How to recognize a true $\Sigma^0_3$ set

by

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Abstract. Let $X$ be a Polish space, and let $(A_p)_{p \in \omega}$ be a sequence of $G_\delta$ hereditary subsets of $K(X)$ (the space of compact subsets of $X$). We give a general criterion which allows one to decide whether $\bigcup_{p \in \omega} A_p$ is a true $\Sigma^0_3$ subset of $K(X)$. We apply this criterion to show that several natural families of thin sets from harmonic analysis are true $\Sigma^0_3$.

1. Introduction. In this paper, we are interested in a particular instance of the following problem: let $X$ be a separable metric space, and denote by $\Sigma^0_\xi(X)$ (resp. $\Pi^0_\xi(X)$) the additive (resp. multiplicative) Borel classes of $X$. The problem is to find some simple criterion allowing one to decide whether a given $\Sigma^0_\xi$ set $A \subseteq X$ is a “true” $\Sigma^0_\xi$, that is, a $\Sigma^0_\xi$ set which is not $\Pi^0_\xi$.

As a matter of fact, we will limit ourselves to the third level of the Borel hierarchy ($G_\delta$ and $F_{\sigma\delta}$ sets). Moreover, since the examples we have in mind are ideals of compact sets coming from harmonic analysis, we will concentrate on proving criteria of “true-$\Sigma^0_3$-ness” for ideals of compact subsets of some Polish space $X$. We denote by $K(X)$ the space of all compact subsets of $X$, equipped with its natural (Polish) topology, generated by the sets $\{K \in K(X) : K \cap V \neq \emptyset\}$ and $\{K \in K(X) : K \subseteq V\}$, where $V$ is an open subset of $X$. For any subset $M$ of $X$, we let $K(M) = \{K \in K(X) : K \subseteq M\}$.

In this particular setting, it turns out that the simplest nontriviality condition is enough to ensure true-$\Sigma^0_3$-ness. To be precise, let $(A_p)_{p \in \omega}$ be a sequence of dense $G_\delta$ hereditary subsets of $K(X)$, and let $A = \bigcup_{p \in \omega} A_p$. Assume that $A$ is an ideal of $K(X)$, and that the union is “nontrivial” in the following sense: for each nonempty open set $V \subseteq X$ and for each $p \in \omega$, $A_p \cap K(V)$ is a proper subset of $A \cap K(V)$. Then one can conclude that $A$ is a true $\Sigma^0_3$ set.

We prove this in the first part of the paper together with some related results. We apply these results in the second part to show that quite a lot of

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natural families of thin sets from harmonic analysis happen to be true $\Sigma^0_3$ (a rather curious descriptive phenomenon). In particular, we show that if $G$ is a second-countable nondiscrete locally compact abelian group, then the family $H$ of compact Helson subsets of $G$ is a true $\Sigma^0_3$. The same result holds within any $M_0$ set, and $H$ is also a true $\Sigma^0_3$ inside the countable sets. In the case of the circle group, we already proved these results in [M]. However, the proofs given there were somewhat obscured by an immoderate use of constructions which are very classical in harmonic analysis, but still rather technical. In the present paper, we actually use very little harmonic analysis.

2. General results. In this section, $X$ is a Polish space, $K(X)$ is the space of compact subsets of $X$, and $K_\omega(X)$ is the family of countable compact subsets of $X$.

**Definition 1.** Let $\mathcal{A}$ be a subset of $K(X)$.

(a) $\mathcal{A}$ is said to be **hereditary** if it is downward closed under inclusion.

(b) $\mathcal{A}$ is an **ideal** of $K(X)$ if it is hereditary and stable under finite unions.

(c) If $\mathcal{A}$ is hereditary, we say that $\mathcal{A}$ is a **big** subset of $K(X)$ if it contains a dense $G_\delta$ hereditary subset of $K(X)$.

It is quite possible that any comeager hereditary subset of $K(X)$ is big, but we are unable to prove it.

**Definition 2.** Let $\mathcal{M}_1$ and $\mathcal{M}_2$ be two subsets of $K(X)$. We say that $\mathcal{M}_1$ is nowhere contained in $\mathcal{M}_2$ if for each nonempty open set $V \subseteq X$, $\mathcal{M}_1 \cap K(V)$ is not contained in $\mathcal{M}_2$.

We can now state the main results of this section.

**Theorem A.** Let $(A_p)_{p \in \omega}$ be a sequence of nonempty hereditary subsets of $K(X)$, and let $\mathcal{A} = \bigcup_{p \in \omega} A_p$. Assume that $\mathcal{A}$ is a big ideal of $K(X)$.

(a) If $\mathcal{A}$ is nowhere contained in any $A_p$, then $\mathcal{A}$ is not $\Pi^0_3$ in $K(X)$.

(b) If the perfect sets in $\mathcal{A}$ are nowhere contained in any $A_p$, then the family of perfect sets in $\mathcal{A}$ is not $\Pi^0_3$ in $K(X)$.

(c) If the finite sets in $\mathcal{A}$ are nowhere contained in any $A_p$, then $\mathcal{A} \cap K_\omega(X)$ is not relatively $\Pi^0_3$ in $K_\omega(X)$.

Theorem A follows immediately from a more precise and less readable result, Theorem B below, which we will state after a few definitions.

In the sequel, we will use the notations $B$ and $B_1$ for the following families of compact sets: either $B = B_1 = K(X)$, or $B = B_1 = \{K \subseteq X : K \text{ of the form } \{x\} \cup \{x_n : n \in \omega\}, \text{ where } (x_n) \subseteq X \text{ and } x_n \to x\}$. Notice that in each case, $\emptyset \in B$ and $B$ is a $G_\delta$ subset of $K(X)$.
If $\mathcal{M}$ is a subset of $\mathcal{K}(X)$, we denote by $\mathcal{M}^f$ the family of compact subsets of $X$ which are finite unions of elements of $\mathcal{M}$.

We denote by $2^\omega$ the Cantor space of all infinite 0-1 sequences, endowed with its usual (compact, metrizable) topology.

If $\alpha \in 2^\omega$ and $p \in \omega$, we define $\alpha_p \in 2^\omega$ by $\alpha_p(q) = \alpha((p, q))$, where $(p, q) \mapsto (p, q)$ is any fixed bijection from $\omega^2$ onto $\omega$.

Finally, let $Q = \{ \alpha \in 2^\omega : \exists k \forall n \geq k \alpha(n) = 0 \}$, and $W = \{ \alpha \in 2^\omega : \exists p \alpha_p \notin Q \}$. It is well known that $W$ is a true $\Sigma^0_3$ subset of $2^\omega$ (see [Ke2]).

Our slightly more precise version of Theorem A now reads as follows. To deduce Theorem A from it, one just has to take $B = \mathcal{K}(X)$ in case (a), $B = \text{the perfect subsets of } X$ in case (b), and $B = \{ \emptyset \} \cup \{ \{ x \} : x \in X \}$ in case (c).

**Theorem B.** Let $(\mathcal{A}_p)_{p \in \omega}$ be a sequence of (nonempty) hereditary subsets of $\mathcal{K}(X)$, and let $\mathcal{A}$ be any big subset of $\mathcal{K}(X)$. Assume that $(A \cap B)^f$ is nowhere contained in any $\mathcal{A}_p$. Then there exists a continuous map $\alpha \mapsto E(\alpha)$ from $2^\omega$ into $\mathcal{K}(X)$ such that:

- For each $\alpha \in 2^\omega$, $E(\alpha) \in \mathcal{B}_1$.
- If $\alpha \in W$, then $E(\alpha) \in A^f$.
- If $\alpha \notin W$, then $E(\alpha) \notin \bigcup_{p \in \omega} A_p$.

In particular, there is no $\Pi^0_3$ set $\mathcal{M} \subseteq \mathcal{K}(X)$ such that $A^f \cap \mathcal{B}_1 \subseteq \mathcal{M} \cap \mathcal{B}_1 \subseteq \bigcup_{p \in \omega} A_p$.

As an immediate consequence, we get a kind of “Baire category theorem” for big $\Pi^0_3$ ideals:

**Corollary.** Let $\mathcal{A} \subseteq \mathcal{K}(X)$ be a big $\Pi^0_3$ ideal. If $(\mathcal{A}_p)_{p \in \omega}$ is a sequence of hereditary subsets of $\mathcal{K}(X)$ such that $\mathcal{A} \subseteq \bigcup_{p \in \omega} A_p$, then there exist an integer $p$ and a nonempty open set $V \subseteq X$ such that $\mathcal{A} \cap \mathcal{K}(V) \subseteq \mathcal{A}_p$.

Some simple remarks may help to justify the hypotheses of Theorem B. Assume that $X$ is perfect.

1) If $\mathcal{A}$ is not big, the result is not true. For example, let $\mathcal{A}$ be the ideal of finite sets and $A_p = \{ K \in \mathcal{K}(X) : \text{card}(K) \leq p \}$ ($p \in \omega$); then $\mathcal{A}$ is nowhere contained in any $\mathcal{A}_p$, but it is obviously an $F_\sigma$ set.

2) One cannot drop the hypothesis that $\mathcal{A}$ is an ideal. For example, let $D = \{ x_n : n \in \omega \}$ be a countable dense subset of $X$, and let $G = X \setminus D$. Define $\mathcal{A} = \mathcal{K}(G) \cup \bigcup_{n \in \omega} \mathcal{K}\{x_n\}$ and $A_p = \mathcal{K}(G) \cup \bigcup_{n \leq p} \mathcal{K}\{x_n\}$. Then $\mathcal{A}$ is big, the $A_p$’s are hereditary and $\mathcal{A}$ is nowhere contained in any $\mathcal{A}_p$; yet $\mathcal{A}$ is $\Pi^0_3$ (it is the union of a $G_\delta$ and a countable set).

3) Finally, one cannot remove the hereditary assumption on the $A_p$’s. For example, let $\{ K_p : p \in \omega \}$ be any countable dense subset of $\mathcal{K}(X)$ (the $K_p$’s being pairwise distinct), and let $A_p = \mathcal{K}(X) \setminus \{ K_n : n > p \}$.
In the proof of Theorem B, we will make use of the following two lemmas. The first one is easy; the second one is proved by applying the Baire category theorem in $2^\omega$ (identified with $P(\omega)$).

**LEMMA 1.** The map $\phi : K(X) \times K(X) \to K(X)$ defined by $\phi(K, L) = K \cup L$ is (continuous and) open.

**LEMMA 2 ([Kc1]).** Let $G \subseteq K(X)$ be $G_\delta$, $F \in K(X)$, and let $(F_m)_{m \in \omega}$ be a sequence converging to $F$ in $K(X)$. Assume that for each finite set $I \subseteq \omega$, the set $F \cup \bigcup_{m \in I} F_m$ belongs to $G$. Then the compact set $F \cup \bigcup_{m \in \omega} F_m$ is the union of two elements of $G$.

Next, we introduce some notation.

Recall that we denote by $\langle p, q \rangle$ the image of a pair $(p, q)$ under some fixed bijection from $\omega^2$ onto $\omega$. The image of an integer $n$ under the inverse map will be denoted by $((n)\!_0, (n)\!_1)$.

Let $2^{<\omega}$ be the set of all finite 0-1 sequences (including the empty sequence). We write $|s|$ for the length of a sequence $s \in 2^{<\omega}$. If $s \in 2^{<\omega}$ (and $s \neq \emptyset$), we denote by $s'$ the immediate predecessor of $s$ in the extension ordering.

If $\alpha \in 2^\omega$ and $n \in \omega$, we denote by $\alpha_{\!_n}$ the length-$n$ initial segment of $\alpha$; thus, if $n \geq 1$, then $\alpha_{\!_n} = (\alpha(0), \ldots, \alpha(n-1))$.

Next, we define inductively a sequence $(\theta_p)_{p \in \omega}$ of functions from $2^{<\omega}$ into $\omega \cup \{+\infty\}$ in the following way:

1. $(0)$ $\theta_p(\emptyset) = +\infty$ for all $p \in \omega$.
2. (i) If $|s| = n + 1$ and $(n)\!_0 > p$, then $\theta_p(s) = \theta_p(s')$.
3. (ii) If $|s| = n + 1$, $(n)\!_0 \leq p$ and $s(n) = 0$, then

$$\theta_p(s) = \begin{cases} \theta_p(s') & \text{if } \theta_p(s') < +\infty, \\ n & \text{if } \theta_p(s') = +\infty. \end{cases}$$

4. (iii) If $|s| = n + 1$, $(n)\!_0 \leq p$ and $s(n) = 1$, then $\theta_p(s) = +\infty$.

In other words, if we define $A_p(s) = \{m < |s| : (m)\!_0 \leq p\}$, then $\theta_p(s) = \min\{m \in A_p(s) : \forall m' \in A_p(s), m' \geq m, \ s(m') = 0\}$ (with the convention that $\min(\emptyset) = +\infty$). Thus, if we denote by $s_p : A_p(s) \to \{0, 1\}$ the restriction of $s$ to $A_p(s)$, then $\theta_p(s)$ indicates the beginning of the longest “cofinal” 0-segment in $s_p$.

Finally, we define another sequence of functions from $2^{<\omega}$ into $\omega \cup \{+\infty\}$ by

$$m_p(s) = \max(\langle p, 0 \rangle, \theta_p(s)).$$

The following facts will be useful later.

**CLAIM 1.** Let $p \in \omega$ and $\alpha \in 2^\omega$. Assume that $\alpha_l \in Q$ for all $l \leq p$.

(a) The sequence $(m_p(\alpha_{\!_n}))_{n \in \omega}$ is eventually finite-valued and constant.
(b) If we let \( M_p = \lim_{n \to \infty} m_p(\alpha|n)_n \), then \( \alpha(M_p) = 0 \) and
\[
\forall n \geq M_p \quad m_p(\alpha|n+1) = M_p \quad \text{and} \quad ((n)_0 > p \text{ or } \alpha(n) = 0).
\]

Proof. Since for each \( s \in 2^{\aleph_0} \), \( m_p(S) \geq \theta_p(s) \) and the first coordinate of \( m_p(s) \) is \( \leq p \), we may content ourselves with proving (a) and (b) with \( \theta_p \) in place of \( m_p \).

By definition of \( Q \), there is a smallest integer \( N \) with the following properties:
\[
(N)_0 \leq p, \quad \forall n \geq N \quad (n)_0 \leq p \Rightarrow \alpha(n) = 0.
\]
The claim will be proved if we can show that \( \theta_p(\alpha|n+1) = N \) for all \( n \geq N \).

(i) First, \( \theta_p(\alpha|n+1) = \theta_p(\alpha|N+1) \) for every integer \( n \geq N \). This follows by induction from the definition of the function \( \theta_p \): by the choice of \( N \), we have \( \alpha(N) = 0 \) and \( (N)_0 \leq p \), hence \( \theta_p(\alpha|N+1) < +\infty \); and if \( n > N \), then either \( (n)_0 > p \) or \( \alpha(n) = 0 \), so \( \theta_p(\alpha|n+1) = \theta_p(\alpha|n) \) for all \( n > N \).

(ii) By (i), it is now enough to check that \( \theta_p(\alpha|N+1) = N \). Let
\[
N_1 = \begin{cases} 
\text{the greatest } n < N \text{ such that } (n)_0 \leq p & \text{if there is any}, \\
-1 & \text{if there is no such } n.
\end{cases}
\]
By the choice of \( N \), we have \( \alpha(N_1) = 1 \) if \( N_1 \geq 0 \); thus, in both cases \( \theta_p(\alpha|N_1+1) = +\infty \). Now, by the choice of \( N_1 \), we also have \( (n)_0 > p \) for all \( n \) such \( N_1 < n < N \). This implies that \( \theta_p(\alpha|N) = \theta_p(\alpha|N_1+1) = +\infty \). Thus \( \theta_p(\alpha|N+1) = N \).

Proof of Theorem B. The result is trivial if \( X \) is not perfect (one just has to let \( E(\alpha) \equiv \{x_0\} \), where \( x_0 \) is an isolated point of \( X \)). Hence, from now on, \( X \) will be perfect.

For simplicity, we assume first that \( X \) is compact. We fix some metric compatible with the topology of \( X \) and we choose a complete metric \( \delta \) for \( B \), which is possible since \( B \) is a \( G_δ \) subset of \( K(X) \).

Since each \( A_p \) is hereditary, the hypotheses of Theorem B remain unchanged if we replace \( A_p \) by \( \bigcup_{i \leq p} A_i \). Thus, we may assume that the sequence \( (A_p)_{p \in \omega} \) is nondecreasing.

Finally, let \( G \) be a dense \( G_δ \) hereditary subset of \( K(X) \) contained in \( A \). We can write \( G = \bigcap_{n \in \omega} U^n \), where \( (U^n)_{n \in \omega} \) is a nonincreasing sequence of hereditary open subsets of \( K(X) \) (if \( (V^n)_{n \in \omega} \) is any nonincreasing sequence of open sets such that \( G = \bigcap_{n \in \omega} V^n \), let \( U^n = \{K \in K(X) : \forall L \subseteq K \ L \in V^n\} \).

Claim 2. For each positive integer \( N \), the set \( \{(x, K_1, \ldots, K_N) \in X \times B^N : \exists \{x\} \cup \bigcup_{i=1}^N K_i \in G\} \) is dense in \( X \times B^N \).

Proof. By Lemma 1, the set \( \{(K_0, K_1, \ldots, K_N) \in K(X)^{N+1} : \bigcup_{i=0}^N K_i \in G\} \) is a dense \( G_δ \) subset of \( K(X)^{N+1} \); and since \( G \) is hereditary, this implies that the set \( \{(x, K_1, \ldots, K_N) \in X \times K(X)^N : \exists \{x\} \cup \bigcup_{i=1}^N K_i \in G\} \)
is a dense $G_δ$ subset of $X \times \mathcal{K}(X)^N$. Thus, the claim is true if $\mathcal{B} = \mathcal{K}(X)$. If $\mathcal{B}$ is the family of perfect sets, which is comeager in $\mathcal{K}(X)$ because $X$ is perfect, the claim follows from the Baire category theorem. Finally, if $\mathcal{B} = \{\emptyset\} \cup \{\{x\} : x \in X\}$, we use again the fact that $\mathcal{G}$ is hereditary.

Now, we shall construct inductively a sequence $(j_m)_{m<\omega}$ of positive integers and, for each $s \in 2^{<\omega}$, a compact set $E(s) \subseteq X$ and a nonempty open set $V(s) \subseteq X$.

For $s \neq \emptyset$, $E(s)$ will be written as $E(s) = \bigcup_{m<|s|} E^m(s)$, where each $E^m(s)$ is compact and of the form $E^m(s) = \bigcup_{j_m = 1}^{j_m} E_j^m(s) (E_j^m(s) \text{ compact})$.

We also construct $(s \in 2^{<\omega} \setminus \emptyset; 0 \leq m < |s|, 1 \leq j \leq j_m)$ nonempty open sets $V_j^m(s) \subseteq X$, and we let $V^m(s) = \bigcup_{j_m = 1}^{j_m} V_j^m(s)$.

The closure of any set $A$ involved in the construction will be denoted by $\overline{A}$.

The following requirements have to be fulfilled (to avoid typographic heaviness, we have omitted more often than not obvious information like “if $|s| \geq 1$”, “$m < |s|$” or “$j \leq j_m$”).

1. $E_j^m(s) \in A \cap \mathcal{B}$ and $E^m(s) \neq \emptyset$.
2. $E^m(s) \subseteq V^m(s) \subseteq \overline{V^m(s)} \subseteq V_j^m(s')$,
3. $\delta(E^m_j(s), E_j^m(s')) < 2^{-|s|}$, $\text{diam } V(s) < 2^{-|s|}$.
4. If $|s| = n + 1$, then $E_j^m(s) = E_j^m_p(s) (s')$ for each $p < (n)_0$ (notice that $m_p(s) = m_p(s')$ here, because $p < (n)_0$).
5. If $|s| = n + 1$ and $s(n) = 0$, then $E_j^m(s) = E_j^m(s')$ for all $m < |s'|$.
6. If $|s| = n + 1$ and $s(n) = 0$, then $E^m(s) \notin A_n$.
7. If $|s| = n + 1$ and $s(n) = 1$, then

\[ \overline{V(s)} \cup \bigcup \{\overline{V^m(s)} : m < |s|, \forall p < (n)_0 m \neq m_p(s)\} \in \mathcal{U}^{|s|}. \]

To begin the construction, we choose a nonempty open set $V(\emptyset) \subseteq X$ of diameter $< 1$, and we let $E(\emptyset) = \emptyset$.

Assume that the sets $E(t)$ and $V(t)$ have been constructed for all sequences $t$ of length $\leq n$. We have to define the positive integer $j_n$ and the sets $V(s), E_j^m(s), V_j^m(s) (0 \leq m \leq n, 1 \leq j \leq j_m)$ for every sequence $s$ of length $n + 1$.

(a) First, for each sequence $t$ of length $n$, we choose two nonempty open sets $W_1(t), W_2(t) \subseteq V(t)$ with $W_1(t) \cap W_2(t) = \emptyset$. This is possible because $X$ is perfect.

(b) Next, we define $j_n$ and the sets $E_j^m(s)$ and $V_j^m(s)$ for all sequences $s$ of length $n + 1$ such that $s(n) = 0$. 
(i) By (5), there is nothing to do for an integer \( m < n \).

(ii) Let \( S_0 = \{ s \in 2^\omega : |s| = n + 1, s(n) = 0 \} \). Since \((A \cap B)^f\) is nowhere contained in \( A_n\), we can find a positive integer \( j_n \) and, for all \( s \in S_0\), compact sets \( E^n_j(s), \ldots, E^n_j(m) \subseteq W_1(s') \) such that each \( E^n_j(s) \) belongs to \( A \cap B\), but \( \bigcup_{j=1}^m E^n_j(s) \notin A_n\). Notice that we can choose the same integer \( j_n \) for all sequences \( s \) because, since \( \emptyset \in A \cap B\), we may always add the empty set (as many times as necessary) to the sets \( E^n_j(s) \).

At this point, (4), (5) and (6) are satisfied, as well as (1) for \( s \in S_0\) \((E^n(s) \neq \emptyset)\) is nonempty because \( \emptyset \in A_n\). Then we can choose for each \( s \in S_0\) nonempty open sets \( V(s) \subseteq W_2(s') \) and \( V^n_j(m) \supseteq E^n_j(s) \) in order to get (2) and (3).

(c) Now, let \( s \) be a sequence of length \( n + 1 \) such that \( s(n) = 1 \).

(i) By (4), we must let \( E^{m_p(s)}_n(s) = E^{m_p(s)}_n(s') \) for all the integers \( p < (n)_0 \) such that \( m_p(s) = m_p(s') < |s'| \).

(ii) Let \( I(s) = \{ m < |s| : \forall p < (n)_0 \ m \neq m_p(s) \} \) and \( N = \sum_{m \in I(s)} j_m \).

By Claim 2, the set \( \{ (x, K_1, \ldots, K_N) \in X \times B^N : \{ x \} \cup \bigcup_{i=1}^N K_i \in G \} \) is dense in \( X \times B^N \). Therefore, we can find a point \( x(s) \in X \) and compact sets \( E^n_m(s) \in B \ (m \in I(s), 1 \leq j \leq j_m) \) such that

\[
\begin{align*}
\delta(E^n_m(s), E^n_j(s')) &< 2^{-n-1} \quad \text{and} \quad E^n_j(s) \subseteq V^n_j(s') \quad \text{if} \ m < n, \\
E^n_j(s) &\subseteq W_1(s') \quad \text{and} \quad E^n_j(s) \neq \emptyset, \\
x(s) &\in W_2(s'), \\
\{ x(s) \} \cup \bigcup \{ E^n_m(s) : m \in I(s), 1 \leq j \leq j_m \} &\in G.
\end{align*}
\]

We can also ensure that \( E^n_j(s) \neq \emptyset \) whenever \( E^n_j(s') \neq \emptyset \), because \( \emptyset \) is an isolated point in \( \mathcal{K}(X) \). Moreover, since \( G \) is hereditary (and contained in \( A\)), the last condition implies that each \( E^n_j(s) \) belongs to \( A\); hence (1) is true for \( m \in I(s) \) (of course, it was also true for \( m \notin I(s) \)).

(iii) It is now easy to choose open sets \( V(s) \ni x(s) \) and \( V^n_m(s) \supseteq E^n_j(s) \) in order to get (2), (3) and (7).

This concludes the inductive step.

It follows from (1) and (3) that if \( m \in \omega \) and \( j \leq j_m \) are fixed, then for any \( \alpha \in 2^\omega \), the sequence \( (E^n_j(\alpha_{[n]}))_{n>m} \) converges to a compact set \( E^n_j(\alpha) \in B \).

For each \( \alpha \in 2^\omega \) and each \( m \in \omega \), let \( E^n(\alpha) = \bigcup_{j \leq j_m} E^n_j(\alpha) \). By (1), all the \( E^n(\alpha) \)’s are nonempty (because \( \emptyset \) is isolated in \( \mathcal{K}(X) \)). Moreover, conditions (2) and (3) imply that the sequence \( (E^n(\alpha))_{m \in \omega} \) converges in \( \mathcal{K}(X) \) to the singleton \( \{ x_\alpha \} = \bigcap_{n \in \omega} V(\alpha_{[n]}) \).

Thus, the set \( E(\alpha) = \{ x_\alpha \} \cup \bigcup_{m \in \omega} E^n(\alpha) \) is compact, and in fact it belongs to \( B_1 \). Furthermore, it follows from (2) and (3) that the map \( \alpha \mapsto E(\alpha) \) is continuous.
CLAIM 3. Let $\alpha \in 2^\omega$ and $p \in \omega$. Assume that $\alpha_l \in \mathbb{Q}$ for all $l \leq p$, and let $M_p = \lim_{n \to \infty} m_p(\alpha_{|n})$. Then $E^{M_p}(\alpha) = E^{M_p}(\alpha_{|M_p+1})$.

Proof. By Claim 1, the integer $M_p$ is well defined. Moreover, we know that for each $n \geq M_p$, $m_p(\alpha_{|n+1}) = M_p$, and either $(n)_0 > p$ or $\alpha(n) = 0$.

This implies that $E^{M_p}(\alpha_{|n+1}) = E^{M_p}(\alpha_{|n})$ for each $n > M_p$. Indeed, we can use (4) if $(n)_0 > p$ and (5) if $\alpha(n) = 0$. Thus $E^{M_p}(\alpha) = E^{M_p}(\alpha_{|M_p+1})$.

Let us now fix $\alpha \in 2^\omega$.

CASE 1. Assume $\alpha \notin \mathbb{W}$. By Claims 1 and 3, all sequences $(m_p(\alpha_{|n}))_{n \in \omega}$ are eventually constant, and if we let $M_p = \lim_{n \to \infty} m_p(\alpha_{|n})$ for each $p \in \omega$, then $\alpha(M_p) = 0$ and $E^{M_p}(\alpha) = E^{M_p}(\alpha_{|M_p+1})$. Hence, by (6), $E^{M_p}(\alpha) \notin \mathcal{A}_{M_p}$.

Since each $\mathcal{A}_{M_p}$ is hereditary, this implies that $E(\alpha) \notin \bigcup_{p \in \omega} \mathcal{A}_{M_p}$. Now $M_p \geq (p, 0)$ for all $p$, hence $\lim_{p \to \infty} M_p = +\infty$ (this was the reason for using the functions $m_p$ rather than the $\theta_p$'s), and consequently $E(\alpha) \notin \bigcup_{p \in \omega} \mathcal{A}_p$.

CASE 2. Assume $\alpha \in \mathbb{W}$. We have to show that $E(\alpha) \in \mathcal{A}^I$.

Let $p_0$ be the smallest integer such that $\alpha_{p_0} \notin \mathbb{Q}$. For each $p < p_0$, let as usual $M_p = \lim_{n \to \infty} m_p(\alpha_{|n})$ (which is well defined by Claim 1) and let

$$ E_1(\alpha) = \bigcup_{p < p_0} E^{M_p}(\alpha), $$

$$ E_2(\alpha) = \{x_\alpha\} \cup \bigcup_{m \neq M_p} \{E^m(\alpha) : m \neq M_p, p < p_0\}. $$

Since $E(\alpha) = E_1(\alpha) \cup E_2(\alpha)$, it is enough to check that $E_1(\alpha) \in \mathcal{A}^I$ and $E_2(\alpha) \in \mathcal{G}^I$.

(i) By the choice of $p_0$, $\alpha_{p_0} \in \mathbb{Q}$ for each $p < p_0$. Hence, by Claim 3 and condition (1), $E_1(\alpha) = \bigcup_{p < p_0} E^{M_p}(\alpha_{|M_p+1}) \in \mathcal{A}^I$.

(ii) Let $I(\alpha) = \{m \in \omega : \forall p < p_0 \ m \neq M_p\}$. It follows from (7) that $V(\alpha_{|n+1}) \cup \bigcup \{V^m(\alpha_{|n+1}) : m \in I(\alpha), m < n + 1\} \in \mathcal{U}^n$ for each integer $n > \max\{M_p : p < p_0\}$ such that $(n)_0 = p_0$ and $\alpha(n) = 1$. Since there are infinitely many such $n$'s (because $\alpha_{p_0} \notin \mathbb{Q}$) and each open set $\mathcal{U}^n$ is hereditary, this implies (by (2)) that for any finite set $I \subseteq I(\alpha)$,

$$ \{x_\alpha\} \cup \bigcup_{m \in I} E^m(\alpha) \in \mathcal{G}. $$

Thus, from Lemma 2 we get $E_2(\alpha) \in \mathcal{G}^I$.

The proof of Theorem B is now complete when $X$ is assumed to be compact.

When $X$ is not compact, we may always view it as a dense $G_\delta$ subset of some compact metric space $X$. Write $X = \bigcap_{n \in \omega} W_n$, where the $W_n$'s are open subsets of $X$. Then we can perform our construction in $\hat{X}$ and moreover, we can easily ensure at each step that the open sets $V^m_{|I}(s)$ and $V(s)$ are all contained in $W_{|s}$. Hence, in the end, $E(\alpha) \subseteq X$ for each $\alpha \in 2^\omega$. This concludes the whole proof.
3. Applications. In this section, $G$ is a second-countable nondiscrete locally compact abelian group, with dual group $\Gamma$. We denote by $C_0(G)$, $M(G)$, $A(G)$ and $PM(G)$ respectively: the space of continuous complex-valued functions on $G$ vanishing at infinity, the space of finite (complex) measures on $G$, the Fourier transform of the convolution algebra $L^1(\Gamma)$, and the space of pseudomeasures on $G$ (the dual space of $A(G)$).

Let $q(G) = \sup\{n \in \omega :$ every neighbourhood of $0_G$ contains elements of order $\geq n\}$. We define $G_q = \{x \in G : x$ is of order $\leq q(G)\}$, and $T_q = \{z \in \mathbb{T} : z^q(G) = 1\}$ (with the convention that $z^\infty = 1$ for any $z \in \mathbb{T}$). Notice that $G_q$ is a clopen subgroup of $G$, by definition of $q(G)$.

**Definition 1.** Let $K$ be a compact subset of $G$.

1) $K$ is said to be a *Helson set* if every continuous function on $K$ can be extended to a function in $A(G)$.

2) $K$ is said to be without true pseudomeasure (for short, WTP) if every pseudomeasure supported by $K$ is actually a measure.

3) $K$ is said to be *independent* if there is no nontrivial relation of the form $\sum_{i=1}^n m_i x_i = 0$, where $m_1, \ldots, m_n \in \mathbb{Z}$ and $x_1, \ldots, x_n \in K$ (that is, $\sum m_i x_i = 0 \Rightarrow \forall i$ $m_i, x_i = 0$; when $G = \mathbb{T}$, this is not exactly the usual definition).

4) $K$ is said to be a $K_q$ set if it is totally disconnected, all its elements have order $q(G)$, and the restrictions of characters of $G$ are uniformly dense in $C(K, T_q)$, the space of continuous functions from $K$ into $T_q$ (when $q(G) = +\infty$, $K_q$ sets are usually called Kronecker sets).

5) $K$ is said to be a $U_0'$ set if there is some constant $c$ such that

$$\forall \mu \in M^+(K) \quad \|\mu\|_{PM} \leq c \lim_{\gamma \to \infty} |\hat{\mu}(\gamma)|.$$  

It is clear that $\mathcal{H}$ (the family of Helson subsets of $G$), WTP, $K_q$ and $U_0'$ are hereditary subsets of $\mathcal{K}(G)$ (for $K_q$ sets, this is because they are assumed to be totally disconnected).

There are natural constants associated with a given Helson or $U_0'$ set. Namely, for each $K \in \mathcal{K}(G)$, define

$$\eta_0(K) = \inf\{\lim_{\gamma \to \infty} |\hat{\mu}(\gamma)|/\|\mu\|_{PM} : \mu \in M^+(K), \mu \neq 0\},$$

$$\alpha(K) = \inf\{\|\mu\|_{PM}/\|\mu\|_M : \mu \in M(K), \mu \neq 0\}.$$  

Then (by definition) $K \in U_0' \Leftrightarrow \eta_0(K) > 0$ and (by standard functional-analytic arguments) $K \in \mathcal{H} \Leftrightarrow \alpha(K) > 0$. The number $\alpha(K)$ is the *Helson constant* of $K$. There is also a “WTP constant”, whose definition should be reasonably clear.

**Lemma 1.** $K_q$ is a $G_S$ subset of $\mathcal{K}(G)$, and $\mathcal{H}$, WTP and $U_0'$ are $\Sigma^0_3$. 
The proofs are standard complexity calculations. For Helson sets, for example, the main point is that for each positive \( \varepsilon \), the set \( \mathcal{H}_\varepsilon = \{ K \in \mathcal{K}(X) : \alpha(K) \geq \varepsilon \} \) is \( G_5 \).

**Definition 2.** Let \( E \) be a closed subset of \( G \).

(a) \( E \) is said to be a \( U_0 \) set (or a set of extended uniqueness) if \( \mu(E) = 0 \) for every positive measure on \( G \) whose Fourier transform vanishes at infinity.

(b) \( E \) is an \( M_0 \) set if \( E \not\in U_0 \), and an \( M_0^p \) set if \( E \cap V \in M_0 \) for each open set \( V \subseteq G \) such that \( E \cap V \neq \emptyset \).

**Definition 3.** Let \( p \) be a positive integer. By a net of length \( p \), we mean any set of cardinality \( 2^p \) of the form \( \{ a + \sum_{i=1}^{p} \varepsilon_i l_i : \varepsilon_i = 0, 1 \} \), where \( a, l_1, \ldots, l_p \) are fixed elements of \( G \).

We shall use the following results. Almost all the proofs can be found in [GMG], and a lot of them in [KL] (see also [LP]).

1) \( WTP \subseteq \mathcal{H} \subseteq U_0^0 \subseteq U_0 \).

2) \( \mathcal{H}, WTP \) and \( U_0^0 \) are translation-invariant ideals of \( \mathcal{K}(G) \); \( U_0 \) is a translation-invariant \( \sigma \)-ideal of closed sets.

3) Finite sets are \( WTP \). A finite set is a \( K_q \) set if and only if it is independent and all its elements have order \( q(G) \).

4) \( K_q \) sets are Helson: in fact, \( \alpha_q = \inf \{ \alpha(K) : K \in K_q \} > 0 \).

5) If \( F \subseteq G \) is a \( p \)-net, then \( \alpha(F) \leq (\sqrt{2})^{-p} \). Hence, if a compact set \( K \) contains arbitrarily long nets, then \( K \) is not Helson.

Before applying the results of Section 2, we prove some general facts about \( K_q \) sets.

For each integer \( m \) such that \( 0 < m < q(G) \), we let \( N_m = \{ x \in G : mx = 0 \} \). The \( N_m \)'s are closed subgroups of \( G \) and, by definition of \( q(G) \), they are nowhere dense in \( G \).

We denote by \( I \) the \( \sigma \)-ideal generated by all translates of the \( N_m \)'s, that is, the family of all subsets of \( G \) which can be covered by countably many translates of the \( N_m \)'s.

Finally, we say that a set \( E \subseteq G \) is \( I \)-perfect if no nonempty relatively open subset of \( E \) belongs to \( I \). By the Baire category theorem, every open subset of \( G \) is \( I \)-perfect.

**Lemma 2.** Let \( F \subseteq G \) be a finite \( K_q \) set, and let \( A = \{ x \in G_q : x \in G_q \text{ and } F \cup \{ x \} \not\in K_q \} \). Then \( A \in I \).

**Proof.** We know that \( F \cup \{ x \} \) is a \( K_q \) set if and only if \( x \) has order \( q(G) \) and \( F \cup \{ x \} \) is independent. Moreover, if \( x \in G \) has order \( \leq q(G) \), it is easy to check that for each \( m \in \mathbb{Z} \), there is an integer \( m' \) such that \( |m'| < q(G) \) and \( mx = m'x \).
Now, let $G_p(F)$ be the subgroup of $G$ generated by $F$. From the two preceding remarks, we easily deduce that $A$ is contained in the set $A'$ defined by

$$x \in A' \iff \exists m \ (0 < |m| < q(G) \text{ and } mx \in G_p(F)).$$

If $0 < |m| < q(G)$ and $y \in G$, the set $E_{m,y} = \{x \in G : mx = y\}$ belongs to $I$. Since $G_p(F)$ is countable, it follows that $A' \in I$. This concludes the proof.

**Theorem 1.** Let $E \subseteq G$ be a closed $I$-perfect set contained in $G_q$. Then $K_q \cap \mathcal{K}(E)$ is dense in $\mathcal{K}(E)$. In fact, for any finite $K_q$ set $F \subseteq G$, the set $\mathcal{G}_F = \{K \in \mathcal{K}(E) : K \cup F \in K_q\}$ is a dense $G_δ$ hereditary subset of $\mathcal{K}(E)$.

**Proof.** Since $K_q$ is hereditary, the second statement implies the first. So let us fix a finite $K_q$ set $F \subseteq G$.

It is clear that $\mathcal{G}_F$ is $G_δ$ and hereditary.

Now, let $V_1, \ldots, V_k$ be nonempty open subsets of $E$. Since each $V_i$ is $I$-perfect, we can apply Lemma 2 $k$ times to get $x_1, \ldots, x_k \in G$ such that $x_i \in V_i$ for all $i$ and $F \cup \{x_1, \ldots, x_k\} \in K_q$. This shows that $\mathcal{G}_F$ is dense in $\mathcal{K}(E)$.

In the circle group, Theorem 1 simply says that the Kronecker sets are dense in any perfect subset of $\mathbb{T}$, which is a well known fact. When $q(G) < \infty$, simple examples show that even if $P \subseteq G$ is perfect and all its elements have order $q(G)$, the $K_q$ sets contained in $P$ need not be dense in $\mathcal{K}(P)$.

**Corollary.** Let $E \subseteq G$ be an $I$-perfect set. Then $WTP \cap \mathcal{K}(E)$ is a big subset of $\mathcal{K}(E)$.

**Proof.** It is easy to check (using the Baire category theorem and the separability of $G$) that, given any nonempty closed set $F \subseteq G$, there exist a point $a \in G$ and an open set $V$ such that $V \cap F \neq \emptyset$ and $a + F \cap V \subseteq G_q$. Thus we may and do assume that $E$ is contained in $G_q$ (because $WTP$ is translation-invariant and every open subset of $E$ is $I$-perfect).

Now, let $\alpha_q$ be the constant introduced above, and let $\mathcal{G}$ be the family of all compact sets $K \subseteq E$ with the following property:

$$\forall S \in \mathcal{B}_1(PM(K)) \forall f \in A(G) \ |\langle S, f \rangle| \leq \alpha_q \sup\{|f(x)| : x \in E\}.$$  

Since $A(G)$ is dense in $C_0(G)$, $\mathcal{G}$ is contained in $WTP$. Moreover, using the separability of $A(G)$, one easily checks that $\mathcal{G}$ is a $G_δ$ subset of $\mathcal{K}(E)$, which is obviously hereditary.

Finally, since finite sets are $WTP$, $\mathcal{G}$ contains every finite $K_q$ subset of $E$; hence, by Theorem 1 (and the fact that $K_q$ is hereditary), $\mathcal{G}$ is dense in $\mathcal{K}(E)$. This concludes the proof.
Now we turn to applications of the results of Section 2. We show first that \( \text{WTP}, \ H \) and \( U'_0 \) are true \( \Sigma^0_3 \) within any \( M_0 \) set; then we prove that \( H \) is true \( \Sigma^0_3 \) inside the countable sets.

**Lemma 3.** Every closed set in \( I \) is a \( U_0 \) set.

**Proof.** For each integer \( m \) such that \( 0 < m < q(G) \), \( N_m \) is a closed but nonopen subgroup of \( G \). By a result of V. Tardivel [T], this implies that \( N_m \in U_0 \). Since \( U_0 \) is a translation-invariant \( \sigma \)-ideal of closed sets, the lemma follows.

**Lemma 4.** Let \( E \subseteq G \) be a nonempty \( M_0^p \) set and for \( p \in \omega \), define \( A_p = \{ K \in \mathcal{K}(E) : \eta_0(K) \geq 2^{-p} \} \). Then the perfect WTP sets contained in \( E \) are nowhere contained in any \( A_p \).

**Proof.** By Lemma 3, \( E \) is \( I \)-perfect. Hence, by Theorem 1 (Corollary), \( \text{WTP} \cap \mathcal{K}(E) \) is a big subset of \( \mathcal{K}(E) \). Now, by a result of Kechris [Ke1], if \( F \) is any \( M_0^p \) set and \( G \) is any dense \( G_\delta \) hereditary subset of \( \mathcal{K}(F) \), then, for each integer \( N \geq 1 \), there exist perfect sets \( K_1, \ldots, K_N \in G \) such that \( \eta_0(K_1 \cup \ldots \cup K_N) < 4/N \). This proves the lemma.

Let \( P \) be the family of perfect compact subsets of \( G \).

**Theorem 2.** If \( E \subseteq G \) is an \( M_0 \) set, then there is no \( \Pi^0_3 \) set such that \( P \cap \text{WTP} \cap \mathcal{K}(E) \subseteq \mathcal{M} \subseteq U'_0 \). In particular, the families of perfect WTP sets, perfect Helson sets and perfect \( U'_0 \) sets contained in \( E \) are true \( \Sigma^0_3 \) in \( \mathcal{K}(E) \).

**Proof.** Since \( U_0 \) is a \( \sigma \)-ideal of closed sets, every \( M_0 \) set contains a nonempty \( M_0^p \) set. Hence, Theorem 2 follows from Lemma 4 and Theorem B.

**Lemma 5.** Let \( p \) be a positive integer, and let \( V \) be a nonempty open subset of \( G \) contained in \( G_q \). Then there is a finite set \( F \subseteq G \) such that

- \( F \subseteq V \).
- \( F \) is a net of length \( p \).
- Each element of \( F \) has order \( q(G) \).

**Proof.** By Theorem 1 applied to \( G_q \), the set \( \{(x_0, \ldots, x_p) \in G^{p+1}_q : \{x_0, \ldots, x_p\} \in K_q\} \) is dense in \( G^{p+1}_q \). Now, for each \( \overline{x} = (x_0, \ldots, x_p) \in G^{p+1}_q \), let \( F(\overline{x}) = \{x_0 + \sum_{i=1}^p \varepsilon_i x_i : \varepsilon_i \in \{0, 1\}\} \). The map \( F \) is clearly continuous, so the set \( \{\overline{x} \in G^{p+1}_q : \overline{x} \subseteq V, x_i \neq x_j \text{ for all } i \neq j\} \) is a nonempty open subset of \( G^{p+1}_q \). It follows that there exist pairwise distinct \( a, l_1, \ldots, l_p \in G_q \) such that \( F(a, l_1, \ldots, l_p) \subseteq V \) and \( \{a, l_1, \ldots, l_p\} \) is a \( K_q \) set.

It is then easy to see that \( F = F(a, l_1, \ldots, l_p) \) has cardinality \( 2^p \) and that each element of \( F \) has order \( q(G) \). This proves the lemma.
Theorem 3. There exists a continuous map $\alpha \mapsto E(\alpha)$ from $2^\omega$ into $K(G)$ such that

- For each $\alpha \in 2^\omega$, $E(\alpha)$ is a convergent sequence.
- If $\alpha \in W$, then $E(\alpha)$ is the union of finitely many $K_q$ sets.
- If $\alpha \not\in W$, then $E(\alpha)$ contains arbitrarily long nets.

In particular, the countable Helson sets form a true $\Sigma^0_3$ subset of $K(G)$.

Proof. By Theorem 1 and Lemma 5, we can apply Theorem B with $X = G_q$, $G = K_q \cap K(X)$, $B = \{\emptyset\} \cup \{\{x\} : x \in X\}$ and $A_p = \{K \in K(X) : K$ does not contain any $p$-net$\}$. Notice that $H \cap K_\omega(G)$ is not $\Sigma^0_3$ in $K(G)$. In fact, it is not even Borel. To see this, take an independent $M_0$ set $E \subseteq G$ (a Rudin set, see [LP]). Then $K_\omega(E)$ is not Borel in $K(G)$, since $E$ is uncountable. But every countable independent compact set is Helson (see [KL]). Hence $H \cap K_\omega(E) = K_\omega(E)$ is not Borel in $K(G)$. The same example shows that we cannot “localize” Theorem 3 within an arbitrary $M$-set.

To conclude this paper, we briefly discuss other examples of natural $\Sigma^0_3$ in harmonic analysis.

Very close to the $U'_q$ sets are the $U'$ sets (see [KL]) and the $U_2'$ sets of R. Lyons [Ly], which also form $\Sigma^0_3$ subsets of $K(\mathbb{T})$. In fact, one has the inclusions $WTP \subseteq U' \subseteq U'_2 \subseteq U'_0$, so (by Theorem 2) $U'$ and $U'_2$ are true $\Sigma^0_3$. For $U'$ sets, there is a more precise “local” result: if $E$ is any closed set of multiplicity, then $U' \cap K(E)$ is a true $\Sigma^0_3$ of $K(E)$. This can be deduced from our criteria, using the family of Dirichlet sets rather than the family of $K_q$ sets.

Other examples in the circle group are the $p$-Helson sets introduced by M. Gregory [G]. A compact set $K \subseteq \mathbb{T}$ is $p$-Helson (say $K \in \mathcal{H}(p)$) if every continuous function on $K$ is the restriction of a continuous function on $\mathbb{T}$ whose Fourier series belongs to $l^p(\mathbb{Z})$. Obviously, $\mathcal{H}_{(1)} = \mathcal{H}$, $\mathcal{H}(p) \subseteq \mathcal{H}(p')$ if $p \leq p'$, and $\mathcal{H}_{(2)} = K(\mathbb{T})$.

It is shown in [G] that for $p > 1$, $\mathcal{H}(p)$ is a $\sigma$-ideal of $K(\mathbb{T})$, and that a compact set $K$ is $p$-Helson if and only if for all $\mu \in M(K)$, $\|\mu\|_M = \inf\{\|\mu - \lambda\|_M + \|\hat{\lambda}\|_q : \lambda \in M_q(\mathbb{T})\}$, where $M_q(\mathbb{T}) = \{\lambda \in M(\mathbb{T}) : \hat{\lambda} \in l^q(\mathbb{Z})\}$ (and $q = p/(p - 1)$). Using this, it is not hard to check that $\mathcal{H}(p)$ is $G_\delta$ in $K(\mathbb{T})$.

Now, let $\mathcal{H}$ be the family of compact subsets of $\mathbb{T}$ which are $p$-Helson for some $p \in [1, 2]$. It follows from the preceding remarks that $\mathcal{H}$ is a big $\Sigma^0_3$ ideal of $K(\mathbb{T})$. Moreover, it is also shown in [G] that for any $p < 2$, $\mathcal{H}(p)$ is a proper subset of $\mathcal{H}$; and it is not difficult to deduce from the proof that this is true in any open subset of $\mathbb{T}$. Thus Theorem A applies, and we conclude that $\mathcal{H}$ is a true $\Sigma^0_3$ subset of $K(\mathbb{T})$. 
On the other hand, our criteria do not apply to the family of $H$-sets of the circle group, which is also a true $\Sigma^0_3$ (see [Li]), because $H$ is not an ideal of $\mathcal{K}(\mathbb{T})$.

References


