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Hypercyclic operators failing the Hypercyclicity Criterion on classical Banach spaces

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Abstract

By a recent result of M. De La Rosa and C. Read, there exist hypercyclic Banach space operators which do not satisfy the Hypercyclicity Criterion. In the present paper, we prove that such operators can be constructed on a large class of Banach spaces, including $c_0(\mathbb{N})$ or $\ell^p(\mathbb{N})$.

Keywords: Hypercyclic operators; Hypercyclicity Criterion

1. Introduction

Let X be a topological vector space over $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$. A continuous linear operator $T \in \mathcal{L}(X)$ is said to be *hypercyclic* if there exists some $x \in X$ whose T-orbit $\{T^n(x);\ n \in \mathbb{N}\}$ is dense in X. For example, the derivation operator and the nontrivial translation operators on the space of entire functions are hypercyclic [3,14], and if B is the usual backward shift on $\ell^2(\mathbb{N})$, then $2B$ is hypercyclic [15]. We refer to [7,8] for much more on hypercyclicity.

If X is a separable Fréchet space, it follows from the Baire Category Theorem that an operator $T \in \mathcal{L}(X)$ is hypercyclic if and only if it is topologically transitive, which means that for each

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pair (U, V) of nonempty open subsets of X, one can find $n \in \mathbb{N}$ such that $T^n(U) \cap V \neq \emptyset$. Using this, one can formulate a very useful criterion for hypercyclicity, which is known as *the Hypercyclicity Criterion*. This criterion was first isolated by C. Kitai [12], and then refined by several authors. We state it in the form given in [8], which is the same as in [2].

The Hypercyclicity Criterion. *Let X be a separable Fréchet space, and let $T \in \mathcal{L}(X)$. Assume one can find dense sets $D, D' \subset X$ and an increasing sequence of integers (n_k) such that the following properties hold*:

(1) $T^{n_k}(z) \to 0$ *for all* $z \in D$;
(2) *For each $z' \in D'$, one can find a sequence $(x_k) \subset X$ such that $x_k \to 0$ and $T^{n_k}(x_k) \to z'$.*

Then T is topologically transitive, and hence hypercyclic.

Conditions (1) and (2) give in fact a stronger result: it is not hard to check that if T satisfies the Hypercyclicity Criterion, then the operator $T \oplus T$ (acting on $X \oplus X$) is topologically transitive, i.e. hypercyclic. It was shown by J. Bès and A. Peris [2] that the converse is also true: if $T \oplus T$ is hypercyclic, then T satisfies the Hypercyclicity Criterion. The following problem, originally posed by D. Herrero [10] in the $T \oplus T$ form, has been recognized as one of the most exciting questions in linear dynamics; see e.g. [2,6,8] or [16].

Problem. *Does every hypercyclic operator on a separable Fréchet space satisfy the Hypercyclicity Criterion*? *Equivalently, is $T \oplus T$ hypercyclic whenever T is*?

Very recently, this problem was solved in the negative by M. De La Rosa and C. Read [5], who constructed a Banach space X and a hypercyclic operator $T \in \mathcal{L}(X)$ such that $T \oplus T$ is not hypercyclic. Their construction may be very roughly described as follows. One starts with a vector space F having an algebraic basis $(f_i)_{i \in \mathbb{N}}$, and with the linear operator $S : F \to F$ defined by $Sf_i = f_{i+1}$. Then one defines a norm $\| \cdot \|$ on F in such a way that S is continuous and hypercyclic on $(F, \| \cdot \|)$, and moreover $\| \cdot \|$ is in some sense maximal with respect to these properties. The desired Banach space X is the completion of $(F, \| \cdot \|)$ and T is the extension of S to X.

Although the definition of the norm $\| \cdot \|$ is not extraordinarily complicated, it is not clear whether the Banach space constructed in [5] can be identified with a "classical" space. In the present paper, we show that one can construct hypercyclic operators whose direct sum with themselves are not hypercyclic on many classical spaces, including $c_0(\mathbb{N})$ or $\ell^p(\mathbb{N})$, $1 \leqslant p < \infty$.

We need the following definition: if $(e_i)_{i \in \mathbb{N}}$ is a linearly independent sequence in some vector space, then the *forward shift associated to* (e_i) is the linear operator $S : E \to E$ defined by $S(e_i) = e_{i+1}$, where $E = \operatorname{span}\{e_i;\ i \in \mathbb{N}\}$. Our main result reads as follows.

Theorem 1.1. *Let X be a Banach space. Assume X has a normalized unconditional basis $(e_i)_{i \in \mathbb{N}}$ whose associated forward shift is continuous. Then there exists a hypercyclic operator $T \in \mathcal{L}(X)$ such that $T \oplus T$ is not hypercyclic.*

From this, we deduce immediately:

Corollary 1.2. *There exist hypercyclic operators on $c_0(\mathbb{N})$ or $\ell^p(\mathbb{N})$, $1 \leqslant p < \infty$ which do not satisfy the Hypercyclicity Criterion. In particular, one can find such operators on a Hilbert space.*

2. Algebraic preliminary

To prove Theorem 1.1, we will need a kind of "non-hypercyclicity criterion" for a direct sum $T \oplus T$. There is in fact a very simple algebraic obstruction, which is contained in the following easy lemma. This idea appears in [5], in a rather different formulation.

Lemma 2.1. *Let A be a commutative algebra endowed with a topology τ, and let $\mathfrak{n}$ be a semi-norm on A such that the map $(p,q) \mapsto pq$ is continuous from $(A,\tau) \times (A,\tau)$ into $(A,\mathfrak{n})$. Let also $a, a', b, b' \in A$ and assume there exist three sequences (p_n), (q_n), and (r_n) in A such that $p_n \to a$, $q_n \to b$, $r_n p_n \to a'$ and $r_n q_n \to b'$. Then $\mathfrak{n}(ab' - a'b) = 0$.*

Proof. Just write $p_n(r_n q_n) = (r_n p_n) q_n$ and use the assumptions. $\square$

Corollary 2.2. *Assume the algebra A has a unit, and that for any a, a', b, b', one can find (p_n), (q_n), (r_n) as above. Then $\mathfrak{n} = 0$.*

Proof. Apply 2.1 with $a \in A$ arbitrary, $b' = \mathbf{1}$ and $a' = b = 0$. $\square$

We will apply 2.2 to an algebra A of the form

$$\mathbb{K}[T]e_0 := \bigl\{P(T)e_0;\ P \text{ polynomial}\bigr\} = \operatorname{span}\bigl\{T^i e_0;\ i \in \mathbb{N}\bigr\},$$

where $T : Z \to Z$ is a linear map on some vector space Z, and $e_0 \in Z$ is such that the vectors $T^i e_0$ are linearly independent. This happens in particular if T is a linear operator on some infinite-dimensional Fréchet space X and if e_0 is a cyclic vector for T. The product on $\mathbb{K}[T]e_0$ is defined by

$$P(T)e_0 \cdot Q(T)e_0 = PQ(T)e_0.$$

Notice that $\mathbb{K}[T]e_0$ has a unit, namely e_0. When T is a Fréchet space operator with cyclic vector e_0, the algebra $\mathbb{K}[T]e_0$ inherits the topology of the underlying Fréchet space X.

Corollary 2.3. *Let X be an infinite-dimensional Fréchet space, and let $T \in \mathcal{L}(X)$ be cyclic with cyclic vector e_0. Assume there exists a nonzero linear functional $\phi : \mathbb{K}[T]e_0 \to \mathbb{K}$ such that the map $(x, y) \mapsto \phi(x \cdot y)$ is continuous on $\mathbb{K}[T]e_0 \times \mathbb{K}[T]e_0$. Then $T \oplus T$ is not hypercyclic on $X \times X$.*

Proof. Suppose on the contrary that $T \oplus T$ is hypercyclic. Then the set of hypercyclic vectors for $T \oplus T$ is dense in $X \oplus X$. Let $a, b, a', b' \in \mathbb{K}[T]e_0$ be arbitrary. One can find a sequence of $(T \oplus T)$-hypercyclic vector $(x_n \oplus y_n) \subset X \oplus X$ such that $x_n \oplus y_n \to a \oplus b$. Then one can choose a sequence of integer (k_n) such that $T^{k_n} x_n \oplus T^{k_n} y_n \to a' \oplus b'$. Finally, since e_0 is cyclic for T, we may find polynomials P_n, Q_n such that $T^{k_n} P_n(T)e_0 \oplus T^{k_n} Q_n(T)e_0 - T^{k_n} x_n \oplus T^{k_n} y_n \to 0$. Setting $p_n := P_n(T)e_0$, $q_n := Q_n(T)e_0$ and $r_n := T^{k_n} e_0$, one can apply Corollary 2.2 with the semi-norm $\mathfrak{n}$ defined by $\mathfrak{n}(z) = |\phi(z)|$, and this gives a contradiction since $\phi \neq 0$. $\square$

Remark. The above proof gives in fact the formally stronger conclusion that $T \oplus T$ cannot have a dense set of cyclic vectors: just note that the proof still works if one replaces T^{k_n} by $R_n(T)$, for some polynomial R_n. However, this is not a real strengthening, for if T is a hypercyclic operator such that $T \oplus T$ is cyclic, then $T \oplus T$ is actually hypercyclic by a result of S. Grivaux [6, Proposition 4.1].

From now on, X is a Banach space with a normalized unconditional basis $(e_i)_{i\in\mathbb{N}}$ whose associated forward shift is continuous. We put $c_{00} := \operatorname{span}\{e_i;\ i \in \mathbb{N}\}$. In view of Corollary 2.3, our main result will be proved if we are able to construct a linear operator $T : c_{00} \to c_{00}$ and a nonzero linear functional $\phi : c_{00} \to \mathbb{K}$ such that the following properties hold.

(a) $\operatorname{span}\{T^i e_0;\ i \in \mathbb{N}\} = \operatorname{span}\{e_i;\ i \in \mathbb{N}\}$; in other words $\mathbb{K}[T]e_0 = c_{00}$.
(b) The set $\{T^i e_0;\ i \in \mathbb{N}\}$ is dense in c_{00}.
(c) T is continuous.
(d) The map $(x, y) \mapsto \phi(x \cdot y)$ is continuous on $c_{00} \times c_{00}$.

Indeed, (c) allows to extend T to a continuous linear operator on X, which is hypercyclic with hypercyclic vector e_0 by (b), and whose direct sum with itself is not hypercyclic by (d) and Corollary 2.3.

The operator T and the linear functional ϕ will be constructed in the next two sections. They will both depend on a sequence of positive numbers $(a_n)_{n\geqslant 0}$ tending to infinity and on an increasing sequence of integers $(b_n)_{n\geqslant 0}$. For convenience, we assume that $a_0 = 1$ and $b_0 = 0$. The conditions needed on (a_n) and (b_n) will be specified later.

It will turn out that the sequence (a_n) can be first chosen arbitrarily, but that (b_n) will then have to grow much faster than (a_n). Thus, we could put e.g. $a_n = 2^n$ from the very beginning, but since this would not simplify the proof, we do not specify an explicit value for a_n.

Notations. If P is a polynomial, we denote by $\deg(P)$ the degree of P and by $|P|_1$ the sum of the moduli of the coefficients of P. We choose a countable dense set $\mathbf{Q} \subset \mathbb{K}$, and we fix once and for all an enumeration $(P_n)_{n\in\mathbb{N}}$ of the set of all polynomials with coefficients in $\mathbf{Q}$, with $P_0 = 0$. We will assume from the beginning that $b_n > \deg(P_n)$ for all $n \in \mathbb{N}$.

3. The operator T

One can associate to (a_n) and (b_n) a unique linear map $T : c_{00} \to c_{00}$ satisfying the following two properties:

$$Te_i = 2e_{i+1} \quad \text{if } i \in [b_{n-1}, b_n - 2]; \tag{1}$$

$$T^{b_n}(e_0) = P_n(T)e_0 + \frac{1}{a_n} e_{b_n} \quad \text{for all } n. \tag{2}$$

Indeed, writing

$$\begin{aligned} T^{b_n} e_0 &= T^{b_n - b_{n-1}} T^{b_{n-1}} e_0 \\ &= T^{b_n - b_{n-1}} \left(P_{n-1}(T)e_0 + \frac{1}{a_{n-1}} e_{b_{n-1}} \right) \\ &= \frac{2^{b_n - b_{n-1} - 1}}{a_{n-1}} T(e_{b_n - 1}) + T^{b_n - b_{n-1}} P_{n-1}(T)e_0, \end{aligned}$$

we have to set

$$Te_{b_n-1} = \frac{a_{n-1}}{2^{b_n-b_{n-1}-1}}\left(\frac{1}{a_n}e_{b_n} + P_n(T)e_0 - T^{b_n-b_{n-1}}P_{n-1}(T)e_0\right); \tag{3}$$

and since $\deg(P_n) < b_n$ and $\deg(P_{n-1}) < b_{n-1}$, the operator T is well defined by formulae (1) and (3).

By definition of T, we have $\{P(T)e_0;\ \deg P \leqslant N\} = \operatorname{span}\{e_0;\dots;e_N\}$ for all $N \in \mathbb{N}$, so that $\mathbb{K}[T]e_0 = c_{00}$. It follows that the set $\{P_n(T)e_0;\ n \in \mathbb{N}\}$ is dense in c_{00}, and hence (by (2)) that the set $\{T^i e_0;\ i \in \mathbb{N}\}$ is dense as well. In other words, the first two conditions needed to prove Theorem 1.1 are satisfied, whatever the choice of (a_n) and (b_n) may be.

In the remainder of this section, we intend to show that if the sequences (a_n) and (b_n) are suitably chosen, then the operator T is continuous. We will make use of the ℓ^1-norm on c_{00} associated to the basis (e_i); this norm will be denoted by $\|\cdot\|_1$. Thus, if $x = \sum_i x_i e_i$, then $\|x\|_1 = \sum_i |x_i|$. If E is a finite-dimensional subspace of c_{00}, we denote by $\|T_{|E}\|_{1,1}$ the norm of the operator $T_{|E}: (E, \|\cdot\|_1) \to (c_{00}, \|\cdot\|_1)$.

Notice that formula (3) above can be written as

$$T(e_{b_n-1}) = \varepsilon_n e_{b_n} + f_n, \tag{4}$$

where

$$\varepsilon_n = \frac{a_{n-1}}{2^{b_n-b_{n-1}-1}a_n}, \quad \text{and} \quad f_n = \frac{a_{n-1}}{2^{b_n-b_{n-1}-1}}\big(P_n(T)e_0 - T^{b_n-b_{n-1}}P_{n-1}(T)e_0\big).$$

Since $\deg(P_n) < b_n$ and $\deg(P_{n-1}) < b_{n-1}$, the vector f_n is supported on $[0, b_n)$.

Given a positive integer n, we shall say that T *is convenient up to stage* n if the following properties hold:

- $\varepsilon_k \leqslant \min(1, \frac{2^{-k-3}}{2a_k\|P_k(T)e_0\|_1})$ for all $k \in \{1;\dots;n\}$;
- $\|f_k\|_1 \leqslant 2^{-k}$ for all $k \in \{1;\dots;n\}$.

The continuity of T will be an easy consequence of the next lemma.

Lemma 3.1. *Let $a_1,\dots,a_n$, $b_1,\dots,b_n$ be given, and assume T is convenient up to stage n. Then, for any choice of a_{n+1}, one can find a positive number B such that T is convenient up to stage $n+1$ whenever $b_{n+1} > B$.*

The following simple remark will be useful for the proof of Lemma 3.1.

Remark. If T is convenient up to stage n, then $\|T_{|\operatorname{span}\{e_0;\dots;e_{b_{n+1}-2}\}}\|_{1,1} \leqslant 2$.

Proof of Remark. We have $\|T(e_{b_k-1})\|_1 \leqslant \varepsilon_k + \|f_k\|_1 \leqslant 2$ for all $k \in \{1;\dots;n\}$, so that $\|T(e_i)\|_1 \leqslant 2$ for all $i < b_{n+1}-1$. $\square$

Proof of Lemma 3.1. Recall that $Te_{b_{n+1}-1} = \varepsilon_{n+1} e_{b_{n+1}} + f_{n+1}$, with $\varepsilon_{n+1} = \frac{a_n}{2^{b_{n+1}-b_n-1} a_{n+1}}$ and

$$f_{n+1} = \frac{a_n}{2^{b_{n+1}-b_n-1}} P_{n+1}(T)e_0 - 2a_n \left(\frac{T}{2}\right)^{b_{n+1}-b_n} P_n(T)e_0.$$

If b_{n+1} is large enough, then ε_{n+1} is small and the first term in f_{n+1} has norm less than 2^{-n-2}. Therefore, we just have to check that $\|2a_n(T/2)^{b_{n+1}-b_n} P_n(T)e_0\|_1 \leqslant 2^{-n-2}$ if b_{n+1} is large enough.

Claim 1. *Assume $b_{n+1} > 2b_n$. Let $p \in [0, b_n)$, so that $b_k \leqslant p < b_{k+1}$ for some $k \leqslant n-1$. If r satisfies $b_n - p \leqslant r \leqslant b_n$, then*

$$\frac{T^r(e_p)}{2^r} = \frac{\varepsilon_{k+1} \dots \varepsilon_n}{2^{n-k}} e_{r+p} + \sum_{j=k+1}^{n} \frac{\prod_{k+1 \leqslant s < j} \varepsilon_s}{2^{j-k}} \frac{T^{r+p-b_j}(f_j)}{2^{r+p-b_j}}.$$

Proof of Claim 1. This is proved by reverse induction, starting with $k = n-1$. For $k+1 = n$, one has

$$\begin{aligned}
\frac{T^r(e_p)}{2^r} &= 2^{-r} T^{r-(b_n-p)} T^{b_n-p}(e_p) \\
&= 2^{-r} T^{r+p-b_n} 2^{b_n-p-1}(\varepsilon_n e_{b_n} + f_n) \\
&= \frac{1}{2} \frac{T^{r+p-b_n}}{2^{r+p-b_n}} (\varepsilon_n e_{b_n} + f_n) \\
&= \frac{\varepsilon_n}{2} e_{r+p} + \frac{1}{2} \frac{T^{r+p-b_n}}{2^{r+p-b_n}} (f_n),
\end{aligned}$$

where we have used in the last line the inequalities $r + p < 2b_n < b_{n+1}$. Thus, the formula holds when $k = n-1$.

Assume the result is known for all pairs (p', r') with $p' \in [b_{k+1}, b_{k+2})$ and $b_{k+1} - p' \leqslant r' \leqslant b_{k+1}$. If $p \in [b_k, b_{k+1})$ and $b_k - p \leqslant r \leqslant b_k$, then, writing

$$\frac{T^r(e_p)}{2^r} = \frac{1}{2} \frac{T^{r-b_{k+1}+p}}{2^{r-b_{k+1}+p}} (\varepsilon_{k+1} e_{b_{k+1}} + f_{k+1}),$$

we conclude by using the induction hypothesis with $p' = b_{k+1}$ and $r' = r + p - b_{k+1}$. □

Claim 2. *If $u \in c_{00}$ is supported on $[0, b_n)$ and if i is a positive integer such that $(i+1)b_n \leqslant b_{n+1}$, one can write*

$$\frac{T^{ib_n}(u)}{2^{ib_n}} = v_i + \sum_p \frac{T^{r_p}(u_p)}{2^{r_p}},$$

where the sum is finite, $\sum_p \|u_p\|_1 \leqslant \frac{\|u\|_1}{2^i}$, $\mathrm{supp}(u_p) \subset [0, b_n)$, $r_p < ib_n$, $\mathrm{supp}(v_i) \subset [b_n, (i+1)b_n)$ and $\|v_i\|_1 \leqslant \frac{\varepsilon_n}{2}(1 + \frac{1}{2} + \dots + \frac{1}{2^{i-1}})\|u\|_1$.

Proof of Claim 2. We prove by induction on i that the result holds for all u. If $i=1$, we apply Claim 1 with $r=b_n$. Writing $u=\sum_p x_p e_p$ and defining k_p by $b_{k_p}\leqslant p<b_{k_p+1}$, we get

$$\frac{T^{b_n}(u)}{2^{b_n}}=v_1+\sum_{p<b_n}\frac{T^p(u_p)}{2^p},$$

where

$$v_1=\sum_{p<b_n}x_p\frac{\varepsilon_{k_p+1}\dots\varepsilon_n}{2^{n-k_p}}e_{b_n+p}\quad\text{and}\quad u_p=x_p\sum_{j=k_p+1}^{n}\frac{\prod_{k_p+1\leqslant s<j}\varepsilon_s}{2^{j-k_p}}\,\frac{T^{b_n-b_j}(f_j)}{2^{b_n-b_j}}.$$

Then v_1 is supported on $[b_n,2b_n)$ and

$$\|v_1\|_1\leqslant\frac{\varepsilon_n}{2}\sum_p|x_p|=\frac{\varepsilon_n}{2}\|u\|_1.$$

Moreover, each u_p is supported on $[0,b_n)$ because $\operatorname{supp}(f_j)\subset[0,b_j)$ for all j. Finally, we have $\|u_p\|_1\leqslant\frac{|x_p|}{2}$ for all p, because $\|f_j\|_1\leqslant\frac{1}{2}$ for all j and $T^{b_n-b_j}$ has norm not greater than $2^{b_n-b_j}$ when restricted to $\operatorname{span}\{e_0;\dots;e_{b_j}\}$. Thus, we get

$$\sum_p\|u_p\|_1\leqslant\frac{\|u\|_1}{2}.$$

Assume the result has been proved for i and that $(i+2)b_n\leqslant b_{n+1}$. Applying the case $i=1$ and then the induction hypothesis to each u_p, we get

$$\begin{aligned}\frac{T^{(i+1)b_n}(u)}{2^{(i+1)b_n}}&=\frac{T^{ib_n}(v_1)}{2^{ib_n}}+\sum_{p<b_n}\frac{T^p}{2^p}\big(T^{ib_n}(u_p)\big)\\&=\frac{T^{ib_n}(v_1)}{2^{ib_n}}+\sum_{p<b_n}\frac{T^p}{2^p}\left(v_{p,i}+\sum_q\frac{T^{r_{p,q}}(u_{p,q})}{2^{r_{p,q}}}\right)\\&=\frac{T^{ib_n}(v_1)}{2^{ib_n}}+\sum_{p<b_n}\frac{T^p(v_{p,i})}{2^p}+\sum_{p<b_n}\sum_q\frac{T^{p+r_{p,q}}(u_{p,q})}{2^{p+r_{p,q}}}.\end{aligned}$$

Since $\|T_{|\operatorname{span}\{e_0;\dots;e_{b_{n+1}-2}\}}\|_{1,1}\leqslant 2$, we have

$$\begin{aligned}\left\|\frac{T^{ib_n}(v_1)}{2^{ib_n}}+\sum_{p<b_n}\frac{T^p(v_{p,i})}{2^p}\right\|_1&\leqslant\frac{\varepsilon_n\|u\|_1}{2}+\sum_{p<b_n}\frac{\varepsilon_n\|u_p\|_1}{2}\left(1+\frac{1}{2}+\dots+\frac{1}{2^{i-1}}\right)\\&\leqslant\frac{\varepsilon_n\|u\|_1}{2}\left(1+\frac{1}{2}+\dots+\frac{1}{2^i}\right).\end{aligned}$$

Moreover,

$$\sum_{p,q}\|u_{p,q}\|_1 \leqslant \sum_{p<b_n}\frac{\|u_p\|_1}{2^i} \leqslant \frac{\|u\|_1}{2^{i+1}}.$$

Finally, we have $p + r_{p,q} < b_n + (i+1)b_n = (i+2)b_n$ for all pairs (p,q). This concludes the proof. □

Now, we apply Claim 2 with $u = 2a_n P_n(T)e_0$. Since T is convenient up to stage n, we have $\|v_i\|_1 \leqslant \varepsilon_n\|u\|_1 \leqslant 2^{-n-3}$ for all i such that $(i+1)b_n \leqslant b_{n+1}$. Moreover, since $\|T_{|\mathrm{span}\{e_0;\ldots;e_{b_{n+1}-2}\}}\|_{1,1} \leqslant 2$ and $r_p < ib_n$ for all p, we also have

$$\left\|\sum_p \frac{T^{r_p}(u_p)}{2^{r_p}}\right\|_1 \leqslant \sum_p \|u_p\|_1 \leqslant \frac{\|u\|_1}{2^i}.$$

Thus, fixing i_0 with $\frac{\|u\|_1}{2^{i_0}} \leqslant 2^{-n-3}$, any choice of b_{n+1} with $b_{n+1} - b_n \geqslant i_0 b_n$ yields the desired result. □

Corollary 3.2. *There exist functions $F_n : \mathbb{N}^n \times \mathbb{N}^{n+1} \to \mathbb{N}$ such that the following holds: if $b_{n+1} \geqslant F_n(b_1,\ldots,b_n,a_1,\ldots,a_{n+1})$ for all n, then T is continuous on c_{00} with respect to the topology of X.*

Proof. By the lemma, it is enough to show that if T is convenient up to every stage n, then T is continuous.

We can decompose T as $T = R + K$, where R is a forward weighted shift with a bounded weight sequence, and K is defined by: $K(e_{b_n-1}) = f_n$ for all n and $K(e_i) = 0$ otherwise. Since the forward shift associated to (e_i) is continuous and since the basis (e_i) is unconditional, the operator R is continuous.

Since f_n is supported on $[0, b_n)$, we have

$$\|K(e_{b_n-1})\|_X \leqslant \max\{\|e_j\|_X;\ j \leqslant b_n - 1\} \cdot \|f_n\|_1$$

for all $n \geqslant 1$. Since the sequence (e_i) is bounded, it follows that $\sum_0^\infty \|K(e_i)\|_X < \infty$; and since the sequence of coordinate functionals (e_i^*) is also bounded (because $\inf_i \|e_i\|_X > 0$), we conclude that the operator K is continuous. □

Remark. It is not difficult to show that the operator K is compact, as a uniform limit of finite rank operators. Hence, T is a compact perturbation of a weighted shift operator.

4. The linear functional ϕ

In this part, we view c_{00} as $\mathrm{span}\{T^i e_0;\ i \in \mathbb{N}\}$ rather than $\mathrm{span}\{e_i;\ i \in \mathbb{N}\}$. In particular, we will say that a vector $z \in c_{00}$ is supported on some set $I \subset \mathbb{N}$ if $z \in \mathrm{span}\{T^i e_0;\ i \in I\}$.

We denote by $|\cdot|_1$ the ℓ^1-norm associated to the basis $(T^i e_0)$. Thus, if $z = P(T)e_0 \in c_{00}$, then $|z|_1 = |P|_1$.

From now on, we fix some positive number $\varepsilon \in (0,1)$, and we assume that $\deg(P_n) + \varepsilon b_n < b_n$ and $b_{n+1} \geqslant (2+\varepsilon)b_n$ for all $n \in \mathbb{N}$.

Using the same idea as in [5], we define a linear functional $\phi : c_{00} \to \mathbb{K}$ as follows. We put $\phi(e_0) = 1$ and $\phi(T^i e_0) = 0$ if $i \in (0, b_1)$. If $i \in [b_k, b_{k+1})$ for some $k \geqslant 1$, we set:

$$\phi(T^i e_0) = \begin{cases} \phi(P_k(T)T^{i-b_k} e_0) & \text{if } i \in [b_k, (1+\varepsilon)b_k) \cup [2b_k, (2+\varepsilon)b_k), \\ 0 & \text{otherwise.} \end{cases}$$

Notice that $\phi(T^i e_0)$ is indeed well defined if $\phi(T^j e_0)$ is known for all $j < i$, because $\deg(P_k) + i - b_k < i$ and hence $P_k(T)T^{i-b_k} e_0$ is supported on $[0, i)$.

The next lemma collects the properties of ϕ which will be needed below.

Lemma 4.1. *The following properties hold for all $k \geqslant 1$.*

(1) $\phi((T^{b_k} - P_k(T))z) = 0$ *whenever z is supported on* $[0, \varepsilon b_k) \cup [b_k, (1+\varepsilon)b_k)$.
(2) $\max_{i \in [0, b_k)} |\phi(T^i e_0)| \leqslant N_k := \prod_{0<j<k} \max(1, |P_k|_1)^2$.

Proof. Part (1) is obvious from the definition of ϕ. The proof of part (2) is the same as in [5], but we give the details for the sake of completeness. The result is true for $k = 1$ if we give the value 1 to an empty product. Assume the inequality holds for k. Setting $\phi_i := |\phi(T^i e_0)|$ we have

$$\begin{aligned} \max_{i \in [b_k, 2b_k)} \phi_i &= \max_{i \in [b_k, (1+\varepsilon)b_k)} \bigl|\phi\bigl(P_k(T)T^{i-b_k} e_0\bigr)\bigr| \\ &\leqslant |P_k|_1 \cdot \max_{j < \varepsilon b_k + \deg(P_k)} \phi_j \\ &\leqslant |P_k|_1 \cdot N_k, \end{aligned}$$

because $\varepsilon b_k + \deg(P_k) < b_k$. Similarly, we have

$$\max_{i \in [2b_k, b_{k+1})} \phi_i = \max_{i \in [2b_k, (2+\varepsilon)b_k)} \phi_i \leqslant |P_k|_1 \max_{j < 2b_k} \phi_j \leqslant |P_k|_1^2 N_k.$$

We conclude that $\max_{i \in [b_k, b_{k+1})} \phi_i \leqslant N_{k+1}$, and the result follows by induction. □

We now prove that the map $(x, y) \mapsto \phi(x \cdot y)$ is continuous if (a_n) and (b_n) are suitably chosen.

Lemma 4.2. *There exist functions $G_n : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ such that the following holds*:

$$\textit{if } b_n \geqslant G_n(b_{n-1}, a_n) \textit{ for all } n \geqslant 1, \textit{ then } \sum_{p,q} \bigl|\phi(e_p \cdot e_q)\bigr| < +\infty.$$

Proof. If $p \in [b_k, b_{k+1})$ and $q \in [b_l, b_{l+1})$, then the definition of T gives

$$\begin{aligned} e_p &= \frac{a_k}{2^u}\bigl(T^{b_k} - P_k(T)\bigr)T^u(e_0), \\ e_q &= \frac{a_l}{2^v}\bigl(T^{b_l} - P_l(T)\bigr)T^v(e_0), \end{aligned}$$

where $u = p - b_k \in [0, b_{k+1} - b_k)$ and $v = q - b_l \in [0, b_{l+1} - b_l)$. Thus, setting $\Lambda := \{(m, w) \in \mathbb{N} \times \mathbb{N};\ w < b_{m+1} - b_m\}$ and

$$y_{(k,u)(l,v)} = \bigl(T^{b_k} - P_k(T)\bigr)\bigl(T^{b_l} - P_l(T)\bigr)T^{u+v}(e_0),$$

we have to prove that

$$\Sigma := \sum_{((k,u),(l,v))\in\Lambda\times\Lambda} \frac{a_k a_l}{2^{u+v}} \bigl|\phi(y_{(k,u)(l,v)})\bigr| < +\infty.$$

We will need the following facts.

Claim 1. *If $k \leqslant l$, then we always have $|\phi(y_{(k,u)(l,v)})| \leqslant M_l := N_{l+2} \cdot \max_{j\leqslant l}(1 + |P_j|_1)^2$.*

Proof of Claim 1. Observe that $\operatorname{supp}(y_{(k,u)(l,v)}) \subset [0, b_k + u + b_l + v] \subset [0, b_{k+1} + b_{l+1})$, hence $y_{(k,u)(l,v)}$ is supported on $[0, b_{l+2})$ because $k \leqslant l$ and $b_{l+2} > 2b_{l+1}$. Moreover, one has $|y_{(k,u)(l,v)}|_1 \leqslant (1 + |P_k|_1)(1 + |P_l|_1)$. By Lemma 4.1, the result follows. $\square$

Claim 2. *In each of the following two cases, we have $\phi(y_{(k,u)(l,v)}) = 0$:*

- *$k = l$ and $u + v + \deg(P_k) < \varepsilon b_k$;*
- *$k < l$ and $u + v + b_k < \varepsilon b_l$.*

Proof of Claim 2. When $k = l \geqslant 1$, we write

$$\begin{aligned} y_{(k,u)(k,v)} &= \bigl(T^{b_k} - P_k(T)\bigr)T^{b_k+u+v}e_0 - \bigl(T^{b_k} - P_k(T)\bigr)P_k(T)T^{u+v}e_0 \\ &= \bigl(T^{b_k} - P_k(T)\bigr)(z_1) - \bigl(T^{b_k} - P_k(T)\bigr)(z_2). \end{aligned}$$

Then z_1 is supported on $[b_k, (1+\varepsilon)b_k)$ if $u + v < \varepsilon b_k$, and z_2 is supported on $[0, \varepsilon b_k)$ if $u + v + \deg(P_k) < \varepsilon b_k$. By Lemma 4.1, this gives the first part of the claim. When $l > k$, we just write

$$y_{(k,u)(l,v)} = \bigl(T^{b_l} - P_l(T)\bigr)(z),$$

where $z = (T^{b_k} - P_k(T))T^{u+v}e_0$ is supported on $[0, u + v + b_k) \subset [0, \varepsilon b_l)$. $\square$

Now, we write

$$\Sigma = \Sigma_1 + 2\Sigma_2,$$

where Σ_1 is the sum over all pairs $((k,u),(l,v))$ with $k = l$, and Σ_2 is the sum over the pairs with $l > k$.

By the two above claims, we have

$$\Sigma_1 \leqslant \sum_{k=0}^{\infty} a_k^2 M_k \sum_{\substack{(u,v)\in\mathbb{N}\times\mathbb{N} \\ u+v\geqslant\varepsilon b_k-\deg(P_k)}} \frac{1}{2^{u+v}} \leqslant \sum_{k=0}^{\infty} a_k^2 M_k \sum_{i\geqslant\varepsilon b_k-\deg(P_k)} \frac{i+1}{2^i},$$

so that $\Sigma_1 < \infty$ provided b_n is always large enough with respect to a_n.

To estimate Σ_2, we use the claims to get

$$\Sigma_2 \leqslant \sum_{k=0}^{\infty}\sum_{l>k} a_l^2 M_l \sum_{u+v\geqslant \varepsilon b_l - b_k} \frac{1}{2^{u+v}}$$
$$\leqslant \sum_{k=0}^{\infty} 2^{b_k} \sum_{l>k} a_l^2 M_l \sum_{i\geqslant \varepsilon b_l} \frac{i+1}{2^i}.$$

Thus, we have $\Sigma_2 < \infty$ provided (b_n) is rapidly increasing and b_n is large enough with respect to a_n. This concludes the proof of the lemma. □

Corollary 4.3. *If (b_n) is rapidly increasing and b_n is large enough with respect to a_n, then the map $(x, y) \mapsto \phi(x \cdot y)$ is continuous on $c_{00} \times c_{00}$.*

Proof. Writing $x = \sum_p x_p e_p$ and $y = \sum_q y_q e_q$, we get

$$\left|\phi(x\cdot y)\right| \leqslant \sum_{p,q} |x_p|\,|y_q|\left|\phi(e_p\cdot e_q)\right| \leqslant C^2 \sum_{p,q}\left|\phi(e_p\cdot e_q)\right|\|x\|\,\|y\|$$

for all $(x, y) \in c_{00} \times c_{00}$, where $C = \sup_i \|e_i^*\|$. □

Putting together Corollaries 3.2 and 4.3, the proof of our main theorem is now complete.

5. Variations on the main result

It should be clear from the proof that Theorem 1.1 can be formulated in a Fréchet space setting. More precisely, the result remains valid if X is a separable Fréchet space with an unconditional basis (e_i) such that the following properties hold true, where (e_i^*) is the sequence of coordinate functionals: the forward shift associated to (e_i) is continuous, the sequence (e_i) is bounded, and the sequence (e_i^*) is equicontinuous. However, this does not seem to apply to the most interesting non-Banach examples.

Nevertheless, we do have the following result. Let us denote by $\mathcal{H}(\Omega)$ the space of all holomorphic functions on an open set $\Omega \subset \mathbb{C}$.

Proposition 5.1. *If $\Omega \subset \mathbb{C}$ is a simply connected domain, then there exists a hypercyclic operator on $\mathcal{H}(\Omega)$ which does not satisfy the Hypercyclicity Criterion.*

Proof. We may assume that Ω is a disk $D(0, R)$, where $1 < R \leqslant \infty$. We will mimic the proof of Theorem 1.1, but the operator has to be slightly modified. Let us denote by $(e_i)_{i\in\mathbb{N}}$ the "canonical basis" of $\mathcal{H}(\Omega)$, $e_i(z) = z^i$. If one wants to imitate the proof of Theorem 1.1, one difficulty comes to mind: the operator T defined above is hypercyclic because $\frac{1}{a_n} e_{b_n} \to 0$. But this is no longer true for an *arbitrary* sequence (a_n) tending to infinity in the present setting: a_n must grow faster than r^{b_n} for any $r < R$. Thus, a_n grows in fact much faster than b_n, so that one cannot simply reproduce the proof of Theorem 1.1. On the other hand, the highly non-Banach structure of $\mathcal{H}(\Omega)$ allows continuous shifts with unbounded weights, and ensures a fast decay of the coordinate functionals associated to (e_i), so one can hope to overcome this difficulty.

Let us fix some increasing sequence $(\rho_n) \subset (1, R)$ with $\lim_{n\to\infty} \rho_n = R$, and let (b_n), (P_n) be as in the proof of Theorem 1.1, with $b_n > 1 + \deg(P_n)$ for all n. According to the above remarks, we define our linear map $T : c_{00} \to c_{00}$ by

$$Te_i = (i+1)e_{i+1} \quad \text{if } i \in [b_{n-1}, b_n - 2]; \tag{5}$$

$$T^{b_n}(e_0) = P_n(T)e_0 + \frac{1}{a_n} e_{b_n} \quad \text{for all } n, \text{ where } a_n = \rho_n^{b_n}. \tag{6}$$

An easy calculation gives

$$T(e_{b_n - 1}) = \varepsilon_n e_{b_n} + f_n,$$

with

$$\varepsilon_n = \frac{\rho_{n-1}^{b_{n-1}}}{(b_{n-1}+1)\dots(b_n-1)\rho_n^{b_n}} \quad \text{and} \tag{7}$$

$$f_n = \frac{\rho_{n-1}^{b_{n-1}}}{(b_{n-1}+1)\dots(b_n-1)} \big(P_n(T)e_0 - T^{b_n - b_{n-1}} P_{n-1}(T)e_0\big). \tag{8}$$

As in the proof of Theorem 1.1, we will show that if the sequence (b_n) is sufficiently fast increasing, then T extends to a continuous linear operator on $\mathcal{H}(\Omega)$ (which is hypercyclic by the choice of (a_n)) and one can construct a linear functional $\phi : c_{00} \to \mathbb{C}$ satisfying the required property.

The proof of continuity is simpler than the corresponding one in the Banach space case. Let us say that T is convenient up to stage n if $\varepsilon_k \leqslant 1$ and $\|f_k\|_1 \leqslant 1$ for all $k \in \{1; \dots; n\}$.

We first show that if T is convenient up to all stages n, then it is continuous. One can write $T = R + K$, where R is a weighted forward shift whose weights have polynomial growth, and K is defined by $K(e_{b_n-1}) = f_n$ and $K(e_i) = 0$ otherwise. The continuity of R is clear, so we just have to show that K is continuous. For each $r \in (0, R)$, let N_r be the semi-norm on $\mathcal{H}(\Omega)$ defined by

$$N_r(f) = \sup\big\{|f(z)|;\ |z| \leqslant r\big\}.$$

The main point is the following: if $N \in \mathbb{N}$ and $g = \sum_i c_i(g)e_i \in \operatorname{span}\{e_0; \dots; e_N\}$, then $N_r(g) \leqslant r^N \sum_i |c_i(g)| = r^N \|g\|_1$ for all $r \geqslant 1$. Since $\|f_n\|_1 \leqslant 1$ for all n and f_n is supported on $[0, b_n)$, it follows that if $f = \sum_i c_i(f)e_i \in c_{00}$ and $r \in [1, R)$ then

$$\begin{aligned} N_r\big(K(f)\big) &\leqslant \sum_n \big|c_{b_n-1}(f)\big| N_r(f_n) \leqslant \sum_n \big|c_{b_n-1}(f)\big| r^{b_n-1} \\ &\leqslant C_{r'} N_{r'}(f), \end{aligned}$$

where r' has been chosen with $r < r' < R$. This shows that K is continuous.

We now prove that if $b_1, \dots, b_{n-1}$ have been fixed and if T is convenient up to stage $n-1$, then T is convenient up to stage n provided b_n is large enough. From (7) and (8), it is clear that ε_n and the first term in f_n are small if b_n is large. It remains to estimate

$$\left\| \frac{\rho_{n-1}^{b_{n-1}}}{(b_{n-1}+1)\dots(b_n-1)} T^{b_n-b_{n-1}}(u) \right\|_1, \quad \text{where } u = P_{n-1}(T)e_0.$$

Since T is convenient up to stage $n-1$, we have $\|T_{|\mathrm{span}\{e_0;\dots;e_k\}}\|_{1,1} \leqslant k+1$ for all $k \leqslant b_n-2$. Since u is supported on $[0, \deg(P_{n-1})]$ and $d := \deg(P_{n-1}) \leqslant b_{n-1}-2$, it follows that

$$\begin{aligned} \left\|T^{b_n-b_{n-1}}(u)\right\|_1 &\leqslant (d+1)\dots(d+b_n-b_{n-1})\|u\|_1 \\ &\leqslant (d+1)(d+2)(b_{n-1}+1)\dots(b_n-2)\,\|u\|_1. \end{aligned}$$

Thus, we get

$$\left\| \frac{\rho_{n-1}^{b_{n-1}}}{(b_{n-1}+1)\dots(b_n-1)} T^{b_n-b_{n-1}}(u) \right\|_1 \leqslant \frac{(d+1)(d+2)\rho_{n-1}^{b_{n-1}}\|u\|_1}{b_n-1},$$

and this is small if b_n is large enough.

We now turn to the linear functional $\phi : c_{00} \to \mathbb{C}$, whose definition is the same as in the proof of Theorem 1.1. In the present setting, it will be enough to show that if the sequence (b_n) is sufficiently fast increasing, then the scalars $\phi(e_p \cdot e_q)$ are bounded. Indeed, once this is done, one can estimate $\phi(f \cdot g)$ as follows, for $f = \sum_p c_p(f)\, e_p$ and $g = \sum_q c_q(g) e_q$:

$$\begin{aligned} \left|\phi(f\cdot g)\right| &\leqslant \sum_{p,q} \left|c_p(f)\right|\left|c_q(g)\right|\left|\phi(e_p\cdot e_q)\right| \\ &\leqslant C\sum_{p,q}\left|c_p(f)\right|\left|c_q(g)\right| \\ &\leqslant C_r N_r(f) N_r(g), \end{aligned}$$

where r has been chosen with $1 < r < R$.

We proceed as in the proof of 1.1. Writing $p = b_k + u$ and $q = b_l + v$, where $u \in [0, b_{k+1} - b_k)$ and $v \in [0, b_{l+1} - b_l)$, we have

$$\begin{aligned} e_p &= \frac{\rho_k^{b_k}}{(b_k+1)\dots(b_k+u)}\big(T^{b_k} - P_k(T)\big)T^u(e_0), \\ e_q &= \frac{\rho_l^{b_l}}{(b_l+1)\dots(b_l+v)}\big(T^{b_l} - P_l(T)\big)T^v(e_0). \end{aligned}$$

Thus, we have to estimate the following quantities:

$$\alpha_{(k,u)(l,v)} := \frac{\rho_k^{b_k}\rho_l^{b_l} M_l}{(b_k+1)\dots(b_k+u)(b_l+1)\dots(b_l+v)},$$

where $k \leqslant l$ and M_l is defined as in Claim 1 above. Moreover, assuming $\deg(P_n) \ll b_n \ll b_{n+1}$, it follows from Claim 2 that we only have to consider two cases:

- $k = l$ and $u + v \geqslant cb_k$;
- $k < l$ and $u + v \geqslant cb_l$,

where c is some positive constant.

In the first case, we have $\max(u, v) \geqslant \frac{c}{2} b_k$, hence

$$\alpha_{(k,u)(k,v)} \leqslant \frac{\rho_k^{2b_k} M_k}{(b_k + 1)^{\frac{c}{2} b_k}},$$

and this is clearly bounded, say less than 1, if b_k is large enough. In the second case, we have $\max(u, v) \geqslant \frac{c}{2} b_l$. Since $u < b_{k+1} - b_k$ (which was not used in the proof of 1.1) and since we are of course assuming $b_{l-1} << b_l$, two subcases can occur: either $k = l - 1$ and $u \geqslant \frac{c}{2} b_l$, or $v \geqslant \frac{c}{2} b_l$. In either subcase, we can write

$$\alpha_{(k,u)(l,v)} \leqslant \frac{\rho_l^{2b_l} M_l}{(b_{l-1} + 1)^{\frac{c}{2} b_l}},$$

which is less than 1 if b_{l-1} is large enough. This concludes the proof. □

In another direction, we now show that Theorem 1.1 can be extended to a larger class of Banach spaces.

Theorem 5.2. *Let X be a separable Banach space. Assume X admits a complemented subspace X_0 having a normalized unconditional basis with continuous forward shift. Then there exists a hypercyclic operator on X whose direct sum with itself is not hypercyclic.*

We may clearly assume that the codimension of X_0 is infinite. Then Theorem 5.2 follows immediately from Theorem 1.1 and the following lemma, which is a variant of a well-known result of S. Ansari [1]. The very simple proof below is due to the referee.

Lemma 5.3. *Let X_0, Y be two separable, infinite-dimensional Banach spaces. If T_0 is a hypercyclic operator on X_0, then there exists an operator $R \in \mathcal{L}(Y)$ such that $T := T_0 \oplus R$ is hypercyclic on $X := X_0 \oplus Y$.*

Proof. Recall that an operator $R \in \mathcal{L}(Y)$ is said to be *mixing* if, for each pair (U, V) of nonempty open subsets of Y, one has $T^n(U) \cap V \neq \emptyset$ for all but finitely many $n \in \mathbb{N}$. It is obvious that if $R \in \mathcal{L}(Y)$ is mixing and $T_0 \in \mathcal{L}(X_0)$ is hypercyclic, then $T_0 \oplus R$ is topologically transitive, i.e. hypercyclic. Thus, it is enough to show that each separable, infinite-dimensional Banach space Y supports a mixing operator. This is done in [6] (see Lemma 2.3 and Theorem 2.6 therein). □

Corollary 5.4. *If X is a separable Banach space containing a complemented copy of some $\ell^p(\mathbb{N})$ or a copy of $c_0(\mathbb{N})$, then there exists a hypercyclic operator on X which does not satisfy the Hypercyclicity Criterion. In particular, this holds for $X = L^1([0, 1])$ or $\mathcal{C}([0, 1])$.*

Proof. The ℓ^p part is obvious, and the c_0 part follows from the fact that if c_0 embeds in a separable Banach space X, then it actually embeds as a complemented subspace. This is a classical result due to A. Sobczyk ([17]; see also [13]). $\square$

6. Concluding remarks

To conclude this paper, it should be observed that there exist Fréchet spaces on which every hypercyclic operator satisfies the Hypercyclicity Criterion. The following result has been obtained independently by K.-G. Grosse-Erdmann [9] using results of J.A. Conejero [4] and G. Herzog and R. Lemmert [11].

Proposition 6.1. *Let X be the sequence space $\mathbb{K}^{\mathbb{N}}$, equipped with the product topology. Then every hypercyclic operator on X satisfies the Hypercyclity Criterion.*

Proof. Let T be such an operator. By a result of S. Grivaux [6, Proposition 4.1], it is enough to prove that given nonempty open sets U_1, U_2, V_1, V_2 in $\mathbb{K}^{\mathbb{N}}$, one can find a polynomial P such that $P(T)(U_1) \cap V_1$ and $P(T)(U_2) \cap V_2$ are both nonempty. We may assume that U_i has the form $J_i^1 \times \cdots \times J_i^q \times \mathbb{K} \times \mathbb{K} \times \cdots$ and that V_i has the form $I_i^1 \times \cdots \times I_i^q \times \mathbb{K} \times \mathbb{K} \times \cdots$, where J_i^j and I_i^j are open subsets of $\mathbb{K}$. For dimensional reasons, there exists a nonzero polynomial P, of degree at most $q^2 + 1$, such that $P(T)$ is represented by a matrix of the form

$$\begin{pmatrix} 0 & B \\ C & D \end{pmatrix},$$

where B is a matrix with q rows and C is a matrix with q columns. Moreover, since T is hypercyclic, $P(T)$ has dense range, and it follows that the rows of B are independent. This ensures that each $P(T)(U_i)$ has the form

$$P(T)(U_i) = \underbrace{\mathbb{K} \times \cdots \times \mathbb{K}}_{q \text{ times}} \times \cdots.$$

Hence, $P(T)(U_1) \cap V_1$ and $P(T)(U_2) \cap V_2$ are both nonempty. $\square$

Of course, the results presented in this paper leave open the problem of characterizing those separable Fréchet spaces on which every hypercyclic operator satisfies the Hypercyclicity Criterion. In particular, it would be very nice to know if there exists a *Banach* space with that property.

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