Indiana University Mathematics Journal ©
Article electronically published on March 21st, 2007

How to Get Common Universal Vectors

F. BAYART & É. MATHERON

ABSTRACT. We prove the existence of common universal vectors for various uncountable families of universal sequence of linear operators. In particular, we give a criterion for a one-parameter family of operators on a Banach space to have a common hypercyclic vector. This criterion relies on some tools from Probability Theory and depends on the geometry of the underlying Banach space. We also study several specific examples, such as shift operators or translation-dilation operators.

1. INTRODUCTION

Let X be a separable Fréchet space. A sequence $\mathbf{T} = (T_n)_{n\in\mathbb{N}}$ of continuous linear operators on X is said to be *universal* if there exists some $x \in X$ such that the set $\{T_n(x) \mid n \in \mathbb{N}\}$ is dense in X. Such a vector x is said to be universal for $\mathbf{T}$, and the set of universal vectors for $\mathbf{T}$ is denoted by $\mathrm{Univ}(\mathbf{T})$. A single operator T is said to be *hypercyclic* if the sequence $\mathbf{T} := (T^n)_{n\in\mathbb{N}}$ is universal, and the set of hypercyclic vectors for T is denoted by $HC(T)$. See [13] for a nice survey on universality and hypercyclicity.

If $\mathbf{T} = (T_n)$ is a universal sequence of operators and if, in addition, all operators T_n have dense range and commute with each other, then $\mathrm{Univ}(\mathbf{T})$ is dense in X since, together with a vector x, it contains the whole set $\{T_n(x) \mid n \in \mathbb{N}\}$. Moreover, without any assumption on $\mathbf{T}$, it is easy to check that $\mathrm{Univ}(\mathbf{T})$ is a G_δ subset of X. In particular, if $T \in \mathcal{L}(X)$ is hypercyclic, then $HC(T)$ is a dense G_δ subset of X. By the Baire Category Theorem, it follows that if $(T_\lambda)_{\lambda\in\Lambda}$ is a *countable* family of hypercyclic operators, then the T_λ's have common hypercyclic vectors, in fact a dense G_δ set of common hypercyclic vectors.

Of course, the situation is less simple in the case of an uncountable family of operators. For example, it is rather easy to check that there is no vector $x \in \ell^2(\mathbb{N})$ which is hypercyclic for all hypercyclic weighted backward shifts on $\ell^2(\mathbb{N})$. More precisely, for each sequence $\mathbf{w} = (w_n) \in \{1; 2\}^{\mathbb{N}^*}$, let $T_{\mathbf{w}}$ be the weighted backward shift associated to $\mathbf{w}$. By a well known result of Salas ([21]), the weighted shift $T_{\mathbf{w}}$ is hypercyclic if and only if $\sup_n w_1 \times \cdots \times w_n = \infty$, which in the present case means that the sequence $\mathbf{w}$ has infinitely many 2's. Now, given any vector $x \in \ell^2(\mathbb{N})$, it is easy to construct inductively, using the fact that $x_n \to 0$, a sequence $\mathbf{w}$ with infinitely many 2's such that $w_1 \times \cdots \times w_n |x_n| \leq 1$ for all $n \in \mathbb{N}^*$. Then $T_{\mathbf{w}}$ is hypercyclic, but $x \notin HC(T_{\mathbf{w}})$ because the first coordinate of $T_{\mathbf{w}}^n(x)$ never exceeds 1.

Nevertheless, several positive results concerning common hypercyclicity have been proved recently. The first example is due to E. Abakumov and J. Gordon ([1]), who showed that if B is the usual backward shift on $\ell^2(\mathbb{N})$, then $\bigcap_{\lambda>1} HC(\lambda B) \neq \varnothing$. Many other natural examples can be found in [2], [5] or [9]. In particular, a general criterion for common universality has been proved by G. Costakis and M. Sambarino ([9]). This criterion is used to prove the existence of common hypercyclic vectors for all operators λD, $\lambda > 0$, where D is the differentiation operator on the space of entire functions $\mathcal{H}(\mathbb{C})$, and it is also applied to the family $(T_\lambda)_{\lambda>1}$, where T_λ is the weighted backward shift on $\ell^2(\mathbb{N})$ associated to the weight sequence $\mathbf{w}(\lambda) = (1 + \lambda/n)_{n\geq 1}$. By very similar arguments, yet without applying the general criterion, it is also proved in [9] that if T_λ is the operator of translation by the complex number λ on $\mathcal{H}(\mathbb{C})$, then $\bigcap_{\lambda\in\mathbb{T}} HC(T_\lambda) \neq \varnothing$.

Notice that in all of these examples, it is shown that there is in fact a residual set of common hypercyclic vectors. This is not accidental. Indeed, it is not hard to check that if $(T_\lambda)_{\lambda\in\Lambda}$ is a family of operator parameterized by some topological space Λ, then $\bigcap_{\lambda\in\Lambda} HC(T_\lambda)$ is always a G_δ set provided Λ is σ-compact and $T_\lambda(x)$ depends continuously on the pair (λ, x). If, in addition, all operators T_λ commute with some hypercyclic operator T, then $\bigcap_{\lambda\in\Lambda} HC(T_\lambda)$ is dense in X whenever it is nonempty, since together with any vector x, it contains the T-orbit of x. Thus, under rather mild hypotheses, the set $\bigcap_{\lambda\in\Lambda} HC(T_\lambda)$ is residual if it is nonempty. Actually, we do not know any natural example where this set is nonempty yet not residual.

In the present paper, we obtain some further positive results concerning common universal vectors for parameterized sequences of operators.

In Section 3, we consider one-parameter families of operators on a Banach space X, and we prove a criterion for common hypercyclicity in that setting. This criterion is formally very similar to the Costakis-Sambarino criterion, but some of its hypotheses are weaker, so that it enables us to treat at least two examples which cannot be attained by the Costakis-Sambarino criterion, namely translation operators on some weighted L^p spaces and composition operators on the Hardy space $H^2(\mathbb{D})$. The price to pay is that our criterion can be applied in the Banach space setting only, since it depends on the *type* of the underlying Banach space X. An interesting feature of the proof is that it mixes Baire Category arguments with a nontrivial result from Probability Theory, namely Dudley's majorization theorem for subgaussian stochastic processes.

Sections 4 and 5 deal with specific examples. In Section 4, we consider one-parameter families of weighted backward shifts; we prove the existence of common hypercyclic vectors under some rather general assumptions. We also prove a common hypercyclicity result for multiples of "generalized backward shifts" in the sense of [12]. In Section 5, we consider operators of translation-dilation type on the space of entire functions $\mathcal{H}(\mathbb{C})$; that is, operators of the form $T_{n,(s,t)}u(z) = a_n(s), u(z + b_n(t))$, where s, t are real parameters.

We conclude the paper by some remarks on the size or the shape of a set of parameters allowing common universality. These are simple observations, but they may indicate that interesting phenomena are to be discovered in that direction.

We tried to put our results into a general framework, wide enough to encompass our examples as well as the aforementioned examples from [9]. This is explained in Section 2.

2. General Facts

In this section, we consider parameterized sequences of operators on some separable Fréchet space X. Thus, we have a parameter space Λ, and to each $\lambda \in \Lambda$ is associated a sequence $\mathbf{T}_\lambda = (T_{n,\lambda})_{n\in\mathbb{N}} \subset \mathcal{L}(X)$. We assume that the parameter space Λ is a topological space, and that $T_{n,\lambda}(x)$ depends continuously on the pair (λ, x), for each $n \in \mathbb{N}$. Finally, we also assume that there exists a dense set $\mathcal{D} \subset X$ such that each operator $T_{n,\lambda}$ has a right inverse $S_{n,\lambda} : \mathcal{D} \to X$. When each $\mathbf{T}_\lambda$ is the sequence of iterates of a single operator T_λ, it is assumed that each T_λ has a right inverse S_λ mapping $\mathcal{D}$ into $\mathcal{D}$, and that $S_{n,\lambda} = S_\lambda^n$ for all $n \in \mathbb{N}$.

2.1. The basic criterion It follows from the Baire Category Theorem that a single sequence of operators $\mathbf{T} = (T_n) \subset \mathcal{L}(X)$ is universal if and only if for each nonempty open set $U \subset X$, the open set set $\{x \mid \exists\, n : T_n(x) \in U\}$ is dense in X. In other words, $\mathbf{T}$ is universal iff for each pair (U, V) of nonempty open subsets of X, one can find a point $p \in X$ and an integer n such that $p \in V$ and $T_n(p) \in U$. The following criterion for common universality is a "parameterized" version of this simple observation. It may look a bit artificial, but all our results ultimately rely on it.

Basic Criterion. *Assume that Λ is a countable union of compact sets K satisfying the following property. For each pair $(u, v) \in \mathcal{D} \times \mathcal{D}$ and each O, open neighbourhood of 0 in X, one can find a point $p \in X$, parameters $\lambda_1, \dots, \lambda_q \in \Lambda$, sets of parameters $\Lambda_1, \dots, \Lambda_q \subset \Lambda$ with $\lambda_i \in \Lambda_i$ for all i, and integers $n_1, \dots, n_q \in \mathbb{N}$, such that*

(i) $\bigcup_i \Lambda_i \supset K$;
(ii) $p - v \in O$;
(iii) *for each $i \in \{1, \dots, q\}$ and all $\lambda \in \Lambda_i$:*
  - $T_{n_i,\lambda} S_{n_i,\lambda_i}(u) - u \in O$,
  - $T_{n_i,\lambda}(p) - T_{n_i,\lambda} S_{n_i,\lambda_i}(u) \in O$.

Then $\bigcap_{\lambda\in\Lambda} \mathrm{Univ}(\mathbf{T}_\lambda)$ is a dense G_δ subset of X.

Proof. First, by the Baire Category Theorem, we may assume that Λ is a compact set satisfying the property described above. Then, since $T_{n,\lambda}(x)$ depends continuously on $(\lambda, x) \in \Lambda \times X$, it follows from the compactness of Λ that for each nonempty open set $U \subset X$, the set

$$U_\Lambda := \{x \in X \mid \forall\, \lambda \in \Lambda\ \exists\, n : T_{n,\lambda}(x) \in U\}$$

is open in X. By the Baire Category Theorem, it is enough to show that all sets U_Λ are dense in X. In other words, we must check that $V \cap U_\Lambda \neq \varnothing$, for each pair (U, V) of nonempty open subsets of X; let us fix such a pair (U, V). Since $\mathcal{D}$ is dense in X, we may assume that $U = u + N$ and $V = v + N$, where u, $v \in \mathcal{D}$ and N is a neighbourhood of 0. Let $O \subset X$ be a neighbourhood of 0 such that $O + O \subset N$, and choose p, n_i, λ_i, Λ_i satisfying the assumptions of the basic criterion. Since $O \subset N$, we already know that $p \in V$; so we just have to check that $p \in U_\Lambda$. Let us fix $\lambda \in \Lambda$, choose $i \in \{1, \dots, q\}$ such that $\lambda \in \Lambda_i$, and put $n = n_i$. Writing

$$T_{n,\lambda}(p) - u = (T_{n,\lambda}(p) - T_{n,\lambda}S_{n\lambda_i}(u)) + (T_{n,\lambda}S_{n,\lambda_i}(u) - u),$$

we immediately get $p \in U_\Lambda$. □

In the hypercyclicity setting, the following refinement of the basic criterion will be used in a crucial way in Section 3. Notice that, unlike the general basic criterion, this refined criterion does not seem to follow directly from the Baire Category Theorem, since our proof uses a nontrivial result due to Costakis and Peris ([7], [19]). Alternatively, one could use the Bourdon-Feldman Theorem ([6]).

Remark 2.1. Assume that for each $\lambda \in \Lambda$, the operators $T_{n,\lambda}$ are the iterates of a single operator T_λ. Then, in the basic criterion, one can replace the second condition in item (iii) by the following one:

$$\textit{there exists } \varepsilon_i \in \{-1; 1\} \textit{ such that } T_\lambda^{n_i}(p) - \varepsilon_i T_\lambda^{n_i} S_{\lambda_i}^{n_i}(u) \in O.$$

Proof. Replacing U_Λ by $\tilde{U}_\Lambda := \{x \in X \mid \forall\, \lambda \in \Lambda\ \exists\, n : T_\lambda^n(x) \in U$ or $T_\lambda^n(x) \in -U\}$, the proof above shows that there exists a dense G_δ set of vectors $x \in X$ such that $\{T_\lambda^n(x) \mid n \in \mathbb{N}\} \cup \{T_\lambda^n(-x) \mid n \in \mathbb{N}\}$ is dense in X for all $\lambda \in \Lambda$. By the Costakis-Peris Theorem, such a vector x is hypercyclic for all operators T_λ. □

2.2. A one-dimensional criterion We now wish to state a "one-dimensional" criterion, which will apply when the parameter set Λ is an interval of $\mathbb{R}$. This criterion, which follows quite easily from the basic criterion, will be used in Section 4. To state it conveniently, we need to introduce the following notation.

◇ For each $n \in \mathbb{N}$, each $u \in \mathcal{D}$ and each O, neighbourhood of 0 in X, we put

$$\delta_n(u, O) = \sup\{\delta \in \mathbb{R}_+ \mid 0 \le \mu - \lambda \le \delta \ \Rightarrow\ T_{n,\lambda}S_{n,\mu}(u) - u \in O\}.$$

If (n_i) is a finite or infinite sequence of integers, the number

$$\sum_i \delta_{n_i}(u,O)$$

will be called the *weighted length* of (n_i) relative to (u,O).

◇ For each $n\in\mathbb{N}$, For each pair $(u,v)\in\mathcal{D}\times\mathcal{D}$ and each O, neighbourhood of 0 in X, we denote by $\mathcal{T}(u,v,O)$ the family of all finite increasing sequences of integers $s=(n_1,\dots,n_q)$ satisfying the following property: given $\lambda_0<\lambda_1<\dots<\lambda_q\in\Lambda$, one can find $p\in X$ such that $p-v\in O$ and $\forall\, i\in\{1,\dots,q\}$, $\forall\,\lambda\in[\lambda_{i-1};\lambda_i]$, $T_{n_i,\lambda}(p)-T_{n_i,\lambda}S_{n_i,\lambda_i}(u)\in O$.

The notation is due to the fact that $\mathcal{T}(u,v,O)$ is a tree of finite sequences of integers: if s is an extension of t and $s\in\mathcal{T}(u,v,O)$, then $t\in\mathcal{T}(u,v,O)$.

One-dimensional Criterion. *If each tree $\mathcal{T}(u,v,O)$ has elements of arbitrarily large weighted length, then $\bigcap_{\lambda\in\Lambda}\mathrm{Univ}(\mathbf{T}_\lambda)$ is a dense G_δ subset of X. In particular, this happens if each tree $\mathcal{T}(u,v,O)$ has an infinite branch with infinite weighted length.*

Proof. By the Baire Category Theorem, we may assume that Λ is a compact interval $[a;b]$. For each triple (u,v,O) one can find $(n_1,\dots,n_q)\in\mathcal{T}(u,v,O)$ such that $\sum_1^q\delta_{n_i}(u,O)\geq b-a$. Let $a=\lambda_0<\dots<\lambda_q=b$ be a subdivision of K such that $\lambda_i-\lambda_{i-1}\leq\delta_{n_i}(u,O)$ for all $i\in\{1,\dots,q\}$. Putting $\Lambda_i=[\lambda_{i-1};\lambda_i]$, we see at once that the basic criterion applies. □

In Section 4, we will use the one-dimensional criterion via the following remark.

Remark 2.2. The one-dimensional criterion criterion can be applied if, for each continuous semi-norm $\|\,.\,\|$ on X, the two following properties are satisfied.

(i) For each $u\in\mathcal{D}$ and $\lambda\leq\mu\in K$ one can write

$$\|T_{n,\lambda}S_{n,\mu}(u)-u\|\leq\omega_u(C_n(u)(\mu-\lambda)),$$

where $\lim_{t\to0}\omega_u(t)=0$.

(ii) For each pair $(u,v)\in\mathcal{D}\times\mathcal{D}$, one can find an infinite set $A\subset\mathbb{N}$ and a positive integer N such that
   - ◇ $\sum_{n\in A}1/C_n(u)=\infty$;
   - ◇ Putting $O=\{\|x\|<1\}$, the tree $\mathcal{T}(u,v,O)$ contains all sequences $(n_1,\dots,n_q)\subset A$ with $n_1\geq N$ and $n_i-n_{i-1}\geq N$ for all $i>1$.

Proof. Let us fix a triple (u,v,O) as in the one-dimensional criterion, where we may assume O has the form $\{\|x\|<1\}$, for some continuous semi-norm on X. Condition (i) implies that one can write $\delta_n(u,O)\geq\eta/C_n(u)$, for some positive constant $\eta=\eta(u,O)$. Let $A=\{a_0,a_1,\dots\}$ be given by (ii), where the enumeration is increasing. Then, for each $r\in\{0,\dots,N-1\}$, the sequence $(n_i^{(r)})=(a_{r+Ni})_{i\geq1}$ is an infinite branch of $\mathcal{T}(u,v,O)$, and one of these branches must have infinite weighted length. □

2.3. The Costakis-Sambarino criterion To make any use of the one-dimensional criterion, it is clear that one must know how to check that a sequence $(n_1, \dots, n_q)$ belongs to some given tree $\mathcal{T}(u, v, O)$. One way to do this is by using the following lemma.

Lemma 2.3. *Let $(n_1, \dots, n_q)$ be a finite increasing sequence of integers, and let O' be a neighbourhood of 0 in X such that $O' + O' + O' \subset O$. Assume that for all $\lambda_1 < \cdots < \lambda_q \in \Lambda$, the following properties hold true:*

(a) $\sum_{i=1}^q S_{n_i,\lambda_i}(u) \in O$;
(b_1) $T_{n_i,\lambda}(v) \in O'$ *for all* $i \in \{1, \dots, q\}$ *and* $\lambda \in \Lambda$;
(b_2) $\sum_{j<i} T_{n_i,\lambda} S_{n_j,\lambda_j}(u) \in O'$ *whenever* $i \in \{1, \dots, q\}$ *and* $\lambda \geq \lambda_j$ *for all* $j < i$;
(b_3) $\sum_{j>i} T_{n_i,\lambda} S_{n_j,\lambda_j}(u) \in O'$ *whenever* $i \in \{1, \dots, q\}$ *and* $\lambda \leq \lambda_j$ *for all* $j > i$.

Then, $(n_1, \dots, n_q) \in \mathcal{T}(u, v, O)$.

Proof. This follows easily from the hypotheses and the definition of $\mathcal{T}(u, v, O)$: given $\lambda_1 < \cdots < \lambda_q \in \Lambda$, just put

$$p = v + \sum_{i=1}^{q} S_{n_i,\lambda_i}(u). \qquad \square$$

From this and Remark 2.2, one can deduce the general criterion for common universality proved in [9]. Here, the parameter space Λ is an interval of $\mathbb{R}$.

Costakis-Sambarino's Criterion. *Assume that for each continuous semi-norm $\| \, . \, \|$ on X, each point $f \in \mathcal{D}$ and each compact set $K \subset \Lambda$, the following properties hold true.*

(i) *There exists a sequence of positive numbers $(c_k)_{k \in \mathbb{N}}$ such that*
- ◇ $\sum_0^\infty c_k < \infty$;
- ◇ $\|T_{n+k,\lambda} S_{n,\alpha}(f)\| \leq c_k$ *for any* $n, k \in \mathbb{N}$ *and* $\alpha \leq \lambda \in K$;
- ◇ $\|T_{n,\lambda} S_{n+k,\alpha}(f)\| \leq c_k$ *for any* $n, k \in \mathbb{N}$ *and* $\lambda \leq \alpha \in K$.

(ii) *Given $\varepsilon > 0$, there exists $\eta > 0$ such that*

$$0 \leq \mu - \lambda < \frac{\eta}{n} \Rightarrow \|T_{n,\lambda} S_{n,\mu}(f) - f\| < \varepsilon.$$

Then $\bigcap_{\lambda \in \Lambda} \mathrm{Univ}(\mathbf{T}_\lambda)$ is a dense G_δ subset of X.

Proof. This follows from the above lemma and the one-dimensional criterion as stated in Remark 2.2. Indeed, condition (2) implies (i) in Remark 2.2, with $C_n(u) = O(n)$, and thanks to Lemma 2.3, condition (1) implies (ii) in Remark 2.2, with $A = \mathbb{N}$. $\square$

The Costakis-Sambarino is quite general, and it does apply to many natural examples. However, it turns out that in some cases, this criterion cannot be applied because the strong summability assumption on the sequence (c_k) is not satisfied. In Section 3, we will show how to relax this assumption in a Banach space setting.

2.4. Additional remarks To conclude this section, let us mention briefly some possible extensions of the one-dimensional criterion.

(1) The definitions of the quantities $\delta_n(u, O)$ and the trees $\mathcal{T}(u, v, O)$ make sense if Λ is a subset of some partially ordered metric space $(E, d, \leq)$. For example, E could be $\mathbb{R}^k$ with the product ordering. In this setting, the one-dimensional criterion remains valid as stated, if one assumes that Λ is a *monotonic Lipschitz curve*, that is, the range of some monotonic Lipschitz map $\varphi : I \to E$, where I is an interval of $\mathbb{R}$. The proof is exactly the same. Remark 2.2 also holds as stated in this more general setting, and so does Lemma 2.3. An application will be given in Section 6.

(2) Still in the partially ordered metric space setting, one can formulate a "d-dimensional" criterion where the parameter space is a *d-dimensional monotonic Lipschitz surface*, that is, the monotonic Lipschitz image of a d-dimensional cube. Here, we put the product ordering on $\mathbb{R}^d$. The only change is the definition of the weighted length, where $\delta_{n_i}(u, O)$ has to be replaced by $\delta_{n_i}(u, O)^d$. Remark 2.2 also holds as stated, provided we replace the condition $\sum_{n\in A} 1/C_n(u) = \infty$ by $\sum_{n\in A} 1/C_n(u)^d = \infty$. However, we have no interesting application of that.

3. Operators on Banach Spaces with Nontrivial Type

In this section, X is a Banach space. As announced at the end of Section ***2.3***, our aim is to relax the very strong summability assumption made on the sequence (c_k) in the Costakis-Sambarino criterion. This will be done with the help of some classical tools from Probability Theory.

3.1. On Dudley's Theorem In this section, we are going to prove a variant of *Dudley's Theorem*, a very useful result from Probability Theory which gives an estimate for the expectation of the supremum of a subgaussian random process. For convenience of the reader, we briefly review some well-known facts. Our main references are [18] and [17].

If $(\Omega, \mathcal{A}, \mathbb{P})$ is a probability space, and if $\psi : \overline{\mathbb{R}^+} \to \overline{\mathbb{R}^+}$ is an increasing convex function with $\psi(0) = 0$ and $\psi(\infty) = \infty$, the *Orlicz space* $L_X^\psi = L^\Psi(\Omega, \mathbb{P}, X)$ is the Banach space of all random variables $Z : \Omega \to X$ such that $\mathbb{E}(\psi(\|Z\|/a)) < +\infty$ for some finite constant $a > 0$. The norm of the space L_X^ψ is the so-called *Luxemburg norm*, which is defined by

$$\|Z\|_\psi = \inf\left\{a > 0 \mid \mathbb{E}\left(\psi\left(\frac{\|Z\|}{a}\right)\right) \leq 1\right\}.$$

We will be concerned with the case $\psi(x) = \psi_2(x) := e^{x^2} - 1$. Elementary computations using Stirling's formula show that there exist two numerical constants a and b such that

$$a\|Z\|_{\psi_2} \leq \sup_{p\geq 1} \frac{\|Z\|_p}{\sqrt{p}} \leq b\|Z\|_{\psi_2} \tag{3.1}$$

for all $Z \in L_X^{\psi_2}$.

If (T, d) is a compact semi-metric space, we denote by $N_d(\varepsilon)$, $\varepsilon > 0$, the ε-covering number of (T, d), that is, the minimal number of d-open balls of radius ε which are needed to cover T. The function N_d is the *entropy function* of the semi-metric space (T, d). The *entropy integral* $J(d)$ is defined by

$$J(d) = \int_0^{+\infty} \sqrt{\log(N_d(\varepsilon))}\, \mathrm{d}\varepsilon.$$

We will use the following form of Dudley's Theorem, which is due to Pisier ([20]).

Theorem 3.1 (Dudley's Theorem)**.** *Let (T, d) be a compact semi-metric space with finite entropy integral $J(d)$, and let $(Z_t)_{t\in T}$ be an X-valued random process such that $Z_t \in L_X^{\psi_2}$ for all $t \in T$. Assume that the process satisfies the Lipschitz condition*

$$\|Z_s - Z_t\|_{\psi_2} \le d(s, t), \quad s,\ t \in T.$$

Then one has the following estimate (t_0 being an arbitrary point in T):

$$\mathbb{E}(\sup_t \|Z_t\|) \le b_1(J(d) + \|Z_{t_0}\|_{\psi_2}),$$

for some numerical constant $b_1 < \infty$.

For our application in hypercyclicity, we will need a variant of Dudley's Theorem, where one deals with finitely many subgaussian processes.

Corollary 3.2. *Let q be a positive integer, and for each $i \in \{1, \dots, q\}$, let $(Z_{i,\lambda})_{\lambda\in\Lambda_i}$ be a random process with values in X, where Λ_i is a compact interval of $\mathbb{R}$. Assume that $Z_{i,\lambda} \in L_X^{\psi_2}$ for each i and all $\lambda \in \Lambda_i$, and that for each $i \in \{1, \dots, q\}$, the process $(Z_{i,\lambda})$ satisfies the Lipschitz condition*

$$\|Z_{i,\lambda} - Z_{i,\mu}\|_{\psi_2} \le c_i|\lambda - \mu|, \quad \lambda,\ \mu \in \Lambda_i.$$

Then one has the following estimate:

$$\mathbb{E}(\sup_{i,\lambda} \|Z_{i,\lambda}\|) \le b_2(\sup_i c_i|\Lambda_i| + \sup_{i,\lambda} \|Z_{i,\lambda}\|_{\psi_2})\sqrt{\log(q + 1)},$$

where $|\Lambda_i|$ is the length of the interval Λ_i and b_2 is a numerical constant.

Proof. We define a compact metric space (T, d) and a random process on it to deduce Corollary 3.2 from Dudley's Theorem. Put $M_\Lambda := \sup_i c_i|\Lambda_i|$, $M_Z := 2\sup_{i,\lambda} \|Z_{i,\lambda}\|_{\psi_2}$, $T := \{(i, \lambda) \mid i \in \{1, \dots, q\},\ \lambda \in \Lambda_i\}$, and let us define a metric d on T by setting

$$d((i, \lambda), (j, \mu)) := \begin{cases} c_i|\lambda - \mu| & \text{if } i = j, \\ M_Z & \text{otherwise.} \end{cases}$$

The entropy function $N_d(\varepsilon)$ is dominated by $2M_\Lambda q/\varepsilon$, and the diameter of (T,d) is not greater than $M_\lambda + M_Z$. Hence, we may estimate the entropy integral as follows:

$$\begin{aligned} J(d) &\le \int_0^{M_\Lambda + M_Z} \sqrt{\log(2M_\Lambda q/\varepsilon)}\,\mathrm{d}\varepsilon \\ &\le \left(\sqrt{\log 2} + \sqrt{\log q} + \int_0^1 \sqrt{\log(1/t)}\,\mathrm{d}t\right)(M_\Lambda + M_Z). \end{aligned}$$

Corollary 3.2 now follows directly from Dudley's Theorem. □

We intend to apply Corollary 3.2 to Rademacher processes. This will be possible via Kahane's inequality. Indeed, for a random variable of the form $Z(\omega) = \sum_k \varepsilon_k(\omega)x_k$, where (ε_k) is a sequence of independent Rademacher variables ($\varepsilon_k = \pm 1$ with probability $\frac{1}{2}$), Kahane's inequality, in a version due to Kwapien ([16]), reads $\|Z\|_p \le C\sqrt{p}\|Z\|_2$. Using (3.1), this shows that one may estimate the Orlicz-norm $\|Z\|_{\psi_2}$ of a Rademacher process by computing its L^2 norm.

3.2. A new criterion We shall now derive from Corollary 3.2 a criterion for common hypercyclicity where the geometry of the underlying Banach space plays a crucial role. In what follows, we consider a family $(T_\lambda)_{\lambda\in\Lambda}$ of bounded operators on X, where the parameter space Λ is an interval of $\mathbb{R}$. We assume that $T_\lambda(x)$ depends continuously on the pair (λ, x), and that there exists a dense set $\mathcal{D} \subset E$ such that each operator T_λ has a right inverse $S_\lambda : \mathcal{D} \to \mathcal{D}$.

Recall that the Banach space X is said to have *type* p ($p \in [1;2]$) if there exists some finite constant C such that

$$\Big\|\sum_{i\in I} \varepsilon_i x_i\Big\|_2 \le C\Big(\sum_{i\in I} \|x_i\|^p\Big)^{1/p}$$

for each finite family $(x_i)_{i\in I} \subset X$ and each finite family of independent Rademacher variables $(\varepsilon_i)_{i\in I}$. It is well known that every Banach space has type 1, and that L^p spaces have type $\min(2,p)$.

Theorem 3.3. *Assume the Banach space X has type $p \in [1;2]$, and that for each $f \in \mathcal{D}$ and any compact set $K \subset \Lambda$, there exists a sequence of positive numbers $(c_k)_{k\in\mathbb{N}}$ such that the following conditions are satisfied:*

(a) *(c_k) is nonincreasing, and $\sum_{k=0}^\infty c_k^p < \infty$;*
(b$_1$) *$\|T_\lambda^{n+k}S_\alpha^n(f)\| \le c_k$, for any n, $k \in \mathbb{N}$ and λ, $\alpha \in K$, $\lambda \ge \alpha$;*
(b$_2$) *$\|T_\lambda^n S_\alpha^{n+k}(f)\| \le c_k$ for any n, $k \in \mathbb{N}$ and λ, $\alpha \in K$, $\lambda \le \alpha$;*
(c$_1$) *$\|(T_\lambda^{n+k} - T_\mu^{n+k})(S_\alpha^n f)\| \le (n+k)|\lambda-\mu|c_k$, for n, $k \in \mathbb{N}$ and λ, $\mu \ge \alpha \in K$;*
(c$_2$) *$\|(T_\lambda^n - T_\mu^n)(S_\alpha^{n+k}(f))\| \le n|\lambda-\mu|c_k$, for n, $k \in \mathbb{N}$ and λ, $\mu \le \alpha \in K$.*

Then $\bigcap_{\lambda\in\Lambda} HC(T_\lambda)$ is a dense G_δ subset of E.

Remark 3.4. Since every Banach space has type 1, this criterion can be applied with $p = 1$ in an arbitrary Banach space X. However, in that case the hypotheses are stronger than in the Costakis-Sambarino criterion, so this does not help at all.

Remark 3.5. In the two examples below, we will use Theorem 3.3 with a sequence (c_k) of the form

$$c_k = \frac{M}{1 + k^s},$$

for some finite constant M and some real number $s > 1/p$.

Proof of Theorem 3.3. We apply the basic criterion, in the form mentioned in Remark 2.1. So let us fix a compact interval $K = [a; b] \subset \Lambda$, a pair $(u, v) \in \mathcal{D} \times \mathcal{D}$ and an open ball $O = B(0, \eta)$. Let $(c_k)_{k\in\mathbb{N}}$ be a sequence of positive numbers satisfying (a), such that (c_1), (c_2) hold for $f = u$, and (b_1), (b_2) hold for both u and v.

We choose some large positive integer N and some small positive constant τ. Since the series $\sum 1/(Ni)$ is divergent, one can find a subdivision $a = \lambda_0 < \cdots < \lambda_q = b$ of K such that $\lambda_i - \lambda_{i-1} < \tau/(Ni)$ for all $i \in \{1, \dots, q\}$. We put $\Lambda_i = [\lambda_{i-1}; \lambda_i]$, $i \in \{1, \dots, q\}$, so that condition (i) in the basic criterion is satisfied.

Now, we consider the random point

$$p(\omega) = v + \sum_{i=1}^{q} \varepsilon_i(\omega) S_{\lambda_i}^{Ni}(u) = v + y(\omega),$$

where (ε_i) is a sequence of independent Rademacher variables. We show that with large probability, $p(\omega)$ satisfies conditions (ii) and (iii) in the basic criterion. In what follows, the symbol C will always stand for a constant whose value may change from line to line and depends only on X and u, v, K.

By condition (b_2) with $n = 0$ and $k = Ni$, we have $\|S_{\lambda_i}^{Ni}(u)\| \le c_{Ni}$ for all $i \in \{1, \dots, q\}$. Since X has type p and $s > 1/p$, it follows that

$$\mathbb{E}(\|y(\omega)\|) \le C\Big(\sum_{i=1}^{q} c_{Ni}^p\Big)^p \le C\Big(\sum_{k\ge N} c_k^p\Big)^{1/p},$$

so that if N is large enough, then $\|y(\omega)\| < \eta$ with large probability. In other words, condition (ii) in the basic criterion is satisfied with large probability if N is large enough.

Next, it follows from (c_2) (with $k = 0$, $n = Ni$ and $\alpha = \mu = \lambda_i$) that for each $i \in \{1, \dots, q\}$ and all $\lambda \in \Lambda_i$, we have

$$\|T_\lambda^{Ni} S_{\lambda_i}^{Ni}(u) - u\| \le CNi|\lambda - \lambda_i| \le C\tau,$$

so that, whatever the choice of N may be, the (non-random) first part of condition (iii) is satisfied if τ is small enough.

Now, we use Corollary 3.2 to check the second part of condition (iii) in the basic criterion, in the form stated in Remark 2.1. For each $i \in \{1, \dots, q\}$ and $\lambda \in \Lambda_i$, we have

$$T_\lambda^{Ni}(p(\omega)) - \varepsilon_i(\omega) T_\lambda^{Ni} S_{\lambda_i}^{Ni}(u) = T_\lambda^{Ni}(v) + T_\lambda^{Ni}(y_{i,\lambda}(\omega)),$$

where $y_{i,\lambda}(\omega) = \sum_{j \neq i} \varepsilon_j(\omega) S_{\lambda_j}^{Nj}(u)$. By condition ($b_1$) with $n = 0$ and $k = Ni$, the non-random term $T_\lambda^{Ni}(v)$ is small if N is large. To estimate the second term, we put

$$Z_{i,\lambda}(\omega) = T_\lambda^{Ni}(y_{i,\lambda}(\omega)),$$

and we check that the hypotheses of Corollary 3.2 are satisfied.

Since X has type p, we have

$$\begin{aligned} \|Z_{i,\lambda} - Z_{i,\mu}\|_{\psi_2} &\leq C \|Z_{i,\lambda} - Z_{i,\mu}\|_2 \\ &\leq C \Big(\sum_{j \neq i} \|(T_\lambda^{Ni} - T_\mu^{Ni})(S_{\lambda_j}^{Nj}(u))\|^p \Big)^{1/p}. \end{aligned}$$

Using (c_1), (c_2), it follows that

$$\begin{aligned} &\|Z_{i,\lambda} - Z_{i,\mu}\|_{\psi_2} \\ &\quad \leq C \Big(\sum_{j<i} (Ni)^p |\lambda - \mu|^p c_{N(i-j)}^p \Big)^{1/p} + C \Big(\sum_{j>i} (Ni)^p |\lambda - \mu|^p c_{N(j-i)}^p \Big)^{1/p} \\ &\quad \leq c_i |\lambda - \mu|, \end{aligned}$$

where $c_i = CNi(\sum_{\ell \geq 1} c_{N\ell}^p)^{1/p}$. Similarly, conditions (b_1), (b_2) give

$$\|Z_{i,\lambda}\|_{\psi_2} \leq C \Big(\sum_{j \neq i} \|T_\lambda^{Ni} S_{\lambda_j}^{Nj}(u)\|^p \Big)^{1/p} \leq C \Big(\sum_{\ell \geq 1} c_{N\ell}^p \Big)^{1/p}.$$

Observe now that $c_i |\lambda_i - \lambda_{i-1}| \leq C\tau(\sum_{\ell \geq 1} c_{N\ell}^p)^{1/p}$ and $\log q \leq CN/\tau$. Using Corollary 3.2, it follows that

$$\mathbb{E}(\sup_{i,\lambda} \|Z_{i,\lambda}\|) \leq \frac{C}{\tau^{1/2}} \sqrt{N} \Big(\sum_{\ell \geq 1} c_{N\ell}^p \Big)^{1/p}.$$

Assume that N is an even integer, $N = 2N'$. Since the sequence (c_k) is nonincreasing, we have

$$c_{N\ell}^p \leq \frac{1}{N'} \sum_{k=(2\ell-1)N'}^{2N'\ell-1} c_k^p \quad \text{for each } \ell \geq 1.$$

Thus, keeping in mind that $p \leq 2$, we get

$$\mathbb{E}(\sup_{i,\lambda} \|Z_{i,\lambda}\|) \leq \frac{C}{\tau^{1/2}} N^{1/2-1/p} \Big(\sum_{k\geq N'} c_k^p \Big)^{1/p} \leq \frac{C}{\tau^{1/2}} \Big(\sum_{k\geq N'} c_k^p \Big)^{1/p}.$$

We conclude that if $N = N(\tau)$ is large enough, then, with large probability, $T_\lambda^{Ni}(y_{i,\lambda}(\omega))$ is close to 0 for each $i \in \{1, \dots, q\}$ and all $\lambda \in \Lambda_i$, and hence

$$\|T_\lambda^{Ni}(p(\omega)) - \varepsilon_i(\omega) T_\lambda^{Ni} S_{\Lambda_i}^{Ni}(u)\| < \eta$$

for each i and all $\lambda \in \Lambda_i$.

Putting everything together, we have shown that if τ is small enough and $N = N(\tau)$ is large enough, then many points $p(\omega)$ satisfy conditions (ii), (iii) in the basic criterion. This concludes the proof. □

Remark 3.6. The use of the Baire Category Theorem to produce hypercyclic vectors has become standard since the 1986 paper of Gethner and Shapiro ([11]). The introduction of probabilistic arguments (via ergodic theory) is much more recent (see [4]). It seems likely that other hypercyclicity results may be obtained by mixing these two types of arguments, as in the proof of Theorem 3.3.

3.3. Two examples We now illustrate Theorem 3.3 with two common hypercyclicity results which cannot be deduced from the Costakis-Sambarino criterion.

Example 3.7. We consider translation operators on some weighted L^p spaces. Let $w : \mathbb{R} \to \mathbb{R}$ be a positive, bounded continuous function such that $w(t-\lambda)/w(t)$ is bounded for each fixed $\lambda \in \mathbb{R}$. For each $p \in [1; \infty[$, we put $X_p = L^p(\mathbb{R}, w(t)\,\mathrm{d}t)$, so that X_p has type $\tilde{p} = \min(2, p)$. For each real number λ, let T_λ be the translation-operator on X_p defined by $T_\lambda f(t) = f(t + \lambda)$. Then T_λ is invertible with inverse $S_\lambda = T_{-\lambda}$. Using Kitai's criterion (see [13]), it is not hard to check that all operators T_λ, $\lambda \neq 0$ are hypercyclic on X_p provided $\lim_{|x|\to\infty} \int_{x-A}^{x+A} w(t)\,\mathrm{d}t = 0$ for each fixed $A > 0$; for example, this happens if $w \in L^1(\mathbb{R})$, or if $w \in C_0(\mathbb{R})$. We now show that if w satisfies an estimate of the form

$$w(t) \leq \frac{C}{1 + |t|^r}$$

for some $r > p/\tilde{p}$, then $\bigcap_{\lambda\in\mathbb{R}^*} HC(T_\lambda)$ is a dense G_δ subset of X_p. In other words, if $w(t) \leq C/(1 + |t|^r)$ for some $r > 1$, then $\bigcap_{\lambda\in\mathbb{R}^*} HC(T_\lambda) \neq \varnothing$ on X_p for all $p < 2r$.

Proof. By symmetry, we may consider only positive λ's. Let $\mathcal{D} \subset X_p$ be the set of all compactly supported smooth functions. We show that the hypotheses of Theorem 3.3 are satisfied with $c_k = M/(1 + k^s)$, where $s = r/p > 1/\tilde{p}$. So let us fix a compact interval $K = [a; b] \subset]0; \infty[$ and a smooth function $f \in \mathcal{D}$ supported on some interval $[-A; A]$.

If $\lambda \geq \alpha \in K$ and $n, k \in \mathbb{N}$, then

$$\begin{aligned} \|T_\lambda^{n+k} S_\alpha^n(f)\|^p &= \int_{\mathbb{R}} |f(t + (n+k)\lambda - n\alpha)|^p w(t)\,\mathrm{d}t \\ &\leq C \int_{-A-(n+k)\lambda+n\alpha}^{A-(n+k)\lambda+n\alpha} w(t)\,\mathrm{d}t \\ &\leq \frac{C}{1 + (ka - A)^r}, \end{aligned}$$

provided k is large enough. This gives (b_1), and the same proof works for (b_2).

To prove (c_1), let us fix $\alpha \leq \lambda, \mu \in K$. If $(n+k)|\lambda - \mu| \geq 1$, then ($c_1$) follows from ($b_1$) and the triangle inequality, so we may assume that $(n+k)(\lambda - \mu) \leq 1$ and $\lambda \geq \mu$. By the Mean Value Theorem, we have

$$|f(t + (n+k)\lambda - n\alpha) - f(t + (n+k)\mu - n\alpha)| \leq C(n+k)(\lambda - \mu)$$

for all $t \in \mathbb{R}$, so that

$$\begin{aligned} \|(T_\lambda^{n+k} - T_\mu^{n+k})(S_\alpha^n(f))\|^p &\leq C(n+k)^p(\lambda-\mu)^p \int_{-A-(n+k)\lambda+n\alpha}^{A-(n+k)\mu+n\alpha} w(t)\,\mathrm{d}t \\ &\leq \frac{C(n+k)^p(\lambda-\mu)^p(2A + (n+k)(\lambda-\mu))}{1 + (ka - A)^r} \\ &\leq \frac{C(n+k)^p(\lambda-\mu)^p}{1 + (ka - A)^r}, \end{aligned}$$

provided k is large enough. This gives (c_1), and the proof is the same for (c_2). □

Example 3.8. We consider composition operators on the Hardy space $H^p(\mathbb{D})$, $1 \leq p < \infty$. If φ is an automorphism of $\mathbb{D}$, the composition operator associated to φ is the operator on $H^p(\mathbb{D})$ defined by $C_\varphi f = f \circ \varphi$. Then C_φ is hypercyclic on H^p if and only if the automorphism φ is either hyperbolic (one attractive fixed point on $\mathbb{T}$, and a second fixed point on $\mathbb{T}$) or parabolic (a single, attractive, fixed point on $\mathbb{T}$).

Proof. Let Λ be the set of all automorphisms of $\mathbb{D}$ having 1 as an attractive fixed point. It is proved in [5] that $\bigcap_{\varphi \in \Lambda} HC(C_\varphi)$ is a residual subset of $H^2(\mathbb{D})$. In the hyperbolic case, one can apply the Costakis-Sambarino criterion, but this turns out to be impossible in the parabolic case; see [5] for details. We now show that for $p \in [1; 4[$, one can apply Theorem 3.3 in the parabolic case.

Let $\mathbb{C}_+$ be the right half-plane $\{w \in \mathbb{C} \mid \operatorname{Re} w > 0\}$, and let $\sigma : \mathbb{D} \to \mathbb{C}_+$ be the Cayley map, $\sigma(z) = (1+z)/(1-z)$. Then $H^p(\mathbb{D})$ can be identified via σ

with a space of entire functions on $\mathbb{C}_+$, namely $\mathcal{H}^p := \{f \circ \sigma^{-1} \mid f \in H^p\}$; the norm of a function $g \in \mathcal{H}^p$ is given by

$$\|g\|^p = \int_{\mathbb{R}} |g(it)|^p \frac{\mathrm{d}t}{1+t^2}.$$

Moreover, if φ is a parabolic automorphism of $\mathbb{D}$ with $+1$ as attractive fixed point, then $\varphi := \sigma \circ \varphi \circ \sigma^{-1}$ is a translation: $\varphi(w) = w + i\lambda$, where λ is a nonzero real number. Thus, C_φ acts as a translation operator T_λ on $\mathcal{H}^p$, and it is enough to show that $\bigcap_{\lambda>0} HC(T_\lambda)$ is a residual subset of $\mathcal{H}^p$. As usual, we denote by S_λ the inverse of T_λ, namely $S_\lambda = T_{-\lambda}$.

Let $\mathcal{P}$ be the set of all holomorphic polynomials P satisfying $P(1) = P'(1) = 0$, and let $\mathcal{D} := \{P \circ \sigma^{-1} \mid P \in \mathcal{P}\}$. It is easy to check that $\mathcal{D}$ is dense in $\mathcal{H}^2$. We show that the hypotheses of Theorem 3.3 are satisfied with a sequence (c_k) of the form $c_k = M/(1+k^{2/p})$. This will be enough since $\mathcal{H}^p$ has type $\tilde{p} = \min(2,p)$ and $2/p > 1/\tilde{p}$ if $p < 4$.

Let us fix $Q = P \circ \sigma^{-1} \in \mathcal{D}$ and some compact interval $K = [a;b] \subset]0;\infty[$. For notational simplicity, we shall write $Q(t)$ instead of $Q(it)$, $t \in \mathbb{R}$. Notice that, by the definition of $\mathcal{P}$, we have $|P(z)| \le C|z-1|^2$ and $|P'(z)| \le C|z-1|$ on $\mathbb{T}$, which implies $|Q(t)| \le C/(1+t^2)$ and $|Q'(t)| \le C/(1+t^2)^{3/2}$ on $\mathbb{R}$. For the second inequality, just compute the derivative of σ^{-1}.

If $\lambda \le \mu \le \alpha \in K$ and $n, k \in \mathbb{N}$, then

$$\begin{aligned} &\|T_\lambda^n S_\alpha^{n+k}(Q) - T_\mu^n S_\alpha^{n+k}(Q)\|^p \\ &\qquad \le \int_{\mathbb{R}} \left(\int_{t+n\lambda-(n+k)\alpha}^{t+n\mu-(n+k)\alpha} |Q'(s)|\,\mathrm{d}s \right)^p \frac{\mathrm{d}t}{1+t^2} \\ &\qquad \le C \int_{\mathbb{R}} \left(\int_{t+n\lambda-(n+k)\alpha}^{t+n\mu-(n+k)\alpha} \frac{\mathrm{d}s}{(1+s^2)^{3/2}} \right)^p \frac{\mathrm{d}t}{1+t^2}. \end{aligned}$$

Now, if $t \le k\alpha/2$, then

$$\left(\int_{t+n\lambda-(n+k)\alpha}^{t+n\mu-(n+k)\alpha} \frac{\mathrm{d}s}{(1+s^2)^{3/2}} \right)^p \le C \frac{n^p |\lambda-\mu|^p}{(1+k^2\alpha^2/4)^{3p/2}},$$

and if $t \ge k\alpha/2$, we can still write

$$\left(\int_{t+n\lambda-(n+k)\alpha}^{t+n\mu-(n+k)\alpha} \frac{\mathrm{d}s}{(1+s^2)^{3/2}} \right)^p \le n^{p/q} |\lambda-\mu|^{p/q} \int_{t+n\lambda-(n+k)\alpha}^{t+n\mu-(n+k)\alpha} \frac{\mathrm{d}s}{(1+s^2)^{3p/2}},$$

by Hölder's inequality ($1/p + 1/q = 1$). Using Fubini's Theorem, it follows that

$$
\begin{aligned}
&\|T_\lambda^n S_\alpha^{n+k}(Q) - T_\mu^n S_\alpha^{n+k}(Q)\|^p \\
&\quad\le C\frac{n^p|\lambda-\mu|^p}{(1+k^2\alpha^2/4)^{3p/2}} \\
&\qquad\qquad + n^{p/q}|\lambda-\mu|^{p/q} \times \int_{\mathbb{R}} \left(\int_{\max(k\alpha/2, s+(n+k)\alpha-n\mu)}^{\max(k\alpha/2, s+(n+k)\alpha-n\lambda)} \frac{\mathrm{d}t}{1+t^2} \right) \frac{\mathrm{d}s}{(1+s^2)^{3p/2}} \\
&\quad\le C\frac{n^p|\lambda-\mu|^p}{(1+k^2\alpha^2/4)^{3p/2}} + n^{p/q}|\lambda-\mu|^{p/q} \times \frac{Cn|\lambda-\mu|}{1+k^2\alpha^2/4} \\
&\quad\le C\,\frac{n^p|\lambda-\mu|^p}{1+k^2a^2/4}.
\end{aligned}
$$

This proves (c_2). The same proof works for (c_1), and the proofs of (b_1), (b_2) are simpler. □

4. Shift-like Operators

In this section, we consider one-parameter families of "shift-like" operators on the Fréchet space X. We first prove a simple result on multiples of a single shift-like operator. Then, we prove a rather general criterion for common hypercyclicity of a one-parameter family of weighted backward shifts.

Both results are proved by using the one-dimensional criterion, as stated in Remark 2.2. We shall need the following lemma, which is a special case of Lemma 2.3. The notation are the same.

Lemma 4.1. *Let $(n_1,\dots,n_q)$ be a finite increasing sequence of integers. Assume that the following properties hold true.*

(a) *$T_{n_i,\lambda}(v) = 0$ for all i and $T_{n_i,\lambda}S_{n_j,\mu}(u) = 0$ if $i > j$, for all $\lambda, \mu \in \Lambda$;*
(b) *$\sum_{i=1}^q S_{n_i,\lambda_i}(u) \in O$, for all $\lambda_1 < \cdots < \lambda_q \in \Lambda$, and $\sum_{j>i} T_{n_i,\lambda}S_{n_j,\lambda_j}(u) \in O$ whenever $i \in \{1,\dots,q\}$ and $\lambda \le \lambda_j$ for all $j > i$.*

Then, $(n_1,\dots,n_q) \in \mathcal{T}(u,v,O)$.

In the Banach space case, the following result was proved in [2] using some ideas from [1].

Proposition 4.2. *Let X be a separable Fréchet space, and let $T \in \mathcal{L}(X)$. Assume that*

(1) *$\mathcal{D} := \bigcup_n \mathrm{Ker}(T^n)$ is dense in X and T has a right inverse $S : \mathcal{D} \to X$;*
(2) *there exists some number $\lambda_0 \ge 0$ such that, for $\lambda > \lambda_0$ and each $u \in \mathcal{D}$, the set $\{\lambda^{-n}S^n(u) \mid n \in \mathbb{N}\}$ is bounded in X.*

Then $\bigcap_{\lambda>\lambda_0} HC(\lambda T)$ is a dense G_δ subset of X.

Proof. The whole parameter space is $]\lambda_0;\infty[$, but we need only consider a compact interval $\Lambda \subset]\lambda_0;\infty[$, by Baire's category theorem. Then one can write $\inf\Lambda = C\alpha$, where $\alpha > \lambda_0$ and $C > 1$. By definition of $\mathcal{D}$, the right inverse S maps $\mathcal{D}$ into itself. We apply the one-dimensional criterion as stated in Remark 2.2, with $T_{n,\lambda} = (\lambda T)^n$ and $S_{n,\lambda} = (\lambda^{-1}S)^n$. So let us fix a continuous semi-norm $\|\,.\,\|$ on X.

For all $u \in \mathcal{D}$ and $\lambda,\,\mu \in \Lambda$, one can write

$$\|T_{n,\lambda}S_{n,\mu}(u) - u\| = |\lambda^n\mu^{-n} - 1|\,\|u\| = |e^{n(\log(\lambda)-\log(\mu))} - 1|\,\|u\|.$$

Thus, condition (i) in Remark 2.2 is satisfied with $C_n(u) = O(n)$.

To see that (ii) is also true with $A = \mathbb{N}$, let us fix $(u,v) \in \mathcal{D}\times\mathcal{D}$, let N be a positive integer, and let $(n_1,\dots,n_q)$ be a finite sequence of integers satisfying $n_1 \geq N$ and $n_i - n_{i-1} \geq N$ for all $i > 1$. Putting $O = \{\|x\| < 1\}$, we show that the hypotheses of Lemma 4.1 are fulfilled if N is large enough.

Since $T_{n,\lambda}S_{n',\mu}(u) = \lambda^n\mu^{-n'}T^{n-n'}(u)$ if $n \geq n'$ and since $u,\ v \in \mathcal{D}$, property (a) in Lemma 4.1 is satisfied if N is large enough. Moreover, we have

$$\|S_{n,\lambda}(u)\| \leq \lambda^{-n}\|S^n(u)\| \leq C^{-n}\|\alpha^{-n}S^n(u)\|$$

for all λ, n, and

$$\|T_{n,\lambda}S_{n',\mu}(u)\| \leq \lambda^{-(n'-n)}\|S^{n'-n}(u)\| \leq C^{-(n'-n)}\|\alpha^{-(n'-n)}S^{n'-n}(u)\|$$

if $\lambda \leq \mu$ and $n' > n$. Thus, thanks to property (2), we see that (b) in Lemma 4.1 will also hold provided N is large enough. This concludes the proof. ◻

Remark 4.3. It follows from the above proof that condition (2) above can be weakened: it is enough to assume that there exists some set $A \subset \mathbb{N}$ such that $\sum_{n\in A} 1/n = \infty$ and the set $\{\lambda^{-m}S^m(u) \mid m \in B\}$ is bounded in X for all $\lambda > \lambda_0$, where B is the "difference set" $\{n' - n \mid n,\, n' \in A,\ n' > n\}$.

Remark 4.4. If $\mathrm{Ker}(T)$ has dimension 1, then the operator T is a "generalized backward shift" in the sense of [12]. In that case, condition (1) is automatically fulfilled; see [12] for details.

Corollary 4.5. *Let X be a Banach space, and let $T \in \mathfrak{L}(X)$. Assume that $\bigcup_n \mathrm{Ker}(T^n)$ is dense in X, and that T is moreover onto. Then there exists some finite $C \geq 0$ such that $\bigcap_{\lambda>C} HC(\lambda T)$ is a dense G_δ subset of X.*

Proof. By the open mapping theorem, T has a right inverse $S : X \to X$ such that $\|S(u)\| \leq C\|u\|$ for all $u \in X$, for some finite constant C. So one can apply Proposition 4.2. ◻

Example 4.6. If B is the usual backward shift on $X = c_0(\mathbb{N})$ or $\ell^p(\mathbb{N})$, $1 \leq p < \infty$, then $\bigcap_{\lambda>1} HC(\lambda B)$ is a dense G_δ subset of X. If D is the differentiation operator on $\mathcal{H}(\mathbb{C})$, then $\bigcap_{\lambda>0} HC(\lambda D)$ is a dense G_δ subset of $\mathcal{H}(\mathbb{C})$.

Proof. In the first case, $\mathcal{D}$ is the space of finitely supported vectors and S is the restriction of the forward shift; in the second case, $\mathcal{D}$ is the space of polynomials and S is the "primitivation" operator defined by $Su(z) = \int_0^z u(s)\,\mathrm{d}s$. We just have to check that property (2) is satisfied in both cases. In the first case, this is trivial because the forward shift has norm 1, and in the second case, this is also clear because $S^n(u)$ tends to 0 very rapidly, for each polynomial u. □

We shall now prove a result similar to Proposition 4.2 for a large class of weighted backward shifts. In what follows, X is a Fréchet space with an unconditional basis $(e_n)_{n\in\mathbb{N}}$, and we put $\mathcal{D} = \operatorname{span}\{e_n \mid n \in \mathbb{N}\}$. By a *weight sequence*, we mean any sequence of positive numbers $\mathbf{w} = (w_n)_{n\geq 1}$. The linear map $T_{\mathbf{w}} : \mathcal{D} \to \mathcal{D}$ defined by $T(e_0) = 0$ and $T(e_n) = w_n e_{n-1}$, $n \geq 1$, is called the *weighted shift* associated to $\mathbf{w}$. We say that a weight sequence $\mathbf{w}$ is *admissible for X* if $T_{\mathbf{w}}$ extends to a continuous linear operator on X. By unconditionality of the basis (e_n), all sequences $\mathbf{w}'$ with $w'_n = O(w_n)$ are then also admissible for X. Each weighted shift $T_{\mathbf{w}}$ has a linear right inverse $S_{\mathbf{w}} : \mathcal{D} \to \mathcal{D}$ defined by the relations $S_{\mathbf{w}}(e_n) = (1/w_{n+1})e_{n+1}$, $n \in \mathbb{N}$.

The hypotheses of the following theorem may look a bit technical, but we think the generality could be useful in some situations. The reader should look first at Corollaries 4.10 and 4.11 for more intuitive statements.

Theorem 4.7. *Let $(\mathbf{w}(\lambda))_{\lambda\in\Lambda}$ be a family of admissible weight sequences parameterized by some interval $\Lambda \subset \mathbb{R}$. For each $\lambda \in \Lambda$, let T_λ be the weighted shift on X associated to $\mathbf{w}(\lambda)$. Assume that the following properties hold true.*

(1) *All functions $w_n(\lambda)$ are nondecreasing, and Lipschitz on compact sets.*
(2) *For each compact interval $K \subset \Lambda$ and each $p \in \mathbb{N}$ there exist two sets of integers $A, B \subset \mathbb{N}$ such that:*
    (2i) *the set B contains all differences $n' - n$, where $n, n' \in A$ and $n' > n$;*
    (2ii) *for each $j \in \{0, \dots, p\}$, all series*

$$\sum_{m\in B} \frac{1}{w_1(\lambda) \times \cdots \times w_{m+j}(\lambda)} e_{m+j}$$

are convergent;
    (2iii) $\sum_{n\in A} 1/\sum_{k=1}^{n+p} L_k = \infty$, *where L_k is the Lipschitz constant of the function $\log(w_k)$ on K.*

Then $\bigcap_{\lambda\in\Lambda} HC(T_\lambda)$ is a dense G_δ subset of X.

Condition (2ii) above can be formally weakened under additional assumptions; this is the content of the next remark, where the canonical backward shift is the backward shift associated to the constant weight sequence $\mathbf{w} = \mathbf{1}$, and the canonical forward shift is its right inverse $S_{\mathbf{1}}$.

Remark 4.8. If all weight sequences $\mathbf{w}(\lambda)$ are bounded and the canonical backward shift is continuous, then condition (2ii) above is equivalent to

$$\text{(2ii)}' \qquad \text{all series} \quad \sum_{m\in B} \frac{1}{w_1(\lambda)\times\cdots\times w_{m+p}(\lambda)} e_{m+p} \quad \text{are convergent.}$$

If all weight sequences $\mathbf{w}(\lambda)$ are bounded below and the canonical forward shift is continuous, then (2ii) is equivalent to

$$\text{(2ii)}'' \qquad \text{all series} \quad \sum_{m\in B} \frac{1}{w_1(\lambda)\times\cdots\times w_{m}(\lambda)} e_{m} \quad \text{are convergent.}$$

Proof. Assume that the canonical forward shift is continuous and all sequences $\mathbf{w}(\lambda)$ are bounded below. If $p\in\mathbb{N}$ is fixed, then, for each $j\in\{0,\dots,p\}$ and for all $m\in B$, one can write

$$\frac{1}{w_1(\lambda)\times\cdots\times w_{m+j}(\lambda)} \leq \frac{C(\lambda)^j}{w_1(\lambda)\times\times w_m(\lambda)}$$

for some finite constant $C(\lambda)$. By unconditionality of the basis (e_n) and continuity of the forward shift, it follows that (2ii)″ is indeed equivalent to (2ii). The proof for (2ii)′ is the same. □

Remark 4.9. Condition (1) can be weakened: instead of assuming that each function $w_n(\lambda)$ is nondecreasing, it is enough to assume that the interval Λ can be partitioned into countably many subintervals on which either all functions w_n are nondecreasing, or all functions w_n are nonincreasing. This follows again from Baire's category theorem.

In the proof of Theorem 4.7, we use the following notation: for $n\in\mathbb{N}^*$ and $j\in\mathbb{N}$, we put

$$\theta_{n,j}(\lambda) = w_{j+1}(\lambda)\times\cdots\times w_{j+n}(\lambda),$$

and

$$\theta_n(\lambda) = \theta_{n,0}(\lambda) = w_1(\lambda)\times\cdots\times w_n(\lambda).$$

Proof of Theorem 4.7. As usual, we may assume that Λ is a compact interval $[a;b]$.

Since all functions $w_n(\lambda)$ are nondecreasing, it follows from the unconditionality of the basis (e_n) that the convergence of the series $\sum_{n\geq 1} w_n(\lambda)e_{n-1}$ is uniform with respect to $\lambda\in\Lambda$. Since all functions $w_n(\lambda)$ are moreover continuous, this implies that $T_\lambda(x)$ depends continuously on the pair $(\lambda,x)\in\Lambda\times X$.

We wish to apply the one-dimensional criterion as stated in Remark 2.2, with $T_{n,\lambda} = T_\lambda^n$ and $S_{n,\lambda} = S_\lambda^n$. So let us fix a continuous semi-norm $\|\,.\,\|$ on X, and put $O := \{\|x\| < 1\}$. Let us also fix $(u, v) \in \mathcal{D}\times\mathcal{D}$, and choose a positive integer p such that u, v are supported on $\{0, \dots, p\}$.

First, we check property (i) in Remark 2.2. For all $j \in \{0, \dots, p\}$ and λ, $\mu \in \Lambda$, we have

$$T_\lambda^n S_\mu^n(e_j) - e_j = \left(\frac{\theta_{n,j}(\lambda)}{\theta_{n,j}(\mu)} - 1\right) e_j.$$

Writing

$$\frac{\theta_{n,j}(\lambda)}{\theta_{n,j}(\mu)} = \exp\Big[\sum_{k=j+1}^{j+n} (\log w_k(\lambda) - \log w_k(\mu))\Big],$$

we get

$$\left|1 - \frac{\theta_{n,j}(\lambda)}{\theta_{n,j}(\mu)}\right| \le 1 - \exp\Big(-|\lambda - \mu| \sum_{k=j+1}^{j+n} L_k\Big).$$

Thus, property (i) holds with $C_n(u) = \sum_{k=1}^{p+n} L_k$ and some easily specified function ω_u.

Now, we check property (ii) in Remark 2.2. Let A, B be the two sets of integers given by (2). Replacing A by $A - \min A$, which will not alter (2iii), we may assume that, in addition to (2ii), all series

$$\sum_{n\in A} \frac{1}{w_1(\lambda) \times \cdots \times w_{n+j}(\lambda)} e_{n+j}, \quad j \in \{0, \dots, p\}$$

are convergent.

Let N be a positive integer to be chosen later, and let $(n_1, \dots, n_q)$ be a finite sequence from A such that $n_1 \ge N$ and $n_i - n_{i-1} \ge N$ for all $i > 1$. It is enough to show that if N is large enough, then $(n_1, \dots, n_q)$ fulfills the hypotheses (a), (b) of Lemma 4.1.

Since $S_{n,\lambda}(u)$ is supported on $\{0, \dots, n + p\}$ for each $n \in \mathbb{N}$ and all λ, we have $T_{n',\lambda} S_{n,\mu} = 0$ for all λ, $\mu \in \Lambda$ if $n' - n > p$. Thus, condition (a) in Lemma 4.1 is satisfied if N is large enough.

Let us fix λ, λ_1, …, $\lambda_q \in \Lambda$. For all $j \in \{0, \dots, p\}$ and $i \in \{1, \dots, q\}$, we have

$$S_{n_i,\lambda_i}(e_j) = \frac{\theta_j(\lambda_i)}{\theta_{n_i+j}(\lambda_i)} e_{n_i+j}.$$

By condition (1), one can write

$$\frac{\theta_j(\lambda_i)}{\theta_{n_i+j}(\lambda_i)} \le \frac{M_p}{\theta_{n_i+j}(a)}$$

for some constant M_p depending only on p. Since, by our additional assumption in (2ii), all series $\sum_{n\in A}(1/\theta_{n+j}(a))e_{n+j}$, $j\in\{0,\dots,p\}$ are unconditionally convergent, it follows that $\sum_1^q S_{n_i,\lambda_i}(u)\in O$ if N is large enough. This proves the first half of (b) in Lemma 4.1.

Similarly, if $i'>i$, then

$$T_{n_i,\lambda}S_{n_{i'},\lambda_{i'}}(e_j)=\frac{\theta_{n_i,j+n_{i'}-n_i}(\lambda)}{\theta_{n_{i'},j}(\lambda_{i'})}e_{j+n_{i'}-n_i}.$$

If moreover $\lambda\le\lambda_{i'}$, then

$$\frac{\theta_{n_i,j+n_{i'}-n_i}(\lambda)}{\theta_{n_{i'},j}(\lambda_{i'})}\le\frac{1}{\theta_{n_{i'}-n_i,j}(\lambda_{i'})}\le\frac{\theta_j(a)}{\theta_{j+n_{i'}-n_i}(a)},$$

because all functions w_k are nondecreasing. Using (2), we conclude that condition (b) in Lemma 4.1 is satisfied if N is large enough. □

Corollary 4.10. *The operators T_λ have a residual set of common hypercyclic vectors as soon as the following properties are satisfied.*

(a) *All functions* $\log(w_n)$ *are nondecreasing, and Lipschitz on compact sets with uniformly bounded Lipschitz constants.*
(b) *All series* $\sum_n 1/(w_1(\lambda)\times\cdots\times w_n(\lambda))e_n$ *are convergent.*

Proof. This follows at once from Theorem 4.7: one may take $A=B=\mathbb{N}$. □

From this, one gets the following extension of the Abakumov-Gordon result.

Corollary 4.11. *Let* $\mathbf{w}=(w_n)$ *be an admissible weight sequence, and put*

$$\lambda_{\mathbf{w}}=\inf\left\{\lambda>0\mid \text{the series }\sum_n\frac{\lambda^{-n}}{w_1\times\cdots\times w_n}e_n\text{ converges in }X\right\}$$

If $T_{\mathbf{w}}$ is the weighted shift associated to $\mathbf{w}$*, then $\bigcap_{\lambda>\lambda_{\mathbf{w}}}HC(\lambda T_{\mathbf{w}})$ is a dense G_δ subset of X.*

Example 4.12. $X=\ell^p(\mathbb{N})$ or $c_0(\mathbb{N})$ and Λ is a single point; so, we just consider a single weighted shift $T_{\mathbf{w}}$ on X. Then condition (1) is vacuously satisfied, as well as (2iii) for any set of integers A. It is easy to check that (2i), (2ii) hold if and only if

$$\sup_n w_1\times\cdots\times w_n=\infty,$$

which is Salas' necessary and sufficient condition for hypercyclicity.

Example 4.13. $X=\ell^p(\mathbb{N})$ or $c_0(\mathbb{N})$, $\mathbf{w}_n(\lambda)=1+\lambda/n$, $\lambda>0$. Since

$$\log(w_1(\lambda)\times\cdots\times w_n(\lambda))\sim\lambda\log(n)$$

for each $\lambda>0$, we get common hypercyclic vectors for all $\lambda>1/p$ in the ℓ^p case, and for all $\lambda>0$ in the c_0 case. This improves an example given in [9].

Example 4.14. $X = \mathcal{H}(\mathbb{C})$, $e_n = z^n$, $\mathbf{w}_n(\lambda) = \lambda c_n$, $\lambda > 0$, where (c_n) is a sequence of positive numbers such that $\sup_n c_n^{1/n} < \infty$ and $(c_1 \cdots c_n)^{1/n} \to \infty$. This includes the case λD, where D is the differentiation operator ($c_n = n$).

5. Translation-dilation Operators

In this section, we consider parameterized sequences of translation-dilation operators on the space of entire functions $\mathcal{H}(\mathbb{C})$. The parameter space Λ is a subset of $\mathbb{R}^2$, and for each $\lambda = (s, t) \in \Lambda$, the sequence $\mathbf{T}_\lambda = (T_{n,\lambda})$ is given by

$$T_{n,\lambda} u(z) = a_n(s) u(z + b_n(t)),$$

where $b_n(t) \in \mathbb{C}$ and $a_n(s) > 0$.

In order to motivate the hypotheses of Theorem 5.1 below, let us explain briefly how the basic criterion can be used to show that the translation operators

$$T_\lambda f(z) = f(z + e^{it}), \quad t \in [0; 2\pi[$$

have common hypercyclic vectors (which was proved in [9]). Let $K = [a; b]$ be a compact subinterval of $[0; 2\pi[$. Choose a subdivision $a = \lambda_0 < \lambda_1 < \cdots < \lambda_q = b$ such that $\lambda_i - \lambda_{i-1} < \tau/(Ni)$ for all $i \in \{1, \dots, q\}$, where $\tau > 0$ and $N \in \mathbb{N}^*$ have to be specified; this can be done because the series $\sum 1/i$ is divergent. Then put $\Lambda_i = [\lambda_{i-1}; \lambda_i]$ and $n_i = Ni$. Assume the given neighbourhood of 0 has the form $O = \{u \in \mathcal{H}(\mathbb{C}) \mid \sup_{|z| \le A} < \varepsilon\}$. Then the first condition in (iii) of the basic criterion is fulfilled if τ is small enough. Moreover, if N is large enough, then the sets $\bar{D}(0, A) + NiK$ are pairwise disjoint, so that Runge's theorem can be applied to get a polynomial p with the required properties.

Using in essence the same ideas, we are now going to prove a more general result for translation-dilation operators.

We shall say that a pair of series of nonnegative numbers $(\sum \alpha_n, \sum \beta_n)$ is *strongly divergent* if, for each positive number C, one can find nonempty finite sets of integers $F_1 < F_2 \cdots$ such that

- ◇ $\sum_{k=1}^{\infty} \inf_{n \in F_k} \beta_n = \infty$;
- ◇ $\inf_{k \ge 1} \sum_{n \in F_k} \alpha_n > C$.

For example, the pair $(\sum 1/\log n, \sum 1/n)$ is strongly divergent. To see this, put $F_k =]N_k; N_{k+1}]$, where N_k is the integral part of $2Ck \log k$. On the other hand the pair $(\sum 1/n^\alpha, \sum 1/n)$ is not strongly divergent if $\alpha > 0$.

Theorem 5.1. *Assume that the parameter space Λ is a countable union of rectangles $I \times J$ for which the following properties hold true.*

(1) *All functions a_n, b_n are Lipschitz.*

(2) *For each closed disk* $E = \bar{D}(0,R) \subset \mathbb{C}$, *one can find a positive integer* N *such that*
   - ◇ $b_n(J) \cap E = \varnothing$ *if* $n \geq N$ *and* $(E + b_n(J)) \cap (E + b_m(J)) = \varnothing$ *if* $|n - m| \geq N$;
   - ◇ $\mathbb{C} \setminus (E + b_n(J))$ *is connected, for each* $n \geq N$.

(3) *Putting* $\alpha_n = 1/\operatorname{Lip}(\log a_n)$ *and* $\beta_n = 1/\operatorname{Lip}(b_n)$, *the pair of series* $(\sum \alpha_n, \sum \beta_n)$ *is strongly divergent.*

Then $\bigcap_{\lambda\in\Lambda} \operatorname{Univ}(\mathbf{T}_\lambda)$ *is a dense* G_δ *subset of* $\mathcal{H}(\mathbb{C})$.

Proof. We may assume that Λ is a compact rectangle $I \times J$ for which (1), (2), (3) are satisfied. We apply the basic criterion, with $\mathcal{D} = \mathcal{H}(\mathbb{C})$. So let us fix some neighbourhood of 0 in $\mathcal{H}(\mathbb{C})$, say $O = \{f \in \mathcal{H}(\mathbb{C}) : \|u\|_E < \varepsilon\}$, where E is a closed disk centered at 0 and $\|f\|_E = \sup_E |f|$. Let us also fix $u, v \in \mathcal{H}(\mathbb{C})$.

Let N be the positive integer given by (2), and let η, η' be small positive numbers. Since the pair of series $(\sum \alpha_n, \sum \beta_n)$ is strongly divergent, one can find nonempty finite sets of integers $F_1 < \cdots < F_d$ such that

- ◇ $\sum_{k=1}^{d} \inf_{n\in F_k} \beta_n > |J|/\eta'$,
- ◇ $\sum_{n\in F_k} \alpha_n > |I|/\eta'$ for each $k \in \{1,\dots,d\}$.

It follows that if η' is small enough, then:

(a) one can partition the interval J into intervals $J_1, \dots, J_d$ such that $\operatorname{Lip}(b_n) \times |J_k| \leq \eta$ for each $k \in \{1,\dots,d\}$ and all $n \in F_k$;

(b) for each $k \in \{1,\dots,d\}$, one can partition the interval I into subintervals $I_{k,n}$, $n \in F_k$, such that $\operatorname{Lip}(\log a_n) \times |I_{k,n}| \leq \eta$ for all $n \in F_k$.

Moreover, by taking some arithmetical progression of the sets F_k, subdividing each set F_k into N pieces and replacing η' by $N\eta'$, we may also assume that $\inf F_1 \geq N$ and $|n - m| \geq N$ whenever n, m are distinct integers from $F_1 \cup \cdots \cup F_d$. Here, N is the positive integer given by (2).

Now, let $\Lambda_1, \dots, \Lambda_q$ be an enumeration of all rectangles $I_{k,n} \times J_k$, $k \in \{1,\dots,d\}$, $n \in F_k$. By definition, we have $\bigcup_i \Lambda_i = \Lambda$. For each $i \in \{1,\dots,q\}$, choose any point $\lambda_i = (s_i, t_i) \in \Lambda_i$, and put $n_i = n$, where i corresponds to the pair (k,n). We show that one can find $p \in \mathcal{H}(\mathbb{C})$ such that conditions (ii), (iii) in the basic criterion are satisfied.

Put $F = F_1 \cup \cdots \cup F_d$. By condition (2), one can apply Runge's Theorem to get some function $p \in \mathcal{H}(\mathbb{C})$ such that

- ◇ $|p(z) - v(z)| < \varepsilon$ on E;
- ◇ $|p(z) - S_{n_i,\lambda_i} u(z)| < \varepsilon / \inf a_{n_i}(J)$ on $E + b_{n_i}(J)$, for each $i \in \{1,\dots,q\}$.

By its very definition, this function p satisfies (ii) and the second half of (iii) in the basic criterion.

It remains to show that one can ensure $T_{n_i,\lambda} S_{n_i,\lambda_i}(u) - u \in O$ for each $i \in \{1,\dots,q\}$ and all $\lambda = (s,t) \in \Lambda_i$. Now, we have

$$T_{n_i,\lambda} S_{n_i,\lambda_i} u(z) - u(z) = \left(\frac{a_{n_i}(s)}{a_{n_i}(s_i)} - 1\right) u(z + b_{n_i}(t) - b_{n_i}(t_i)),$$

so that, by (a), (b) above, we get the desired result if η is small enough. ◻

Example 5.2. Theorem 5.1 can be applied to the family of operators

$$T_{n,(s,\zeta)} = n^s \tau_{n\zeta}, \quad s \geq 0,\ \zeta \in \mathbb{T},$$

where τ_w is the operator of translation by w. Identifying $\zeta = e^{it} \in \mathbb{T}$ with $t \in [0; 2\pi[$, the space of parameters is $\Lambda = \mathbb{R}^+ \times [0; 2\pi[$, and $a_n(s) = n^s$, $b_n(t) = ne^{it}$. Conditions (1) and (2) are obviously satisfied, and (3) is satisfied as well since $\alpha_n = 1/\log n$ and $\beta_n = 1/n$. Actually, one can even consider operators of the form $c_n n^s \tau_{n\zeta}$, where (c_n) is an arbitrary (but fixed) sequence of positive numbers. This answers a question of Costakis raised in [8], where the one-parameter family $T_{n,s} = n^s \tau_n$ was considered. Proceeding as in [8], one obtains in fact the existence of a common universal vector for the three parameter family $T_{n,(s,r,\zeta)} = n^s \tau_{nr\zeta}$, with $s \geq 0$, $r > 0$ and $\zeta \in \mathbb{T}$.

6. Concluding Remarks

We conclude this paper by some simple remarks concerning the size or the shape of a set of parameters allowing common universality.

First, we show that under quite general assumptions, there is always a very large set of parameters on which common universality does occur. Here, we are again in the general case of parameterized sequences of operators $(\mathbf{T}_\lambda)_{\lambda \in \Lambda}$, where Λ is a topological space and $T_{n,\lambda}(x)$ depends continuously on (x, λ), for each $n \in \mathbb{N}$.

Remark 6.1. Assume $\mathrm{Univ}(\mathbf{T}_\lambda)$ is residual for each $\lambda \in \Lambda$ (in particular, this holds if each sequence $\mathbf{T}_\lambda$ is the sequence of iterates of a hypercyclic operator T_λ). If the topological space Λ is second countable and is a Baire space, then there exists a dense G_δ set $G \subset \Lambda$ such that $\bigcap_{\lambda \in G} \mathrm{Univ}(\mathbf{T}_\lambda) \neq \emptyset$.

Proof. It is easy to check that the set

$$\mathcal{G} := \{(x, \lambda) \mid x \in \mathrm{Univ}(\mathbf{T}_\lambda)\}$$

is a G_δ subset of $X \times \Lambda$. In particular, $\mathcal{G}$ has the Baire property in $X \times \Lambda$. By assumption, for each $\lambda \in \Lambda$, the λ-section of $\mathcal{G}$ is a dense G_δ subset of X. By the classical Kuratowski-Ulam Theorem (see [14]), this implies that for comeagerly many points $x \in X$, the x-section of $\mathcal{G}$ is comeager in Λ. In particular, there exists at least one point $x \in X$ such that $G := \mathcal{G}_x$ is comeager in Λ, which is the desired result. ◻

Next, we consider the case of weighted shifts whose weight sequence consists only in 1's and 2's. Here, the parameter space is $\mathbf{W} = \{1; 2\}^{\mathbb{N}^*}$. For each $\mathbf{w} \in \mathbf{W}$, we denote by $T_{\mathbf{w}}$ the weighted shift associated to $\mathbf{w}$, acting on $X = c_0(\mathbb{N})$ or $\ell^p(\mathbb{N})$. The following result shows that common hypercyclicity occurs on a set of parameters which is very large both in the Baire Category sense and in the measure sense.

Remark 6.2. Let m the canonical product measure on $\mathbf{W}$. There exists a G_δ set $H \subset \mathbf{W}$ such that $m(\mathbf{W} \setminus H) = 0$ and $\bigcap_{\mathbf{w} \in H} HC(T_{\mathbf{w}}) \neq \emptyset$.

Proof. An elementary computation shows that if $\mathcal{F}$ is a closed subset of $\mathbf{W} \times X$ and $\varepsilon > 0$, then the set

$$\{x \in X \mid m(\{\mathbf{w} \mid (\mathbf{w}, x) \in \mathcal{F}\}) < \varepsilon\}$$

is open in X. It follows that for each nonempty open set $V \subset X$, the set

$$G_V := \{x \in X \mid \forall^{\text{ae}}\, \mathbf{w} \in \mathbf{W}\ \exists n : T_{\mathbf{w}}^n(x) \in V\},$$

is a G_δ subset of X. Here, $\forall^{\text{ae}}$ means "for m-almost all." By the Baire Category Theorem, it is therefore enough to show that each set G_V is dense in X: once this is done, just pick any point $x \in \bigcap_p G_{V_p}$, where (V_p) is a countable basis of open sets for X, and put $H = \{\mathbf{w} \mid x \in HC(T_{\mathbf{w}})\}$. Thus, we have to show that $G_V \cap U \neq \emptyset$ for each pair of nonempty open sets (U, V); and of course, we may assume that $U = B(u, \varepsilon)$ and $V = B(v, \varepsilon)$, where u, v are finitely supported vectors and $\varepsilon > 0$.

Let $(e_i)_{i \in \mathbb{N}}$ be the canonical basis of X. The powers of a weighted shift $T_{\mathbf{w}}$ act on X in the following way: for all $x = \sum_0^\infty x_i e_i \in X$ and $j \in \mathbb{N}$, we have

$$\langle e_j^*, T_{\mathbf{w}}^n(x)\rangle = 2^{p_{n,j}(\mathbf{w})} x_{j+n},$$

where $p_{n,j}(\mathbf{w})$ is the number of 2's in the sequence $\mathbf{w}$ between coordinates $j+1$ and $j+n$.

Let d be a positive integer such that u and v are supported on $\{0, \dots, d\}$, and let $N > d$ be a large positive integer. It follows from the classical theory of random walks (see [10]) that if we put

$$A_k = \{\mathbf{w} \in \mathbf{W} \mid p_{2kN,0}(\mathbf{w}) = \cdots = p_{2kN,d}(\mathbf{w}) = kN\},$$

then almost every point $\mathbf{w} \in \mathbf{W}$ belongs to infinitely many sets A_k, $k \in \mathbb{N}^*$. We put $A = \bigcup_k A_k$, so that $m(\mathbf{W} \setminus A) = 0$. Now, we define $x = \sum_0^\infty x_i e_i \in X$ as follows.

- ◇ $x_j = u_j$ for all $j \in \{0, \dots, d\}$;
- ◇ $x_i = 2^{-kN} v_j$ if $i = j + 2kN$ for some $j \in \{0, \dots, d\}$ and $k \in \mathbb{N}^*$;
- ◇ $x_i = 0$ otherwise.

It is clear that $\|x - u\| < \varepsilon$ if N is large enough, so that $x \in U$. Moreover, it is also clear that if N is large enough and $\mathbf{w} \in A_k$ for some $k \in \mathbb{N}^*$, then $\|T_{\mathbf{w}}^{kN}(x) - v\| < \varepsilon$. Since $m(\mathbf{W} \setminus A) = 0$, it follows that $x \in G_V$ if N is large enough. This concludes the proof. ◻

Finally, let us consider the rather intriguing case of direct sums of shifts. Let B be the usual backward shift on $\ell^2(\mathbb{N})$. By Salas' criterion, the direct sum $B_{s,t} := sB \oplus$

tB is hypercyclic provided $s, t > 1$. Thus, one can look for common hypercyclicity results for those operators B_λ, $\lambda = (s,t) \in \Lambda_0 :=]1;\infty[\times]1;\infty[$.

If one wants to prove that all operators B_λ, $\lambda \in \Lambda_0$ have a common hypercyclic vector, one may try to apply the 2-dimensional criterion described at the end of Section 2. However, it is immediately seen that this does not work: with the notations of Remark 2.2, the order of magnitude of $C_n(u)$ is n, hence $\sum_0^\infty 1/C_n(u)^2 < \infty$. The following result, which is due to A. Borichev, is therefore quite natural.

Remark 6.3. If $\Lambda \subset \Lambda_0$ is such that $\bigcap_{\lambda\in\Lambda} HC(B_\lambda) \neq \emptyset$, then Λ has Lebesgue measure 0.

Proof. Assume that $\bigcap_{\lambda\in\Lambda} HC(B_\lambda) \neq \emptyset$, and choose a common hypercyclic vector $p = x \oplus y \in \ell^2(\mathbb{N}) \oplus \ell^2(\mathbb{N})$. Then, for each $\lambda = (s,t) \in \Lambda$, one can approximate the vector $e_0 \oplus e_0$ by vectors of the form $s^n B^n(x) \oplus t^n B^n(y)$. In particular, looking at the first coordinates, one can find $n \in \mathbb{N}^*$ such that $|s^n x_n - 1|$ and $|t^n y_n - 1|$ are arbitrarily small. Then x_n and y_n have positive real parts, and putting $a_n = -(1/n)\log(\operatorname{Re} x_n)$, $b_n = -(1/n)\log(\operatorname{Re} y_n)$, we see that $n|\log(s) - a_n|$ and $n|\log(t) - b_n|$ are arbitrarily small. Thus, we see that for each $\varepsilon > 0$, the set $\log(\Lambda) := \{(\log(s), \log(t)) \mid (s,t) \in \Lambda\}$ can be covered by a sequence of squares C_{n_i} with respective sides not greater than ε/n_i. It follows that $\log(\Lambda)$ has Lebesgue measure 0, which concludes the proof. ◻

On the other hand, we also have a positive result if the set Λ has a a particular shape. As in Section 2, let us define a *monotonic Lipschitz curve* to be the image of a Lipschitz arc $\gamma : I \to \mathbb{R}^2$ whose coordinates are both nondecreasing or both nonincreasing.

Remark 6.4. If $\Lambda \subset \Lambda_0$ can be covered by countably many monotonic Lipschitz curves, then $\bigcap_{\lambda\in\Lambda} HC(B_\lambda)$ is a dense G_δ subset of $\ell^2(\mathbb{N}) \oplus \ell^2(\mathbb{N})$.

Proof. As usual, it is enough to consider the case of a single monotonic Lipschitz curve. Let us denote by $\leq$ the product ordering $\mathbb{R}^2$. Proceeding exactly as in the proof of Proposition 4.2, one checks that the hypotheses of Lemma 2.3 are satisfied. Therefore, one can apply the generalized form of the one-dimensional criterion mentioned at the end of Section 2. ◻

Thus, we have common hypercyclicity of the operators B_λ when the parameter set Λ is the graph of a nondecreasing Lipschitz function. In particular, the operators $sB \oplus sB$ have common hypercyclic vectors. It is tempting (although this is probably irrelevant) to hope to find a link with one of the most famous open problems in the area: is $T \oplus T$ hypercyclic whenever T is?

On the other hand, we don't know what to say when Λ is the graph of a *nonincreasing* Lipschitz function. For example, we are unable to decide whether $\bigcap_{2\leq s\leq 3} HC(B_{(s,4-s)})$ is nonempty. This is a bit surprising, and rather irritating.

To conclude this paper in a more optimistic fashion, let us point out briefly some possible connections with descriptive set theory. We turn back to the general case of a family of hypercyclic operators $(T_\lambda)_{\lambda\in\Lambda}$ acting on a separable Fréchet space X. We assume that the parameter space Λ is a Polish space, that is, a separable, completely metrizable topological space, and that $T_\lambda(x)$ depends continuously on $(\lambda, x) \in \Lambda \times X$. We also assume that the operators T_λ commute with each other.

Let us denote by $\mathcal{I}$ the family of all compact sets $K \subset \Lambda$ such that

$$\bigcap_{\lambda\in K} HC(T_\lambda) \neq \emptyset.$$

It is easily checked that if $K \subset \Lambda$ is compact, then $\bigcap_{\lambda\in K} HC(T_\lambda)$ is a G_δ subset of X. Moreover, if $K \in \mathcal{I}$ then, by our commutativity assumption, this set is also dense in X. By the Baire Category Theorem, it follows that the family $\mathcal{I}$ is closed under countable unions. In other words, $\mathcal{I}$ is a *σ-ideal* of compact sets, a kind of objects for which there is by now a well-developed theory (see e.g. [15]). It is easy to check that the σ-ideal $\mathcal{I}$ is G_δ in $\mathcal{K}(\Lambda)$, the space of all compact subsets of Λ equipped with the (Polish) topology induced by the Hausdorff metric. Thus, from the point of view of descriptive set theory, $\mathcal{I}$ is a very simple object. Turning back to direct sums of shifts, this allows us to think that it is not completely hopeless to seek for an explicit, "geometrical" characterization of those compact sets $K \subset]1;\infty[\times]1;\infty[$ for which $\bigcap_{(s,t)\in K} HC(sB \oplus tB) \neq \emptyset$.

Acknowledgement The authors would like to thank the anonymous referee for his/her careful reading and pertinent suggestions.

REFERENCES

[1] EVGENY ABAKUMOV and JULIA GORDON, *Common hypercyclic vectors for multiples of backward shift*, J. Funct. Anal. **200** (2003), 494–504, http://dx.doi.org/10.1016/S0022-1236(02)00030-7. MR 1979020 (2004g:47012)

[2] FRÉDÉRIC BAYART, *Common hypercyclic vectors for composition operators*, J. Operator Theory **52** (2004), 353–370. MR 2119275 (2006a:47014)

[3] ______, *Common hypercyclic subspaces*, Integral Equations Operator Theory **53** (2005), 467–476, http://dx.doi.org/10.1007/s00020-004-1316-6. MR 2187432 (2006k:47016)

[4] FRÉDÉRIC BAYART and SOPHIE GRIVAUX, *Frequently hypercyclic operators*, Trans. Amer. Math. Soc. **358** (2006), 5083–5117 (electronic), http://dx.doi.org/10.1090/S0002-9947-06-04019-0. MR 2231886

[5] ______, *Hypercyclicity and unimodular point spectrum*, J. Funct. Anal. **226** (2005), 281–300, http://dx.doi.org/10.1016/j.jfa.2005.06.001. MR 2159459 (2006i:47014)

[6] PAUL S. BOURDON and NATHAN S. FELDMAN, *Somewhere dense orbits are everywhere dense*, Indiana Univ. Math. J. **52** (2003), 811–819, http://dx.doi.org/10.1512/iumj.2003.52.2303. MR 1986898 (2004d:47020)

[7] George Costakis, *On a conjecture of D. Herrero concerning hypercyclic operators*, C. R. Acad. Sci. Paris Sér. I Math. **330** (2000), 179–182. MR 1748304 (2001a:47012) (English, with English and French summaries)

[8] ______, *Common Cesàro hypercyclic vectors* (preprint).

[9] George Costakis and Martin Sambarino, *Genericity of wild holomorphic functions and common hypercyclic vectors*, Adv. Math. **182** (2004), 278–306, http://dx.doi.org/10.1016/S0001-8708(03)00079-3. MR 2032030 (2004k:47009)

[10] William Feller, *An Introduction to Probability Theory and Its Applications. Vol. I*, John Wiley & Sons Inc., New York, N.Y., 1950. MR 0038583 (12,424a)

[11] Robert M. Gethner and Joel H. Shapiro, *Universal vectors for operators on spaces of holomorphic functions*, Proc. Amer. Math. Soc. **100** (1987), 281–288, http://dx.doi.org/10.2307/2045959. MR 884467 (88g:47060)

[12] Gilles Godefroy and Joel H. Shapiro, *Operators with dense, invariant, cyclic vector manifolds*, J. Funct. Anal. **98** (1991), 229–269, http://dx.doi.org/10.1016/0022-1236(91)90078-J. MR 1111569 (92d:47029)

[13] Karl-Goswin Grosse-Erdmann, *Universal families and hypercyclic operators*, Bull. Amer. Math. Soc. (N.S.) **36** (1999), 345–381, http://dx.doi.org/10.1090/S0273-0979-99-00788-0. MR 1685272 (2000c:47001)

[14] Alexander S. Kechris, *Classical Descriptive Set Theory*, Graduate Texts in Mathematics, vol. 156, Springer-Verlag, New York, 1995, ISBN 0-387-94374-9. MR 1321597 (96e:03057)

[15] A. S. Kechris, A. Louveau, and W. H. Woodin, *The structure of σ-ideals of compact sets*, Trans. Amer. Math. Soc. **301** (1987), 263–288. MR 879573 (88f:03042)

[16] Stanisław Kwapień, *A theorem on the Rademacher series with vector valued coefficients*, Proc. Probability in Banach spaces (Proc. First Internat. Conf., Oberwolfach, 1975), Springer, Berlin, 1976, pp. 157–158. Lecture Notes in Math., Vol. 526. MR 0451333 (56 #9620)

[17] Michel Ledoux and Michel Talagrand, *Probability in Banach Spaces*, Ergebnisse der Mathematik und ihrer Grenzgebiete (3) [Results in Mathematics and Related Areas (3)], vol. 23, Springer-Verlag, Berlin, 1991, ISBN 3-540-52013-9, Isoperimetry and processes. MR 1102015 (93c:60001)

[18] Daniel Li and Hervé Queffélec, *Introduction à l'étude des espaces de Banach*, Cours Spécialisés [Specialized Courses], vol. 12, Société Mathématique de France, Paris, 2004, ISBN 2-85629-155-4, Analyse et probabilités. [Analysis and Probability Theory]. MR 2124356 (2006c:46012) (French)

[19] Alfredo Peris, *Multi-hypercyclic operators are hypercyclic*, Math. Z. **236** (2001), 779–786, http://dx.doi.org/10.1007/PL00004850. MR 1827503 (2002a:47008)

[20] G. Pisier, *Conditions d'entropie assurant la continuité de certains processus et applications à l'analyse harmonique*, Proc. Seminar on Functional Analysis, 1979–1980, École Polytech., Palaiseau, 1980, pp. Exp. No. 13-14, 43. MR 604395 (82j:60067) (French)

[21] Héctor N. Salas, *Supercyclicity and weighted shifts*, Studia Math. **135** (1999), 55–74. MR 1686371 (2000b:47020)

E-mail: `Frederic.Bayart@math.u-bordeaux1.fr`

E-mail: `Etienne.Matheron@math.u-bordeaux1.fr`

Laboratoire Bordelais d'Analyse et de Géométrie
UMR 5467, Université Bordeaux 1
351 Cours de la Libération
33405 Talence Cedex, France.

KEY WORDS AND PHRASES: hypercyclic operators, Baire Category, type, subgaussian processes.
2000 MATHEMATICS SUBJECT CLASSIFICATION, 2000: 47A16, 47B20, 43A46.
Received: *September 14th, 2005; revised: March 8th, 2006.*