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ON THE COMPLEXITY OF H SETS OF THE UNIT CIRCLE

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Let $\mathcal{K}(\mathbb{T})$ be the space of all compact subsets of the circle group $\mathbb{T}$ equipped with its natural (metric, compact) topology. Recently T. Linton ([8]) showed that an important class of thin sets from harmonic analysis, the *H sets*, form a true $\mathbf{\Sigma}^0_3$ ($\mathbf{G}_{\delta\sigma}$) subset of $\mathcal{K}(\mathbb{T})$, that is, a $\mathbf{\Sigma}^0_3$ set which is not $\mathbf{\Pi}^0_3$ ($\mathbf{F}_{\sigma\delta}$). In this note, we generalize his result by showing that if $E \in \mathcal{K}(\mathbb{T})$ is an M set (see the definition below), then the H sets contained in E also form a true $\mathbf{\Sigma}^0_3$ subset of $\mathcal{K}(\mathbb{T})$. In fact, the result is somewhat more general and shows that several related classes of thin sets are true $\mathbf{\Sigma}^0_3$ within any M set.

Before stating precisely the result, we have to introduce some definitions. We denote by A the Banach algebra of all continuous complex-valued functions on $\mathbb{T}$ with absolutely convergent Fourier series. The norm of $f \in A$ is given by $\|f\|_A = \sum_{n\in\mathbb{Z}} |\widehat{f}(n)|$. Thus the Fourier transform identifies A with $l^1(\mathbb{Z})$. The dual space of A is the space PM ($\sim l^\infty$) of all distributions on $\mathbb{T}$ with bounded Fourier coefficients, and A itself is the dual of the space PF ($\sim c_0$) of pseudofunctions (distributions with Fourier coefficients tending to 0).

DEFINITION 1. A closed set $E \subseteq \mathbb{T}$ is said to be a *set of uniqueness*, or a *U set* (resp. a *U_0 set*) if it supports no non-zero pseudofunction (resp. no probability measure in PF). E is an *M set* (*M_0 set*) if it is not in U (U_0).

DEFINITION 2. A closed set $E \subseteq \mathbb{T}$ is said to be an *H set* if there exists a non-empty open set $V \subseteq \mathbb{T}$ and an infinite sequence (m_k) of positive integers such that for all k, $m_k E \cap V = \emptyset$ (where $mE = \{mx : x \in E\}$).

Evidently $U \subseteq U_0$ but the converse is not true. It is well known that every H set is a set of uniqueness; in fact, H is only a very small part of U. All these sets have a long history and we refer to [1], [4] or [7] for much more information.

It is not difficult to check that H is $\mathbf{\Sigma}^0_3$ in $\mathcal{K}(\mathbb{T})$ (a proof is given in [8]). Now the easiest way to show that it is not $\mathbf{\Pi}^0_3$ is to produce a continuous

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map $\varphi : X \to \mathcal{K}(\mathbb{T})$ from some Polish space X, such that $\varphi^{-1}(H)$ is not $\mathbf{\Pi}^0_3$ in X. The Polish space we use is the space ω^ω of all infinite sequences of non-negative integers, with the product topology. Let W be the following subset of ω^ω:

$$W = \{\alpha \in \omega^\omega : \alpha(p) \to \infty \text{ as } p \to \infty\}.$$

It is well known that W is a true $\mathbf{\Pi}^0_3$ subset of ω^ω (see [9], pp. 92–96, for a proof). In [8] T. Linton constructs a continuous $\varphi : \omega^\omega \to \mathcal{K}(\mathbb{T})$ such that $\varphi^{-1}(H) = \omega^\omega \setminus W$. We now state a similar but more general result. Below, if $E \in \mathcal{K}(\mathbb{T})$ we let $\mathcal{K}(E) = \{K \in \mathcal{K}(\mathbb{T}) : K \subseteq E\}$.

THEOREM. (1) *Let $E \subseteq \mathbb{T}$ be an M set. Then there is a continuous map $\varphi : \omega^\omega \to \mathcal{K}(E)$ such that*

- *if $\alpha \in W$ then $\varphi(\alpha)$ is an M set,*
- *if $\alpha \notin W$ then $\varphi(\alpha)$ is an H set.*

In particular, there is no $\mathbf{\Pi}^0_3$ set $B \subseteq \mathcal{K}(\mathbb{T})$ such that $H \cap \mathcal{K}(E) \subseteq B \subseteq U$.

(2) *If E is an M_0 set, the same conclusion holds with M replaced by M_0. Hence there is no $\mathbf{\Pi}^0_3$ subset of $\mathcal{K}(\mathbb{T})$ such that $H \cap \mathcal{K}(E) \subseteq B \subseteq U_0$.*

Remark. It follows from a result of N. Bary ([2], Théorème V) that the proof in [8] gives the above conclusion for $E = \mathbb{T}$.

For the proof of our theorem we will make use of two standard lemmas.

LEMMA 1 (see [7], p. 234). *Let h be a function in A and $S \in PF$. For $m \in \mathbb{Z}$ define $h^m \in A$ by $h^m(x) = h(mx)$ and $S^m = h^m \cdot S \in PF$. Then*

(i) $S^m \to \widehat{h}(0) \cdot S$ *weakly in PF as $|m| \to \infty$;*
(ii) $\|S^m\|_{PM} \to \|\widehat{h}\|_{PM} \, \|S\|_{PM}$.

LEMMA 2. *Let ε be any positive number. Then one can find $h \in A$, $h \geq 0$, such that*

- $\widehat{h}(0) = 1$,
- $|\widehat{h}(k)| < \varepsilon$ *if* $k \neq 0$,
- $h \equiv 0$ *in a neighbourhood of* 0.

One can take $1-\tau_\eta$ suitably normalized, where τ_η is the usual trapezoidal function and η is small enough.

Before turning to the proof of the theorem, let us fix some notations. The set of non-negative integers is denoted by ω, and $\omega^{<\omega}$ is the set of all finite sequences of (non-negative) integers. If $s \in \omega^{<\omega}$, $|s|$ is the length of s. If $s = (n_0, \ldots, n_k) \in \omega^{<\omega}$ and $n \in \omega$, $s^\frown n$ is the sequence $(n_0, \ldots, n_k, n)$. If $s \in \omega^{<\omega}$ and $\alpha \in \omega^\omega$, $s \preceq \alpha$ means that $\alpha(i) = s(i)$ for all $i < |s|$. Finally, if $\alpha \in \omega^\omega$ and N is a positive integer, we denote by $\alpha_{\lceil N}$ the sequence $(\alpha(0), \ldots, \alpha(N-1))$.

We can at last begin the proof of the theorem. The two parts will be treated together.

First, according to Lemma 2, we choose for each $n \in \omega$ a non-negative function $h_n \in A$ and an open set $U_n \subseteq \mathbb{T}$ such that $h_n \equiv 0$ on U_n, $\widehat{h}_n(0) = 1$ and $|\widehat{h}(k)| < 1/(2(n+1))$ if $k \neq 0$.

Let now $E \in \mathcal{K}(\mathbb{T})$ be an M set and T be a non-zero pseudofunction with $\operatorname{supp}(T) \subseteq E$, $\|T\|_{PM} = 1 = \widehat{T}(0)$.

We construct for each $s \in \omega^{<\omega}$ a closed set $E_s \subseteq \mathbb{T}$, a pseudofunction T_s and positive integers N_s, m_s satisfying the following conditions:

(0) $T_\emptyset = T$, $E_\emptyset = \operatorname{supp}(T)$;
(1) $N_{s^\frown n} > N_s$, $m_{s^\frown n} > m_s$, $E_{s^\frown n} \subseteq E_s$ for all $n \in \omega$;
(2) E_s is a perfect set and $\operatorname{supp}(T_s) \subseteq E_s$;
(3) $\delta(E_{s^\frown n}, E_s) < 2^{-|s|}$ for all n, where δ is the Hausdorff metric on $\mathcal{K}(\mathbb{T})$;
(4) $\sup\{|\widehat{T}_s(k)| : |k| > N_s\} < 2^{-|s|}$;
(5) • $\|T_s\|_{PM} < 2$,
• $|\widehat{T}_s(k) - \widehat{T}_{s^\frown n}(k)| < 2^{-|s|-1}$ for all $n \in \omega$ and k, $|k| \leq N_s$;
(6) $\|T_s - T_{s^\frown n}\|_{PM} < 2^{-|s|} + 1/(n+1)$ for all n;
(7) $m_{s^\frown n} \cdot E_{s^\frown n} \cap U_n = \emptyset$ for all n.

By condition (0) we must let $T_\emptyset = T$, $E_\emptyset = \operatorname{supp}(T_\emptyset)$. Then $E_\emptyset$ is perfect because $T \in PF$, so that (2) is true. We can also choose $N_\emptyset$ big enough to ensure (4).

Assume E_s, T_s, N_s, m_s have been constructed and fix $n \in \omega$. Let $h = h_n$ and, as in Lemma 1, $S^m = h(mx) \cdot T_s$ for $m \in \mathbb{Z}$.

If we apply Lemma 1 to $h - 1$ and T_s, then by the definition of h and condition (5) (i.e. $\|T_s\|_{PM} < 2$) we obtain

$$\varlimsup_{|m| \to \infty} \|S^m - T_s\|_{PM} < \frac{1}{n+1}.$$

Lemma 1 also gives that $S^m \to T_s$ weakly and $\|S^m\|_{PM} \to \|T_s\|_{PM}$. Thus we can find a positive integer $M > m_s$ such that for every $m \geq M$,

- $|\widehat{S}^m(k) - \widehat{T}_s(k)| < 2^{-|s|-1}$ if $|k| \leq N_s$,
- $\|S^m\|_{PM} < 2$,
- $\|S^m - T_s\|_{PM} < 1/(n+1)$.

Then we almost get what we want, except that perhaps there will be no $m \geq M$ such that $\delta\big(E_s, \operatorname{supp}(S^m)\big) < 2^{-|s|}$. To overcome this difficulty, we introduce another definition: a set $K \in \mathcal{K}(\mathbb{T})$ is said to be a *Kronecker set* if the exponentials e^{int} are uniformly dense in $S(K) = \{f \in C(K) : |f(t)| = 1$ for all $t \in K\}$. We shall use two results about Kronecker sets. The first one is almost obvious: if K is a Kronecker set, then for any non-empty open set

$V \subseteq \mathbb{T}$ and any integer L one can find $l \geq L$ such that $lK \subseteq V$. The second result is essentially due to R. Kaufman (see [5], or [7], pp. 337–338): for any perfect set $F \subseteq \mathbb{T}$, the perfect Kronecker sets contained in F are dense in $\mathcal{K}(F)$.

After this detour we complete the inductive step as follows. Since E_s is perfect (by (2)), we choose a Kronecker set $K \subseteq E_s$ with $\delta(K, E_s) < 2^{-|s|}$. Then we pick $m \geq M$ such that $mK \cap \overline{U}_n = \emptyset$, and let $T_{s\frown n} = S^m$, $E_{s\frown n} = K \cup \operatorname{supp}(T_{s\frown n})$, $m_{s\frown n} = m$. Finally, we take $N_{s\frown n} > N_s$ large enough to ensure (4). Then conditions (1), …, (7) are clearly satisfied and this concludes the inductive step.

Now if $\alpha \in \omega^\omega$ it follows from (1) and (5) that the sequence $(T_{\alpha\lceil N})_{N\geq 1}$ converges w^* to a pseudomeasure T_α. By (5), $\widehat{T}_\alpha(0) \geq \widehat{T}(0) - 1/2 = 1/2$, hence $T_\alpha \neq 0$. If we set $E_\alpha = \bigcap_{N\geq 1} E_{\alpha\lceil N}$ then by (1) and (2), $\operatorname{supp}(T_\alpha) \subseteq E_\alpha \subseteq E$. Moreover, condition (3) implies that the map $\alpha \mapsto E_\alpha$ is continuous.

We claim that if $\alpha(p) \to \infty$ as $p \to \infty$ then $T_\alpha \in PF$, hence E_α is an M set. Indeed, if k is any integer with $|k| > N_\emptyset$ then by (1) there is a unique $(n, s) \in \omega \times \omega^{<\omega}$ such that $s\frown n \preceq \alpha$ and $N_s < |k| \leq N_{s\frown n}$. Now by conditions (4), (5), (6) we get

$$|\widehat{T}_\alpha(k)| \leq |\widehat{T}_\alpha(k) - \widehat{T}_{s\frown n}(k)| + |\widehat{T}_{s\frown n}(k) - \widehat{T}_s(k)| + |\widehat{T}_s(k)| < 3\cdot 2^{-|s|} + \frac{1}{n+1}$$

and the claim follows.

On the other hand, if $\alpha(p) \not\to \infty$ as $p \to \infty$ then conditions (1) and (7) readily imply that E_α is an H set.

Thus we have proved the first part of the theorem.

Now if we assume that E is an M_0 set rather than an M set then the preceding construction begins with a positive measure in PF and since the functions h_n are non-negative we get a positive measure μ_α in the end. This completes the whole proof. ∎

To conclude this note we point out very quickly some consequences of the above result (or of its proof). For all the notions involved below, we refer to [7] (and [10] for the definition of U_2').

(1) Given $T \in PF$ and $\varepsilon > 0$ there exists a pseudomeasure S whose support is an H set contained in $\operatorname{supp}(T)$ such that $\|T - S\|_{PM} < \varepsilon$; if T is a probability measure, then S can be chosen to be a probability measure as well (see [3], or [7], pp. 217, 239, for comparison).

(2) If E is an M set then the class U' and all the classes $H^{(n)}$, $n \geq 1$, are true $\mathbf{\Sigma}_3^0$ in $\mathcal{K}(E)$. If E is an M_0 set the same conclusion holds for the U_0' and U_2' sets contained in E.

(3) (Debs–Saint Raymond [3], Kechris–Louveau [7], p. 242, Kaufman [6]) Let H_σ be the sigma-ideal generated by the H sets. Then if E is an M set there is no $\mathbf{\Sigma}^1_1$ set $B \subseteq \mathcal{K}(E)$ such that $H_\sigma \cap \mathcal{K}(E) \subseteq B \subseteq U$; if E is an M_0 set there is no $\mathbf{\Sigma}^1_1$ set such that $H_\sigma \cap \mathcal{K}(E) \subseteq B \subseteq U_0$.

The proof of (3) is as follows. Let $\mathbf{2}^\omega$ be the space of all infinite sequences of 0's and 1's (with the product topology) and $D = \{\alpha \in \mathbf{2}^\omega : \exists n \forall p > n\ \alpha(p) = 0\}$. Then D is $\mathbf{\Sigma}^0_2$ in $\mathbf{2}^\omega$, hence by the result just proved there is a continuous map $f : \mathbf{2}^\omega \to \mathcal{K}(E)$ such that $f(\alpha) \in H$ if $\alpha \in D$ and $f(\alpha)$ is an M (or M_0) set if $\alpha \notin D$. One can define a continuous map $F : \mathcal{K}(\mathbf{2}^\omega) \to \mathcal{K}(E)$ by setting $F(K) = \bigcup\{f(\alpha) : \alpha \in K\}$. Then $F(K)$ is an H_σ set if $K \subseteq D$ and an M (or M_0) set if $K \not\subseteq D$. This completes the proof since $\mathcal{K}(D)$ is not a $\mathbf{\Sigma}^1_1$ set (for a proof of this last result see e.g. [7], p. 119).

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