbb{R}^d$, where t is is the mapping given by Lemma 4.5 with $B = \{x \in \mathbb{R}^d \colon \|x\| \leqslant R + 1\}$;
- (vi) $\|u_{n+1}\|_\infty \leqslant 2^{-n}\|u_n\|_\infty$ for all n, and if $N_{k-1} < n \leqslant N_k$, then $\|v_n\|_\infty \leqslant \varepsilon_k/4$ and $\operatorname{osc}(s_n, \|u_{n+1}\|_\infty) < \varepsilon_k/4$;
- (vii) for each $k \geqslant 1$, we have

$$\lambda_d\big(\{x \in [0, 1)^d \colon s_{N_k}(x) \notin \partial U_{\varepsilon_k}\}\big) \leqslant 2^{-k}.$$

Inductive step. Let us fix $k \geqslant 1$. Assume N_{k-1} has been defined, and that u_n has been constructed for some $n \geqslant N_{k-1}$. By (i), the function σ_n is constant on each cube $Q \in \mathfrak{Q}_n$; we denote by $\sigma_n(Q)$ the value taken by σ_n on such a cube Q, and we choose some vector $a = a(Q) \in \mathbb{R}^d$ such that $\|a\| = \varepsilon_k/4$ and $\langle t(\sigma_n(Q)), a\rangle = 0$; this can be done because $d \geqslant 2$. Finally, we choose a $\mathbb{Z}^d$-periodic cube partition $\widetilde{\mathfrak{Q}}_n$ of $\mathbb{R}^d$ refining $\mathfrak{Q}_n$, such that the oscillation of s_n on each cube $\tilde{Q} \in \widetilde{\mathfrak{Q}}_n$ is less than $\varepsilon_k/4$. Recall that $u_n \neq 0$. Applying Lemma 4.4 to all cubes $\tilde{Q} \in \widetilde{\mathfrak{Q}}_n$, we see that one can construct a $\mathbb{Z}^d$-periodic, $\mathcal{C}^\infty$-smooth function $\tilde{u} : \mathbb{R}^d \to \mathbb{R}$ and a $\mathbb{Z}^d$-periodic cube partition $\mathfrak{Q}_{n+1}$ refining $\widetilde{\mathfrak{Q}}_n$ (hence a refinement of $\mathfrak{Q}_n$), such that

(a) $\|\tilde{u}\|_\infty < 2^{-n}\|u_n\|_\infty$ and $\operatorname{osc}(s_n, \|\tilde{u}\|_\infty) < \varepsilon_k/4$;
(b) $\lambda_d(\{x \in Q \colon \nabla\tilde{u}(x) = \pm a\}) \geqslant (1 - 2^{-k})\lambda_d(Q)$ for each cube $Q \in \mathfrak{Q}_n$;
(c) one can write $\nabla\tilde{u} = \tilde{v} + \tilde{w}$, where $\|\tilde{w}\| \leqslant \varepsilon_k/2^{n+2}$, the function $\tilde{v}$ is constant on each cube of the partition $\mathfrak{Q}_{n+1}$, and $\tilde{v}\big(\tilde{Q}\big) \subset [-a(Q); a(Q)]$ for each cube $\tilde{Q} \in \widetilde{\mathfrak{Q}}_n$, where Q is the unique cube of the partition $\mathfrak{Q}_n$ containing $\tilde{Q}$;
(d) the function $\tilde{u}$ vanishes on a neighbourhood of $\partial\tilde{Q}$, for each cube $\tilde{Q} \in \widetilde{\mathfrak{Q}}_n$.

Notice that $\tilde{u} \neq 0$. Let $c \in \mathbb{R}^d$ be a non-zero vector such that $\|c\| < \|\tilde{u}\|_\infty$ and $\|\tilde{u}\|_\infty + \|c\| \leqslant 2^{-n}\|u_n\|_\infty$.

Now, we define the function u_{n+1} on each cube $\tilde{Q} \in \widetilde{\mathfrak{Q}}_n$. Let us choose a point $g_{\tilde{Q}} \in \tilde{Q}$ for each such cube $\tilde{Q}$, assuming (as we may) that this choice is compatible with the $\mathbb{Z}^d$-periodicity of $\widetilde{\mathfrak{Q}}_n$.

If $s_n(g_{\tilde{Q}}) \in \partial U_{3\varepsilon_k/4}$, we set $u_{n+1} = c$ on $\tilde{Q}$, and $v_{n+1} = 0 = w_{n+1}$.

If $s_n(g_{\tilde{Q}}) \notin \partial U_{3\varepsilon_k/4}$, we set $u_{n+1} = \tilde{u} + c$ on $\tilde{Q}$; and accordingly, $v_{n+1} = \tilde{v}$ and $w_{n+1} = \tilde{w}$ on $\tilde{Q}$. In this case, we have

$$\lambda_d\big(\{x \in \tilde{Q} \colon \|\nabla u_{n+1}(x)\| = \varepsilon_k/4\}\big) \geqslant (1 - 2^{-k})\lambda_d(\tilde{Q}). \tag{4.2}$$

The function u_{n+1} is $\mathbb{Z}^d$-periodic, and it is C^∞-smooth because the auxiliary function $\tilde{u}$ is smooth and vanishes on a neighbourhood of $\partial\tilde{Q}$, for each cube $\tilde{Q} \in \widetilde{\mathfrak{Q}}_n$. Notice also that $u_{n+1} \neq 0$: this is clear if $s_n(g_{\tilde{Q}}) \in \partial U_{3\varepsilon_k/4}$ for at least one cube $\tilde{Q}$, and otherwise it is also clear because $\|c\|_\infty < \|\tilde{u}\|_\infty$.

Conditions (o), (i) and (ii) for $n+1$ are clearly satisfied, as well as (vi) (though the integer N_k is not yet defined). Condition (iv) for $n+1$ will follow from (iii) and the inequality

$$\|s_{n+1} - \sigma_{n+1}\| \leqslant \sum_{k=1}^{n+1} \|w_k(x)\| \leqslant 1.$$

Let us check condition (iii) for $n+1$. Let $x \in \mathbb{R}^d$, and choose $\tilde{Q} \in \widetilde{\mathfrak{Q}}_n$ such that $x \in \tilde{Q}$. If $s_n(g_{\tilde{Q}}) \in \partial U_{3\varepsilon_k/4}$, then $s_{n+1}(x) = s_n(x) \in U$ by the induction hypothesis. If $s_n(g_{\tilde{Q}}) \notin \partial U_{3\varepsilon_k/4}$, then, since the oscillation of s_n on $\tilde{Q}$ is less than $\varepsilon_k/4$, we have

$$\operatorname{dist}(s_n(x), \partial U) \geqslant 3\varepsilon_k/4 - \|s_n(x) - s_n(g_{\tilde{Q}})\| \geqslant \varepsilon_k/2,$$

and so $s_n(x) \notin \partial U_{\varepsilon_k/2}$. Observing that

$$s_{n+1}(x) = s_n(x) + \nabla u_{n+1}(x)$$

and

$$\|\nabla u_{n+1}(x)\| \leqslant \|v_{n+1}(x)\| + \|w_{n+1}(x)\| < \varepsilon_k/2,$$

we conclude that $s_{n+1}(x) \in U$.

Let us prove (v). If $Q \in \mathfrak{Q}_n$ and $x \in Q$, then $v_{n+1}(x)$ is proportional to $a(Q)$, and hence orthogonal to $t\big(\sigma_n(Q)\big)$. Since $v_{n+1}(x) = \sigma_{n+1}(x) - \sigma_n(x)$ and $\sigma_n(Q) = \sigma_n(x)$, this gives (v).

Now, we show that if we continue this construction, then we will find $N_k > N_{k-1}$ satisfying (vii). Assume by contradiction that for all $n > N_{k-1}$,

$$\lambda_d\big(\{x \in [0,1)^d \colon s_n(x) \notin \partial U_{\varepsilon_k}\}\big) > 2^{-k}. \tag{4.3}$$

If $\tilde{Q} \in \widetilde{\mathfrak{Q}}_n$ is a cube that meets $\{x \in [0,1)^d \colon s_n(x) \notin \partial U_{\varepsilon_k}\}$, then $s_n(g_{\tilde{Q}}) \notin \partial U_{3\varepsilon_k/4}$ because the oscillation of s_n on $\tilde{Q}$ is less than $\varepsilon_k/4$. By (4.2), it follows that for every such cube $\tilde{Q}$, we have

$$\lambda_d\big(\{y \in \tilde{Q} \colon \|s_{n+1}(y) - s_n(y)\| \geqslant \varepsilon_k/4\}\big) \geqslant (1 - 2^{-k})\lambda_d(\tilde{Q}).$$

On the other hand, condition (4.3) implies that the proportion of cubes $\tilde{Q} \in \widetilde{\mathfrak{Q}}_n$ that meet $\{x \in [0,1)^d \colon s_n(x) \notin \partial U_{\varepsilon_k}\}$ is at least 2^{-k}. Therefore,

$$\lambda_d\big(\{y \in [0,1)^d \colon \|s_{n+1}(y) - s_n(y)\| \geqslant \varepsilon_k/4\}\big) \geqslant (1-2^{-k}) \times 2^{-k}. \tag{4.4}$$

This will contradict Lemma 4.6 if we can prove that the sequence (s_n) is pointwise convergent. Now, it follows from (ii) that $s_n(x) - \sigma_n(x) = \sum_{k=1}^n w_k(x)$ converges at each point $x \in \mathbb{R}^d$, so that conditions (iii), (iv), (v) allow us to apply Lemma 4.5 to conclude that (s_n) is indeed pointwise convergent. Thus we have proved by contradiction that there exists $N_k \geqslant N_{k-1}$ satisfying (vii). This concludes the inductive step.

The function u. Let us denote by c_n the constant value of u_n on ∂Q_0. By (vi), we can put $c := \sum_1^\infty c_n$ and define

$$u := -c + \sum_{n=1}^{+\infty} u_n.$$

The function u is $\mathbb{Z}^d$-periodic. To show that it is also differentiable, we check the conditions of Lemma 4.3. For each $n \in \mathbb{N}$, let k_n be the unique positive integer such that $N_{k_n-1} < n \leqslant N_{k_n}$. From (ii) and (vi), we get $\|u_n'\|_\infty \leqslant \|v_n\|_\infty + \|w_n\|_\infty \leqslant \varepsilon_{k_n} + 2^{-n}$, so that $\|u_n'\|_\infty$ tends to 0. By (vi), we have $\|u_{n+1}\|_\infty = o(\|u_n\|_\infty)$, and $\operatorname{osc}(s_n, \|u_{n+1}\|_\infty) < \varepsilon_{k_n}/4 \to 0$. Moreover, it follows as above from (ii)–(v) and Lemma 4.5 that the sequence (s_n) is pointwise convergent; that is, the series $\sum u_n'(x)$ converges at every point $x \in \mathbb{R}^d$. Thus, one can apply Lemma 4.3 to conclude that u is everywhere differentiable, and that ∇u is the pointwise limit of the sequence s_n.

The function u vanishes on ∂Q_0, and by (o) we also have $\nabla u = 0$ on ∂Q_0. It follows from (ii) that $\nabla u(x) \in \overline{U}$ for all $x \in \mathbb{R}^d$. Finally, condition (vii) implies that if $k \leqslant \ell$, then

$$\lambda_d\big(\{x \in [0,1)^d \colon s_{N_\ell}(x) \notin \partial U_{\varepsilon_k}\}\big) \leqslant 2^{-k}.$$

Sending ℓ to ∞, we get $\lambda_d\left(\{x \in [0,1)^d \colon \nabla u(x) \notin \partial U_{\varepsilon_k}\}\right) \leqslant 2^{-k}$ for all $k \in \mathbb{N}$; and sending k to ∞, we conclude that $\nabla u(x) \in \partial U$ for almost every $x \in \mathbb{R}^d$. □

COROLLARY 4.7. *Let Ω be an open subset of $\mathbb{R}^d$, and let $x_0 \in \Omega$. Also let U be a bounded open subset of $\mathbb{R}^d$ containing 0, and put $K := \sup\{\|y\| \colon y \in U\}$. Then, there exists a K-Lipschitz function $u : \overline{\Omega} \to \mathbb{R}$ with the following properties:*
1. *u is bounded and everywhere differentiable in Ω, with $\nabla u(\Omega) \subset \overline{U}$;*
2. *$u(\xi) = 0$ for all $\xi \in \partial\Omega$;*
3. *$\nabla u(x_0) = 0$ and $\nabla u(x) \in \partial U$ for almost every $x \in \Omega$.*

When U is the open unit ball, this gives the result announced in the introduction: there exists a 1-Lipschitz function $u : \overline{\Omega} \to \mathbb{R}$, differentiable on Ω, such that $u(x) = 0$ for all $x \in \partial\Omega$, $\nabla u(x_0) = 0$ and $\|\nabla u(x)\| = 1$ almost everywhere. More generally, Corollary 4.7 gives the existence of non-trivial, differentiable, almost everywhere solutions of the equation $F(\nabla u) = 0$, for any continuous function $F : \mathbb{R}^d \to \mathbb{R}$ such that $F(0) \neq 0$ and the connected component of the set $\{F \neq 0\}$ containing 0 is bounded.

Notice also that the condition $\nabla u(x_0) = 0$ is not really essential when the open set Ω is bounded. Indeed, by Rolle's theorem, the boundary condition (2) forces ∇u to vanish somewhere in Ω.

Proof of Corollary 4.7. Let $\mathfrak{Q}$ be a locally finite cube partition of the open set Ω, with $\partial Q \subset \Omega$ for all $Q \in \mathfrak{Q}$ and $x_0 \in \partial Q_0$ for some cube $Q_0 \in \mathfrak{Q}$. By translation and dilation, it follows from Theorem 4.2 that for each cube $Q \in \mathfrak{Q}$, one can find an everywhere differentiable function $u_Q : \mathbb{R}^d \to \mathbb{R}$ such that $\nabla u_Q(x) \in \overline{U}$ for all $x \in \mathbb{R}^d$, $\nabla u_Q(x) \in \partial U$ almost everywhere, and $u_Q = 0$, $\nabla u_Q = 0$ on ∂Q. We define u on Ω in the obvious way: $u = u_Q$ on each cube $Q \in \mathfrak{Q}$. Then u is everywhere differentiable on Ω. Moreover, we have $\nabla u(x) \in \overline{U}$ for all $x \in \Omega$, so, by the mean value theorem, the restriction of u to the closure of each cube $Q \in \mathfrak{Q}$ is K-Lipschitz. Since $u \equiv 0$ on the boundary of each cube of the partition $\mathfrak{Q}$, it follows that u is in fact K-Lipschitz on Ω. Indeed, given $x_1, x_2 \in \Omega$, one can find $Q_1, Q_2 \in \mathfrak{Q}$ such that $x_i \in Q_i$ and the line segment $[x_1, x_2]$ intersects both ∂Q_1 and ∂Q_2, say at points q_1 and q_2. Since $u(q_1) = 0 = u(q_2)$, we have

$$\begin{aligned} \|u(x_2) - u(x_1)\| &\leqslant \|u(x_2) - u(q_2)\| + 0 + \|u(q_1) - u(x_1)\| \\ &\leqslant K\,(\|x_2 - q_2\| + \|q_1 - x_1\|) \\ &\leqslant K\,\|x_2 - x_1\|. \end{aligned}$$

Therefore, u can be extended to a K-Lipschitz function on $\overline{\Omega}$ with the required properties. Notice that the boundary condition (2) is satisfied because u vanishes on the boundary of each cube of the partition $\mathfrak{Q}$ and the closure of $\bigcup\{\partial Q \colon Q \in \mathfrak{Q}\}$ contains $\partial\Omega$. □

REMARK 4. In addition to the boundary condition $u_{|\partial\Omega} = 0$, one may also impose the condition '$\nabla u = 0$ on the boundary', that is, $u(x) = o(\|x - \xi\|)$ as $x \to \xi \in \partial\Omega$. Actually, given any positive function ϕ on $(0, \infty)$ such that $\inf\{\phi(t) \colon t \geqslant \alpha\} > 0$ for each $\alpha > 0$, one may require that $|u(x)| \leqslant \phi\,(\mathrm{dist}(x, \partial\Omega))$ for all $x \in \Omega$.

Proof. Let the function ϕ be given. If we denote by Q_x the unique cube $Q \in \mathfrak{Q}$ containing $x \in \Omega$, then $|u(x)| \leqslant K\,\mathrm{diam}(Q_x)$, because u is K-Lipschitz and vanishes on ∂Q_x. Thus, it is enough to show that the partition $\mathfrak{Q}$ can be chosen in such a way that $\mathrm{diam}(Q_x) \leqslant \phi\,(\mathrm{dist}(x, \partial\Omega))$ for all $x \to \partial\Omega$. To do this, start with a cube partition $\mathfrak{P}$, and let (δ_k) be a sequence of positive numbers. For each $k \in \mathbb{N}$, set

$$\mathfrak{P}_k = \left\{P \in \mathfrak{P} \colon \frac{1}{k+1} \leqslant \mathrm{dist}(P, \partial\Omega) < \frac{1}{k}\right\}'$$

and subdivide each cube $P \in \mathfrak{P}_k$ into finitely many cubes Q of diameter less than δ_k. This gives a cube partition $\mathfrak{Q}$ with the required additional property if the sequence (δ_k) is suitably chosen. □

REMARK 5. Very recently, M. Zelený was able to construct a differentiable function u on $\mathbb{R}^d$ such that the set $(\nabla u)^{-1}(B(0,1))$ is non-empty and has Hausdorff dimension 1 [**12**]. As far as the Denjoy–Clarkson property is concerned, this may be viewed as the 'optimal' improvement of Buczolich's example. Indeed, Buczolich had shown earlier that for any differentiable function u on $\mathbb{R}^d$, the set $(\nabla u)^{-1}(B(0,1))$ is either empty or has positive 1-dimensional Hausdorff measure [**2**]. Zelený's proof goes along the same lines as in [**9**], but several delicate arguments from geometric measure theory are additionally needed.

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