FOURIER UNIQUENESS (ALMOST) FROM SCRATCH

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ABSTRACT. We give short and direct proofs of the injectivity of the Fourier transforms on the circle and on the real line. Using exactly the same ideas, we also give a new proof of the Stone-Weierstrass Theorem.

1. INTRODUCTION

A basic fact in the theory of Fourier series is the "uniqueness theorem", according to which an integrable function $f : [0, 2\pi) \to \mathbb{C}$ is uniquely determined by its Fourier coefficients

$$\widehat{f}(n) := \int_0^{2\pi} f(t)e^{-int}\,\frac{dt}{2\pi}, \quad n \in \mathbb{Z}.$$

Equivalently, if $f : [0, 2\pi) \to \mathbb{C}$ is an integrable function such that $\widehat{f}(n) = 0$ for all $n \in \mathbb{Z}$, then $f(t) = 0$ almost everywhere.

However basic, this is a *non-trivial* fact. In spite of that (or, perhaps, because of that), it has become standard, in textbooks on Fourier analysis, to deduce the uniqueness theorem from more general results. For example, one may use the L^1 version of Fejér's Theorem; or, one can first show by any means (Stone-Weierstrass Theorem, continuous version of Fejér's Theorem, ...) that the trigonometric polynomials are dense in the space of continuous 2π-periodic functions, and then conclude by a duality argument.

Now, imagining oneself preparing an introductory course on Fourier series for students knowing the basics of Lebesgue integration but "nothing more" (which may happen for real), it is natural to wonder if the uniqueness theorem can be proved "from scratch", immediately after giving the definition of Fourier coefficients. This is not a merely academic question. Indeed, the uniqueness theorem entails the completeness of the trigonometric system in $L^2(0, 2\pi)$, which is the key point in the L^2 theory of Fourier series; so it seems quite desirable to have it at one's disposal as quickly as possible.

As it turns out, a direct proof of the uniqueness theorem does exist. This proof goes back at least to Lebesgue (see [7, Chapter II, 24]), and it can be found in the "classics" [10], [1] and [4]. The idea is to first reduce the result to the case of *continuous*

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functions by observing that if an integrable function $f:[0,2\pi)\to\mathbb{C}$ has all its Fourier coefficients equal to 0, then so does the continuous function $F(t):=c+\int_0^t f(s)\,ds$ for some suitable constant c (since $\widehat{F}(n)=\frac{1}{in}\widehat{f}(n)$ for all $n\neq 0$); and then to prove that a continuous function which is non-zero at some point t cannot have all its Fourier coefficients equal to 0, by using a sequence of trigonometric polynomials peaking at t. The second part of the proof can also be found in some "modern" textbooks, e.g. [9].

In this note, we propose an even more direct proof of the uniqueness theorem. As the one we just outlined, it is completely elementary except for the same non-trivial fact used as a "blackbox", namely that if f is an integrable function on some interval $I\subset\mathbb{R}$ such that $\int_{(a,b)} f(t)\,dt=0$ for all bounded open intervals $(a,b)\subset I$, then $f(t)=0$ almost everywhere on I. The latter can be proved for example by applying the so-called *Monotone Class Theorem*, see Section 5.

Essentially the same proof yields the corresponding uniqueness result for the Fourier transform on $\mathbb{R}$; which is of course not surprising. Perhaps more unexpectedly, the very same ideas also lead to a seemingly new proof of the Stone-Weierstrass Theorem.

2. Fourier coefficients

In this section, we prove the uniqueness theorem for Fourier coefficients:

Theorem 2.1. *If $f:[0,2\pi)\to\mathbb{C}$ is an integrable function such that $\widehat{f}(n)=0$ for all $n\in\mathbb{Z}$, then $f(t)=0$ almost everywhere.*

Proof. It is enough to show that $\int_{(a,b)} f(t)\,dt=0$ for every open interval $(a,b)\subset[0,2\pi)$. This will follow from the next two facts. Here, by a *trigonometric polynomial*, we mean a linear combination of the functions e^{int}, $n\in\mathbb{Z}$. Note that, by assumption on f, we have $\int_0^{2\pi} f(t)P(t)\,dt=0$ for any trigonometric polynomial P.

Fact 2.2. *There exists a real-valued trigonometric polynomial q such that $q(t)<0$ on (a,b) and $q(t)>0$ on $[0,2\pi)\backslash[a,b]$.*

Proof of Fact 2.2. Let

$$q(t):=\sin\left(\frac{t-a}{2}\right)\sin\left(\frac{t-b}{2}\right).$$

Since $\frac{t-a}{2}\in(0,\pi)$ and $\frac{t-b}{2}\in(-\pi,0)$ if $t\in(a,b)$, we have $q(t)<0$ on (a,b). Similarly, $q(t)>0$ on $[0,2\pi)\backslash[a,b]$. Moreover, using the well known formula $\sin(u)\sin(v)=\frac{1}{2}\big(\cos(u-v)-\cos(u+v)\big)$, we see that

$$q(t)=\frac{1}{2}\left[\cos\left(\frac{b-a}{2}\right)-\cos\left(t-\frac{a+b}{2}\right)\right],$$

so that q is indeed a trigonometric polynomial. □

Fact 2.3. *There exists a sequence of real-valued trigonometric polynomials* $(P_k)_{k\in\mathbb{N}}$ *with* $0 \leqslant P_k \leqslant 1$ *for all* k*, such that* $P_k(t) \to 1$ *pointwise on* (a,b) *and* $P_k(t) \to 0$ *on* $[0,2\pi)\backslash[a,b]$.

Proof of Fact 2.3. Let q be the trigonometric polynomial given by Fact 2.2 and let $r(t) := 1 + cq(t)$, where $c > 0$ is chosen in such a way that $0 \leqslant r(t) \leqslant 2$ on $[0,2\pi)$. This is possible since the function q is bounded. Then $0 \leqslant r(t) < 1$ on (a,b) and $1 < r(t) \leqslant 2$ on $[0,2\pi)\backslash[a,b]$. Choose a sequence of positive integers (n_k) such that $n_k > 2^k$ for all k and $2^k/n_k \to 0$, e.g. $n_k = 3^k$, and let

$$P_k(t) := \left(1 - \frac{r(t)^k}{n_k}\right)^{n_k}.$$

Each P_k is a real trigonometric polynomial such that $0 < P_k(t) \leqslant 1$ on $[0,2\pi)$. Moreover, since

$$\log\big(P_k(t)\big) = n_k \log\left(1 - \frac{r(t)^k}{n_k}\right) \sim -r(t)^k,$$

we see that $P_k(t) \to 1$ on (a,b) and $P_k(t) \to 0$ on $[0,2\pi)\backslash[a,b]$. □

It is now easy to show that $\int_{(a,b)} f(t)\,dt = 0$. Let (P_k) be the sequence of trigonometric polynomials given by Fact 2.3. By definition, $P_k(t) \to \mathbf{1}_{(a,b)}(t)$ almost everywhere on $[0,2\pi)$ – in fact, everywhere except at a finite number of points. By the Dominated Convergence Theorem, it follows that $\int_0^{2\pi} P_k(t)f(t)\,dt \to \int_{(a,b)} f(t)\,dt$; which concludes the proof since $\int_0^{2\pi} P_k(t)f(t)\,dt = 0$ for all k. □

Remark 2.1. The above proof can be slightly modified to show that if μ is a complex Borel measure on $\mathbb{T}$ such that $\widehat{\mu}(n) = 0$ for all $n \in \mathbb{Z}$, then $\mu = 0$ (which means, by duality, that the trigonometric polynomials are dense in $\mathcal{C}(\mathbb{T})$). Indeed, considering μ as a measure on $[0,2\pi)$, the proof shows that $\mu((a,b)) = 0$ for every open interval $(a,b) \subset [0,2\pi)$ whose end-points are not atoms of $|\mu|$, the total variation of μ, since in this case the above sequence (P_k) converges $|\mu|$-almost everywhere to $\mathbf{1}_{(a,b)}$. Since $|\mu|$ has only countably many atoms, the family of all such intervals generates the Borel sigma-algebra of $[0,2\pi)$, and the result follows by the Monotone Class Theorem.

3. Fourier transform on the line

The Fourier transform of an integrable function $f : \mathbb{R} \to \mathbb{C}$ may be defined by

$$\widehat{f}(x) := \int_{\mathbb{R}} f(t)e^{-ixt}dt\,, \quad x \in \mathbb{R}.$$

In this section, we give a direct proof of the uniqueness theorem:

Theorem 3.1. *If* $f : \mathbb{R} \to \mathbb{C}$ *is an integrable function such that* $\widehat{f}(x) = 0$ *for all* $x \in \mathbb{R}$*, then* $f(t) = 0$ *almost everywhere.*

Proof. We adapt the proof of Theorem 2.1. So, a trigonometric polynomial is now a linear combination of functions of the form $e^{i\lambda t}$, $\lambda \in \mathbb{R}$, and the assumption means that $\int_{\mathbb{R}} f(t)P(t)\,dt = 0$ for every trigonometric polynomial P. It is enough to show that $\int_{(a,b)} f(t)\,dt = 0$ for every bounded open interval $(a,b) \subset \mathbb{R}$.

Let $\varepsilon > 0$. We choose a bounded interval $I \supset (a,b)$ such that $\int_{\mathbb{R}\setminus I} |f| < \varepsilon$.

Having fixed the interval I, there exists a real trigonometric polynomial q such that $q(t) < 0$ on (a,b) and $q(t) > 0$ on $I\setminus[a,b]$. For example, one may take

$$q(t) := \sin\left(\frac{t-a}{T}\right)\sin\left(\frac{t-b}{T}\right),$$

where $T > 0$ is large enough to ensure that $\frac{1}{T}(I - [a,b]) \subset (-\pi,\pi)$.

As above, it follows that there exists a sequence of real trigonometric polynomials (P_k) such that $0 \leqslant P_k \leqslant 1$ on I and $P_k(t) \to \mathbf{1}_{(a,b)}(t)$ almost everywhere on I. Then $\int_I f(t)P_k(t)\,dt \to \int_{(a,b)} f(t)\,dt$ by the Dominated Convergence Theorem. Now, we have $\int_I f(t)P_k(t)\,dt = -\int_{\mathbb{R}\setminus I} f(t)P_k(t)\,dt$ since $\int_{\mathbb{R}} f(t)P_k(t)\,dt = 0$; so $\left|\int_I f(t)P_k(t)\,dt\right| \leqslant \int_{\mathbb{R}\setminus I} |f(t)|\,dt < \varepsilon$ for all $k \in \mathbb{N}$, and hence $\left|\int_{(a,b)} f(t)\right| dt \leqslant \varepsilon$. Since $\varepsilon > 0$ is arbitrary, this concludes the proof. □

Remark 3.1. The above proof is a rather straightforward modification of that of Theorem 2.1. It does not seem immediately clear that Lebesgue's proof outlined in the introduction can be similarly adapted.

Remark 3.2. It is rather tempting to try to prove along the same lines the uniqueness theorem for the Fourier transform on $L^1(G)$, where G is a locally compact abelian group. In this setting, a trigonometric polynomial is a linear combination of *continuous characters* of G, *i.e.* continuous homomorphisms $\gamma : G \to \mathbb{T}$ (considered as complex-valued functions on G). The most obvious analogue of Fact 2.2 would be the following: for any set O in some basis for the topology of G, one can find a real-valued trigonometric polynomial q such that $q < 0$ on O and $q > 0$ on $G\setminus\overline{O}$. However, this already implies that *the continuous characters separate the points of* G; which is a non-trivial result at this level of generality. Even "worse" from our point of view, it seems that the simplest way to prove the separation property is... to use the uniqueness theorem, as for example in [2, Proposition 3.5.2]. So, it appears that our approach cannot yield the uniqueness theorem " from scratch" in this general setting.

4. Stone-Weierstrass

In this section, we recycle the ideas used in the proof of Theorem 2.1 to get a "new proof" of the Stone-Weierstrass Theorem. Compared with the standard one (see e.g. [8]), this proof cannot be considered as elementary since it makes use of the Hahn-Banach Theorem and the Riesz Representation Theorem; yet, we find it rather amusing. As for "novelty", it seems fair to point out that it looks very similar – albeit more self-contained – to the dazzling one-page proof of the lattice version of the Stone-Weierstrass Theorem due to Khurana [6].

Let us recall the statement of the Stone-Weierstrass Theorem.

Theorem 4.1. *Let X be a compact Hausdorff space, and denote by $\mathcal{C}(X)$ the real or complex space of all continuous function on X. If $\mathcal{A}$ is a subalgebra of $\mathcal{C}(X)$ containing $\mathbf{1}$ such that the real-valued functions in $\mathcal{A}$ separate the points of X, then $\mathcal{A}$ is dense in $\mathcal{C}(X)$.*

Note that in the complex case, the standard version of the Stone-Weierstrass Theorem involves a separating algebra closed under complex conjugation. However, this version follows from Theorem 4.1 as stated, since if a linear subspace $\mathcal{A} \subset \mathcal{C}(X)$ is separating and closed under conjugation, then the real functions in $\mathcal{A}$ are also separating. Moreover, it is in fact enough to prove Theorem 4.1 in the real case, for if $\mathcal{A}$ is a linear subspace of $\mathcal{C}(X)$ such that the real functions in $\mathcal{A}$ are dense in $\mathcal{C}_{\mathbb{R}}(X)$, then $\mathcal{A}$ is dense in $\mathcal{C}(X)$.

The proof of (the real version of) Theorem 4.1 relies on the next two facts, which play the roles of Facts 2.2 and 2.3 above. More precisely, Fact 4.3 is a clear analogue of Fact 2.3, whereas Fact 4.2 says that the family of sets satisfying something like the conclusion of Fact 2.2 is rich enough. To state these two facts, we need to fix some notation.

In what follows, we endow the space X with its *Baire sigma-algebra*, *i.e.* the sigma-algebra generated by the compact G_δ sets. The word "measurable" will refer to this sigma-algebra. Note that every $f \in \mathcal{C}(X)$ is measurable, since for any $\alpha \in \mathbb{R}$, the set $\{f \leqslant \alpha\}$ is compact and G_δ.

If $\mathbf{q} = (q_1, \dots, q_d)$ is a finite family of real-valued functions on X, we set

$$O_{\mathbf{q}} := \big\{x \in X : q_i(x) < 0 \text{ for all } i \in [\![1, d]\!]\big\}.$$

We also define

$$\check{O}_{\mathbf{q}} := \big\{x \in X : q_i(x) > 0 \text{ for some } i \in [\![1, d]\!]\big\}.$$

If ν is a finite positive Baire measure on X, we will say that a measurable function $q : X \to \mathbb{R}$ is "ν-adequate" if $\nu\big(\{q = 0\}\big) = 0$; and

> we denote by $\mathfrak{O}_\nu$ the family of all subsets of X of the form $O_{\mathbf{q}}$, where $\mathbf{q}$ is a finite family of ν-adequate functions in $\mathcal{A}$.

All sets in $\mathfrak{O}_\nu$ are open and measurable; and by definition of "ν-adequate", we have

$$\nu\big(X \backslash (O_{\mathbf{q}} \cup \check{O}_{\mathbf{q}})\big) = 0 \quad \text{for every } O_{\mathbf{q}} \in \mathfrak{O}_\nu.$$

Fact 4.2. *For any finite positive Baire measure ν on X, the family $\mathfrak{O}_\nu$ is a basis for the topology of X.*

Proof. The family $\mathfrak{O}_\nu$ is closed under finite intersections and covers X (since $X = \{-\mathbf{1} < 0\}$); so it is a basis for a topology τ on X, which is coarser than the original topology τ_X since each $O \in \mathfrak{O}_\nu$ is open in X. Since (X, τ_X) is compact, it is enough to show that this topology τ is Hausdorff.

We first observe that for any $q \in \mathcal{A}$, one can find a ν-adequate function $\tilde{q} \in \mathcal{A}$ such that $\|\tilde{q} - q\|_\infty$ is arbitrarily small. Indeed, let $\varepsilon > 0$. For any $\alpha \in (0, \varepsilon)$, the function $q_\alpha := q + \alpha\mathbf{1}$ belongs to $\mathcal{A}$ and $\|q_\alpha - q\|_\infty < \varepsilon$. Moreover, the sets $E_\alpha := \{q_\alpha = 0\}$ are pairwise disjoint. Since ν is a finite measure, it follows that $\nu(E_\alpha) = 0$, *i.e.* q_α is ν-adequate, for all but countably many $\alpha \in (0, \varepsilon)$.

Now, let $x, x' \in X$ with $x \neq x'$. Since $\mathcal{A}$ is separating, one can find a function $f \in \mathcal{A}$ such that $f(x) \neq f(x')$, say $f(x) < f(x')$. Choose α such that $f(x) < \alpha < f(x')$. Then $q := f - \alpha\mathbf{1}$ belongs to $\mathcal{A}$, and $q(x) < 0 < q(x')$. Moreover, by our initial observation, we may also assume that q is ν-adequate. Then $O := \{q < 0\}$ and $O' := \{q > 0\} = \{-q < 0\}$ are τ-open sets separating x and x'. □

Remark. The proof works for any linear subspace $\mathcal{A} \subset \mathcal{C}(X)$ containing $\mathbf{1}$ and separating the points of X.

Fact 4.3. *For any finite family* $\mathbf{q} = (q_1, \dots, q_d) \subset \mathcal{A}$*, one can find a sequence* $(P_k)_{k\in\mathbb{N}} \subset \mathcal{A}$ *such that* $0 \leqslant P_k \leqslant 1$ *for all* k*,* $P_k(x) \to 1$ *pointwise on* $O_\mathbf{q}$ *and* $P_k(x) \to 0$ *pointwise on* $\check{O}_\mathbf{q}$*.*

Proof. The proof of Fact 2.3 shows that for $i = 1, \dots, d$, one can find a sequence $(P_{i,k})_{k\in\mathbb{N}} \subset \mathcal{A}$ such that $0 \leqslant P_{i,k} \leqslant 1$ for all k, $P_{i,k}(x) \to 1$ on $\{q_i < 0\}$ as $k \to \infty$, and $P_{i,k}(x) \to 0$ on $\{q_i > 0\}$. If we set $P_k := \prod_{i=1}^d P_{i,k}$, then the sequence (P_k) does the required job. □

Remark. The proof works for any algebra $\mathcal{A}$ of real-valued bounded functions containing $\mathbf{1}$.

Proof of Theorem 4.1. By the Hahn-Banach Theorem and the "Baire measure version" of the Riesz Representation Theorem (for the latter, see e.g. [8] or the very elegant [3]), it is enough to show that if μ is a real Baire measure on X such that $\int_X P\,d\mu = 0$ for all $P \in \mathcal{A}$, then $\mu = 0$. We fix such a measure μ, and we let $\mathfrak{O} := \mathfrak{O}_{|\mu|}$, where $|\mu|$ is the total variation of μ.

Claim 4.4. *The family* $\mathfrak{O}$ *generates the Baire sigma-algebra of* X*, and we have* $\mu(O) = 0$ *for every* $O \in \mathfrak{O}$*.*

Proof of Claim 4.4. To prove the first part, it is enough to show that any compact G_δ set $E \subset X$ belongs to the sigma-algebra $\sigma(\mathfrak{O})$ generated by $\mathfrak{O}$. Write $E = \bigcap_{n\in\mathbb{N}} V_n$ where the sets V_n are open in X. Since $\mathfrak{O}$ is a basis for the topology of X by Fact 4.2 and since E is compact, we see that for each $n \in \mathbb{N}$, one can find finitely many sets in $\mathfrak{O}$, say $O_{1,n}, \dots, O_{k_n,n}$, such that $E \subset \bigcup_{k=1}^{k_n} O_{k,n} \subset V_n$. Then $E = \bigcap_{n\in\mathbb{N}} \bigcup_{k=1}^{k_n} O_{k,n}$, so $E \in \sigma(\mathfrak{O})$.

As for the second part, let $O = O_{\mathbf{q}} \in \mathfrak{O}$, where $\mathbf{q} = (q_1, \dots, q_d)$ is a finite family of $|\mu|$-adequate functions in $\mathcal{A}$. Then $|\mu|\big(X \backslash (O_{\mathbf{q}} \cup \check{O}_{\mathbf{q}})\big) = 0$. Hence, the sequence $(P_k) \subset \mathcal{A}$ given by Fact 4.3 converges $|\mu|$-almost everywhere to $\mathbf{1}_{O_{\mathbf{q}}}$. By the Dominated Convergence Theorem, it follows that $\int_X P_k \, d\mu \to \mu(O)$ as $k \to \infty$; which concludes the proof since $\int_X P_k \, d\mu = 0$ for all k. □

Since the family $\mathfrak{O}$ is closed under finite intersections, we can now conclude that $\mu = 0$ by Claim 4.4 and the Monotone Class Theorem. □

5. The Monotone Class Theorem

In this "appendix", for the sake of completeness, we prove the version of the Monotone Class Theorem we have used in the previous sections.

Let X be an abstract set, and denote by 2^X the family of all subsets of X. We will say that family $\mathfrak{M} \subset 2^X$ is a *monotone class* if it has the following properties:

- $X \in \mathfrak{M}$;
- if $A, A' \in \mathfrak{M}$ and $A \subset A'$, then $A' \backslash A \in \mathfrak{M}$;
- if $(A_n)_{n \in \mathbb{N}}$ is an increasing sequence of sets in $\mathfrak{M}$, then $\bigcup_{n \in \mathbb{N}} A_n \in \mathfrak{A}$.

The terminology "monotone class" is not universally used for such families; many people say "Dynkin system" instead.

The usefulness of this notion is due to the fact that there exist natural families of sets which are easily seen to be monotone classes but are *a priori* not sigma-algebras. For example, if $(X, \mathfrak{B})$ is a measurable space and if μ_1, μ_2 are two finite positive measures on $(X, \mathfrak{B})$ – or, more generally, two complex measures – such that $\mu_1(X) = \mu_2(X)$, then the family $\mathfrak{M} = \{A \in \mathfrak{B};\ \mu_1(A) = \mu_2(A)\}$ is a monotone class.

As stated for example in [5], the Monotone Class Theorem reads as follows.

Theorem 5.1. *Let X be an abstract set, and let $\mathfrak{O}$ be a family of subsets which is closed under finite intersections. If $\mathfrak{M} \subset 2^X$ is a monotone class containing $\mathfrak{O}$, then $\mathfrak{M}$ contains the sigma-algebra $\sigma(\mathfrak{O})$ generated by $\mathfrak{O}$.*

Proof. The idea is to show that the monotone class $\mathfrak{M}_0$ generated by $\mathfrak{O}$ is a sigma-algebra. For any $C \in \mathfrak{M}_0$, let $\mathfrak{M}_C := \{E \in 2^X : E \cap C \in \mathfrak{M}_0\}$, and observe that $\mathfrak{M}_C$ is a monotone class. Now, the proof proceeds in three steps, as follows.

(i) For any $A \in \mathfrak{O}$, the monotone class $\mathfrak{M}_A$ contains $\mathfrak{O}$ because $\mathfrak{O}$ is stable under finite intersections. It follows that $A \cap B \in \mathfrak{M}_0$ for any $A \in \mathfrak{O}$ and all $B \in \mathfrak{M}_0$.

(ii) By (i), the monotone class $\mathfrak{M}_B$ contains $\mathfrak{O}$ for any $B \in \mathfrak{M}_0$. It follows that $\mathfrak{M}_0$ is closed under finite intersections.

(iii) Since the monotone class $\mathfrak{M}_0$ is closed under complementation, it is closed under finite unions by (ii). So it is in fact closed under countable unions, and hence it is a sigma-algebra.

□

As an immediate consequence, we get

Corollary 5.2. *Let $(X, \mathfrak{B})$ be a measurable space, and let $\mathfrak{O}$ be a family of measurable sets which is closed under finite intersections. If μ is a complex measure on $(X, \mathfrak{B})$ such that $\mu(O) = 0$ for all $O \in \mathfrak{O}$, then $\mu(A) = 0$ for all $A \in \sigma(\mathfrak{O})$.*

Proof. As observed above, the family $\mathfrak{M} = \{A \in \mathfrak{B};\, \mu(A) = 0\}$ is a monotone class. □

And finally, the "blackbox" result mentioned in the introduction:

Corollary 5.3. *Let I be an interval of $\mathbb{R}$, and let $f : I \to \mathbb{C}$ be a Lebesgue integrable function. If $\int_{(a,b)} f(t)\, dt = 0$ for all bounded open intervals $(a, b) \subset I$, then $f(t) = 0$ almost everywhere.*

Proof. We may assume that the function f is Borel; and considering $\mathrm{Re}(f)$ and $\mathrm{Im}(f)$ separately, we may also assume that f is real-valued. By the Monotone Class Theorem and since the family $\mathfrak{O}$ of all bounded open intervals $(a, b) \subset I$ generates the Borel sigma-algebra of I, we see that $\int_A f(t)\, dt = 0$ for every Borel set $A \subset I$. Applying that to the sets $A^+ := \{f > 0\}$ and $A^- := \{f < 0\}$, the result follows. □

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