GENERIC PROPERTIES OF ℓ_p-CONTRACTIONS AND SIMILAR OPERATOR TOPOLOGIES

by

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Abstract. — If X is a separable reflexive Banach space, there are several natural Polish topologies on $\mathcal{B}_1(X)$, the set of contraction operators on X (none of which being clearly "more natural" than the others), and hence several *a priori* different notions of genericity – in the Baire category sense – for properties of contraction operators. So it makes sense to investigate to which extent the generic properties, *i.e.* the comeager sets, really depend on the chosen topology on $\mathcal{B}_1(X)$. In this paper, we focus on ℓ_p-spaces, $1 < p \neq 2 < \infty$. We show that for some pairs of natural Polish topologies on $\mathcal{B}_1(\ell_p)$, the comeager sets are in fact the same; and our main result asserts that for $p = 3$ or $3/2$ and in the real case, *all* topologies on $\mathcal{B}_1(\ell_p)$ lying between the Weak Operator Topology and the Strong* Operator Topology share the same comeager sets. Our study relies on the consideration of continuity points of the identity map for two different topologies on $\mathcal{B}_1(\ell_p)$. The other essential ingredient in the proof of our main result is a careful examination of norming vectors for finite-dimensional contractions of a special type.

1. Introduction and main results

1.1. Polish spaces of operators and typical properties. — Let X be a real or complex infinite-dimensional separable Banach space. Denote by $\mathcal{B}(X)$ the space of all bounded operators on X, and by $\mathcal{B}_1(X)$ the unit ball of $\mathcal{B}(X)$, *i.e.* the set of contraction operators on X. In this paper, we will be interested in *typical properties* of elements of $\mathcal{B}_1(X)$ in the sense of Baire Category. More precisely, the setup is the following: we endow the ball $\mathcal{B}_1(X)$ with a topology τ which turns it into a *Polish* (i.e. separable and completely metrizable) space. A property (P) of elements of $\mathcal{B}_1(X)$ is said to be *typical for* τ, or τ-*typical* if the set of operators satisfying (P) is comeager in $(\mathcal{B}_1(X), \tau)$, i.e. contains a dense G_δ subset of $(\mathcal{B}_1(X), \tau)$. A typical property is thus a property that is satisfied by "quasi-all" contractions (in the Baire Category sense). We will mainly focus on the case where X is the real or complex ℓ_p-space, $1 \leqslant p < \infty$.

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The ball $\mathcal{B}_1(X)$ is usually not a Polish space when endowed with the operator norm topology (as it is not separable), and some weaker topologies are to be considered in order to turn $\mathcal{B}_1(X)$ into a Polish space. We will consider in this paper four natural topologies on $\mathcal{B}(X)$: the *Strong Operator Topology* ($\mathtt{SOT}$), the *Weak Operator Topology* ($\mathtt{WOT}$), the *Strong* Operator Topology* ($\mathtt{SOT}^*$), and a topology that might be called the "Dual Strong Operator Topology", which we denote by $\mathtt{SOT}_*$. Recall that $\mathtt{SOT}$ is the topology of pointwise convergence, that $\mathtt{WOT}$ is the topology of weak pointwise convergence, and that $\mathtt{SOT}^*$ is the topology of pointwise convergence for operators and their adjoints. As to $\mathtt{SOT}_*$, it is the topology of pointwise convergence for the adjoints. In other words, if (T_i) is a net in $\mathcal{B}(X)$ and if $T \in \mathcal{B}(X)$, then

$$\begin{aligned}
T_i \xrightarrow{\mathtt{SOT}} T &\iff T_i x \xrightarrow{\|\cdot\|} Tx \quad \text{for all } x \in X,\\
T_i \xrightarrow{\mathtt{WOT}} T &\iff T_i x \xrightarrow{w} Tx \quad \text{for all } x \in X,\\
T_i \xrightarrow{\mathtt{SOT}^*} T &\iff T_i \xrightarrow{\mathtt{SOT}} T \quad \text{and} \quad T_i^* \xrightarrow{\mathtt{SOT}} T^*,\\
T_i \xrightarrow{\mathtt{SOT}_*} T &\iff T_i^* \xrightarrow{\mathtt{SOT}} T^*.
\end{aligned}$$

The space $\mathcal{B}(X)$ is not a Baire space for any of these topologies, but its unit ball $\mathcal{B}_1(X)$ is nicer: it is well known that since X is separable, $(\mathcal{B}_1(X), \mathtt{SOT})$ is Polish; that if X^* is separable then $(\mathcal{B}_1(X), \mathtt{SOT}^*)$ is Polish; and that if X is reflexive then $(\mathcal{B}_1(X), \mathtt{WOT})$ is compact and metrizable (hence Polish) and $(\mathcal{B}_1(X), \mathtt{SOT}_*)$ is Polish. Note also that obviously $\mathtt{SOT}^* = \mathtt{SOT} \vee \mathtt{SOT}_*$, *i.e.* $\mathtt{SOT}^*$ is the coarsest topology which is finer than both $\mathtt{SOT}$ and $\mathtt{SOT}_*$, and that $\mathtt{WOT} \subset \mathtt{SOT} \cap \mathtt{SOT}_*$. The topology $\mathtt{SOT}_*$ may look a bit artificial, but our results will show that it is indeed natural to consider it.

The study of typical properties of contractions was initiated by Eisner [9] and Eisner-Mátrai [10] in a purely Hilbertian setting, and further developed in [13] and [14]. The setting of the monograph [13] is also mainly Hilbertian, and it is focused on typical properties of operators connected to linear dynamics (existence of orbits with various properties, existence of non-trivial measures...). The works [14] and [12] are set in the broader context of (complex) ℓ_p-spaces, and centre around the following question: is it true that a typical $T \in \mathcal{B}_1(\ell_p)$ has a non-trivial invariant subspace? The underlying motivation of this question is, of course, the famous Invariant Subspace Problem, which asks (for a given Banach space X) whether any bounded operator T on X admits a closed subspace M with $M \neq \{0\}$ and $M \neq X$ such that $T(M) \subseteq M$. The answer is known to be negative in general ([11], [26]), and examples of operators without non-trivial invariant subspaces have been constructed by Read on ℓ_1 or c_0 ([27], [28], see also [15] for a unifying approach to such constructions). Let us also mention the existence, proved by Argyros and Haydon in [2], of separable infinite-dimensional spaces X on which every operator does have a non-trivial invariant subspace. Despite considerable efforts, no answer (positive or negative) to the Invariant Subspace Problem have been obtained so far for any reflexive X, which was a motivation for studying its *a priori* more tractable "generic" version in [14].

One would rather naturally expect typical properties of contractions $T \in (\mathcal{B}_1(X), \tau)$ to depend heavily on the Polish topology τ under consideration. This intuition is quite right in the Hilbertian setting. Indeed, in the case of the complex ℓ_2-space, it is proved in [9] that a typical $T \in (\mathcal{B}_1(\ell_2), \mathtt{WOT})$ is unitary, whereas in [10] the following surprising result is obtained: a typical $T \in (\mathcal{B}_1(\ell_2), \mathtt{SOT})$ is unitarily equivalent to B_∞, the backward shift of infinite multiplicity acting on $\ell_2(\mathbb{Z}_+, \ell_2)$. In particular, the typical situation for $\mathtt{WOT}$ (resp.

SOT) is that T (resp. T^*) is an isometry. As to the topology SOT*, it is proved in [13] that a typical $T \in (\mathcal{B}_1(\ell_2), \mathtt{SOT}^*)$ is such that $2T$ and $2T^*$ are hypercyclic (*i.e.* they admit vectors with dense orbit), so that T and T^* are very far from being isometries.

It is clear from these results that a typical $T \in \mathcal{B}_1(\ell_2)$ for any one of the topologies WOT, SOT, $\mathtt{SOT}_*$ has a non-trivial invariant subspace. It is much less clear that this remains true for the topology SOT*; but it is indeed the case thanks to a deep result of Brown, Chevreau and Pearcy [6], according to which any $T \in \mathcal{B}_1(\ell_2)$ whose spectrum $\sigma(T)$ contains the unit circle $\mathbb{T}$ has a non-trivial invariant subspace. As it can be shown (see [10]) that a typical $T \in (\mathcal{B}_1(\ell_2), \mathtt{SOT}^*)$ is such that $\sigma(T) = \overline{\mathbb{D}}$, the closed unit disk in $\mathbb{C}$, it follows that an SOT*- typical $T \in \mathcal{B}_1(\ell_2)$ does have a non-trivial invariant subspace.

Much less is known on (complex) ℓ_p - spaces: while it is still true that an SOT - typical $T \in \mathcal{B}_1(\ell_1)$ has a non-trivial invariant subspace (see [14]), the question remains widely open for typical operators $T \in \mathcal{B}_1(\ell_p)$, $1 < p \neq 2 < \infty$, whatever the topology we consider on $\mathcal{B}_1(\ell_p)$ among our favourite ones. However, at least for $p > 2$, an interesting link between typical properties of contractions on ℓ_p for *two different topologies* is uncovered in [14]: if $p > 2$, then any comeager subset in $(\mathcal{B}_1(\ell_p), \mathtt{SOT}^*)$ is also comeager in $(\mathcal{B}_1(\ell_p), \mathtt{SOT})$; in other words, a property of ℓ_p - contractions which is typical for SOT* is also typical for SOT. Since it is not too hard to show that an SOT*- typical $T \in \mathcal{B}_1(\ell_p)$ is such that $2T^*$ is hypercyclic (which implies in particular that T has no eigenvalue), it follows that an SOT - typical $T \in \mathcal{B}_1(\ell_p)$ has no eigenvalue [14]. It should be pointed out that we are unable to provide a proof of this result which does not make use of the topology SOT*.

1.2. Main results. — With the above result from [14] in mind, our purpose in the present paper is to undertake a systematic study of the links between comeager subsets of $\mathcal{B}_1(\ell_p)$, $1 < p \neq 2 < \infty$ for different topologies, with a view towards a better understanding of typical properties of contractions on ℓ_p. The basic question we consider is the following:

Question 1.1. — Let X be a real or complex ℓ_p - space, with $1 < p \neq 2 < \infty$. Is it true that the four topologies WOT, SOT, $\mathtt{SOT}_*$, and SOT* on $\mathcal{B}_1(\ell_p)$ have the same comeager sets?

We will in fact consider a formally stronger property than just having the same comeager sets. Following [3], let us say that two topologies τ and τ' on an abstract set $\mathbf{M}$ are *similar* if they have the same dense sets, or equivalently the same sets with empty interior. It is easy to see that similar topologies have the same comeager sets, and simple examples show that the converse is not true (see Section 2). Similar topologies may be extremely different. Consider for example the classical Sorgenfrey topology on $\mathbb{R}$, which is generated by the half-open intervals $[a, b)$, $a, b \in \mathbb{R}$. This topology is zero-dimensional and non-metrizable, and yet it is similar to the usual topology of $\mathbb{R}$.

Our first result regarding similarity of operator topologies is the following.

Theorem 1.2. — *Let X be a real or complex ℓ_p - space with $1 < p \neq 2 < \infty$.*

(1) *If $p > 2$, the topologies* SOT *and* SOT* *on $\mathcal{B}_1(X)$ are similar, and the topologies* WOT *and* $\mathtt{SOT}_*$ *are similar.*
(2) *If $1 < p < 2$, the topologies* $\mathtt{SOT}_*$ *and* SOT* *on $\mathcal{B}_1(X)$ are similar, and the topologies* WOT *and* SOT *are similar.*

It follows immediately that a property (P) of contractions on ℓ_p, $p > 2$ is SOT - typical (resp. WOT - typical) if and only if it is SOT*- typical (resp. $\mathtt{SOT}_*$- typical). Analogous statements hold in the case $1 < p < 2$.

However, Theorem 1.2 falls short of proving, for instance, that an SOT-typical contraction on the complex ℓ_p-space for $1 < p < 2$ has no eigenvalue – which is one of the nagging questions left open in [14] – while a positive answer to the Question 1.1 would yield a weight of interesting corollaries concerning typical properties of contractions on ℓ_p.

Our main result provides a positive answer to Question 1.1 in the particular case where X is a *real* ℓ_p-space with $p = 3$ or $p = 3/2$:

Theorem 1.3. — *If X is a real ℓ_p-space with $p = 3$ or $p = 3/2$, then all topologies on $\mathcal{B}_1(X)$ lying between the two topologies* WOT *and* SOT* *are similar.*

It is tempting to conjecture that the same result holds true for any real ℓ_p-space with $1 < p \neq 2 < \infty$, but we have been unable to prove it for any other value of p than $p = 3$ and $p = 3/2$. We do not take the risk of making the same conjecture for complex ℓ_p-spaces: the situation might well be different.

1.3. About the proofs. — As it turns out, the question of the similarity of two topologies τ and τ' on some abstract set $\mathbf{M}$ is very much enlightened by the consideration of the *points of continuity* of the formal identity map between the spaces $(\mathbf{M}, \tau)$ and $(\mathbf{M}, \tau')$. This point of view will be our starting point for the proofs of Theorem 1.2 and Theorem 1.3. It should be noted that the proof of Theorem 1.2 relies heavily on some results from [14], and that Theorem 1.2 could have been proved using only the methods of [14]; but the "points of continuity approach" makes everything much more transparent. As to the proof of Theorem 1.3, it requires quite a lot of additional work besides the use of points of continuity. In particular, a thorough study of the *norming vectors* for certain classes of contractions on the real ℓ_3-space will be needed.

1.4. Organization of the paper. — We develop the points of continuity approach in Section 2. As a first illustration we give some natural examples of similar topologies. We also describe the continuity points of the identity map between $(\mathbf{M}, \tau)$ and $(\mathbf{M}, \tau')$ when $\mathbf{M} = \mathcal{B}_1(\ell_2)$ and τ, τ' are chosen among WOT, SOT, SOT_*, SOT* (Proposition 2.11). Finally, we show that for certain classes of Banach spaces X, all contractions $T \in \mathcal{B}_1(X)$ with enough norming vectors are points of WOT-SOT continuity (Proposition 2.14); a result that will be essential for the proof of Theorem 1.3. Theorem 1.2 is proved in Section 3, and Theorem 1.3 is proved in Section 4. Thanks to the points of continuity approach, things boil down to showing that the contractions with enough norming vectors are SOT-dense in $\mathcal{B}(\ell_3)$ (Proposition 4.2). The remaining sections are not directly related to Theorem 1.2 and Theorem 1.3, but they fit naturally into the landscape. In Section 5, we show that the points of continuity approach allows to retrieve in a very direct way the main Hilbertian typicality results from [9] and [10]. In Section 6, we show that, in strong contrast to what happens for the operator-norm topology, an SOT*-typical $T \in \mathcal{B}_1(\ell_p)$, $1 < p < \infty$ does not attain its norm. Finally, Section 7 contains additional facts concerning similar topologies, other examples of points of WOT-SOT continuity, and a few typicality results pertaining to Fredholm theory.

1.5. Notation. — We will denote by $\overline{\mathbb{D}}$ the closed unit disk in $\mathbb{C}$, and by $\mathbb{T}$ the unit circle. The letter X will always denote a real or complex infinite-dimensional separable Banach space, and the scalar field will be denoted by $\mathbb{K}$. The closed unit ball of X will be denoted by B_X. The canonical basis of ℓ_p or c_0 is $(e_j)_{j \in \mathbb{Z}_+}$, and (e_j^*) is the associated sequence of coordinate functionals. If $N \in \mathbb{Z}_+$ we set $E_N := \mathrm{span}(e_0, \ldots, e_N)$, and we denote by P_N the

canonical projection of X onto E_N. The letter $\mathbf{M}$ will stand for an abstract set potentially equipped with several topologies. If τ and τ' are two topologies on $\mathbf{M}$, we denote by $\mathcal{C}(\tau,\tau')$ the set of all points of continuity of the identity map $\mathbf{i}_{\tau,\tau'} : (\mathbf{M},\tau) \to (\mathbf{M},\tau')$.

2. Similarity and points of continuity

2.1. Two general facts. — We start by presenting two very simple facts concerning similarity of topologies on an abstract set $\mathbf{M}$. The first one is well known; see e.g. [1, Proposition 9] or [3, Theorem 2.2].

Lemma 2.1. — *Let τ and τ' be two topologies on $\mathbf{M}$. If τ and τ' are similar, then they have the same comeager sets.*

Proof. — It is enough to show that if τ and τ' are similar, then they have the same nowhere dense sets. Now, given a topology on $\mathbf{M}$, a set $E \subseteq \mathbf{M}$ is nowhere dense if and only if the following holds true: for any set $A \subseteq \mathbf{M}$ with non-empty interior, one can find a set B with non-empty interior such that $B \subseteq A$ and $B \cap E = \varnothing$. The lemma follows immediately. □

Remark 2.2. — Topologies sharing the same comeager sets may not be similar. For example, on $\mathbf{M} := \mathbb{R}$, let τ be the usual topology and let τ' be the topology generated by $\tau \cup \{\mathbb{R}\backslash\mathbb{Q}\}$, so that a set $V' \subseteq \mathbb{R}$ is τ' - open if and only if $V' = V \cup \big(W \cap (\mathbb{R}\backslash\mathbb{Q})\big)$ where V, W are τ - open. By the Baire Category Theorem and since $\tau \subset \tau'$, any closed set with empty interior in $(\mathbb{R},\tau)$ is also closed with empty interior in $(\mathbb{R},\tau')$, and hence any τ - meager set is τ'- meager. Conversely, since every τ' - closed set $C' \subseteq \mathbb{R}$ has the form $C' = C \cup D$ where C is τ - closed and $D \subseteq \mathbb{Q}$, any closed set with empty interior in $(\mathbb{R},\tau')$ is τ - meager and hence any τ'- meager set is τ - meager. So τ and τ' share the same comeager sets; but they are not similar since $\mathbb{Q}$ is not dense in $(\mathbb{R},\tau')$. However, for the operator topologies we will be considering, the two properties are in fact equivalent; see Proposition 7.1.

Our second lemma shows the relevance of points of continuity of identity maps when investigating the similarity of two topologies.

Lemma 2.3. — *Let τ and τ' be two topologies on $\mathbf{M}$.*

(a) *For any set $D \subseteq \mathbf{M}$, we have $\overline{D}^{\tau} \cap \mathcal{C}(\tau,\tau') \subseteq \overline{D}^{\tau'}$.*
(b) *If $D \subseteq \mathbf{M}$ is τ - dense and τ' - closed, then $D \supseteq \mathcal{C}(\tau,\tau')$.*
(c) *If $\mathcal{C}(\tau,\tau')$ is τ' - dense in $\mathbf{M}$ and $\mathcal{C}(\tau',\tau)$ is τ - dense, then τ and τ' are similar.*

Proof. — (a) If $x \in \overline{D}^{\tau} \cap \mathcal{C}(\tau,\tau')$, one can find a net $(z_i) \subseteq D$ such that $z_i \xrightarrow{\tau} x$; and then $z_i \xrightarrow{\tau'} x$ because $x \in \mathcal{C}(\tau,\tau')$.

(b) follows immediately from (a).

(c) By symmetry, it is enough to show that if $\mathcal{C}(\tau,\tau')$ is τ' - dense in $\mathbf{M}$, then any τ - dense subset of $\mathbf{M}$ is τ'- dense; which is obvious: if $D \subseteq \mathbf{M}$ is τ - dense, then $\mathcal{C}(\tau,\tau') \subseteq \overline{D}^{\tau'}$ by (a), and hence D is τ'- dense. □

Remark 2.4. — The assumption of part (c) above is satisfied in particular if $\tau \subseteq \tau'$ and $\mathcal{C}(\tau,\tau')$ is τ'- dense in $\mathbf{M}$. This is the situation we will mostly consider in what follows.

2.2. Examples of similar topologies. — The next proposition is a concrete illustration of Lemma 2.3. Recall that a Banach space X is said to have the *Kadec-Klee property* if the following holds true: if (x_n) is a sequence in B_X such that $x_n \xrightarrow{w} x$ where $\|x\| = 1$, then in fact $\|x_n - x\| \to 0$. (This is the "sequential" definition of Huff [17].) For example, ℓ_p has the Kadec-Klee property for every $1 \leqslant p < \infty$. One defines in the same way the w^*- *Kadec-Klee property* for a dual Banach space $X = Z^*$. For example, $\ell_1 = c_0^*$ has the w^*- Kadec-Klee property; and so does ℓ_p for any $1 < p < \infty$ since KK and w^*- KK are the same properties for reflexive spaces.

Proposition 2.5. — *Let X be a Banach space with separable dual (resp. a separable dual space) having the Kadec-Klee property (resp. the w^*- Kadec-Klee property) , and let $\mathbf{M} := B_X$, the closed unit ball of X. If τ is any topology on B_X finer than the weak (resp. the weak*) topology , weaker than the norm topology and such that the unit sphere S_X is τ - dense in B_X, then the topologies τ and w (resp. τ and w^*) are similar.*

Proof. — If X^* is separable (so that (B_X, w) is metrizable) and X has the Kadec-Klee property, then $S_X \subseteq \mathcal{C}(w, \|\cdot\|)$ and hence $S_X \subseteq \mathcal{C}(w, \tau)$; and similarly, if X is a separable dual space with the w^*- Kadec-Klee property then $S_X \subseteq \mathcal{C}(w^*, \tau)$. So the result follows immediately from Lemma 2.3. □

Remark. — It follows in particular from Corollary 2.5 that if X is a separable dual space with the w^*- Kadec-Klee property (e.g. if $X = \ell_1$), then the weak topology and the weak* topology on B_X are similar. This could of course have been proved directly.

The following special case of Proposition 2.5 allows to construct lots of similar Polish topologies on the unit ball of a separable dual space with the w^*- Kadec-Klee property.

Corollary 2.6. — *Let $X = Z^*$ be a separable dual Banach space, and let $\mathbf{M} := B_{Z^*}$. Let also $\mathbf{q} = (q_i)_{i\in I}$ be a separating countable family of continuous and w^*- lower semicontinuous seminorms on Z^*. Denote by τ the topology on B_{Z^*} generated by $\mathbf{q}$.*

(i) *The topology τ is Polish, finer than the weak* topology restricted to B_{Z^*}, and weaker than the norm topology.*
(ii) *If $X = Z^*$ has the w^*- Kadec-Klee property and $\inf_{\|h\|=1} \max_{i\in F} q_i(h) = 0$ for every finite set $F \subseteq I$, then the topologies τ and w^* are similar.*

Proof. — (i) Since the seminorms q_i are continuous, τ is weaker than the norm topology.

Let us show that τ is finer than w^* restricted to B_{Z^*}. Since each seminorm q_i is w^*- lower semicontinuous, we can write for each $i \in I$ the set $\{x \in X \ ; \ q_i(x) \leqslant 1\}$ as

$$\bigcap_{z\in A_i} \{z^* \in Z^*; \ |\langle z^*, z\rangle| \leqslant 1\}$$

for some set $A_i \subseteq Z$. Since the family $\mathbf{q}$ is separating, we then have $\overline{\operatorname{span}} \bigcup_{i\in I} A_i = Z$; and the result follows easily since we are working on the bounded set B_{Z^*}.

Since $\mathbf{q}$ is a countable family of seminorms, the topology τ is metrizable; namely, assuming as we may that $\mathbf{q}$ is uniformly bounded (on bounded sets) and that $I \subseteq \mathbb{N}$, it is generated by the metric $d(x, y) := \sum_{i\in I} 2^{-i} q_i(x - y)$. Since (B_{Z^*}, w^*) is compact and the seminorms q_i are w^*- lower semicontinuous, it is not hard to check that (B_{Z^*}, d) is complete. Hence τ is completely metrizable. Moreover, (B_{Z^*}, τ) is separable since τ is weaker than the norm topology of B_{Z^*}. So (B_{Z^*}, τ) is Polish.

(ii) By Proposition 2.5, in order to show that τ and w^* are similar, it is enough to prove that S_X is τ-dense in B_X. Let $x_0 \in B_X$ be arbitrary, let $\varepsilon > 0$, and let F be any finite subset of I: we need to find a vector $x \in S_X$ such that $q_i(x - x_0) < \varepsilon$ for all $i \in F$. By our assumption on $\mathbf{q}$, one can find some $h \in X$ with $\|h\| = 1$ such that $q_i(h) \leqslant \varepsilon/2$ for all $i \in F$. Then $q_i\big((x_0 + th) - x_0\big) \leqslant t\varepsilon/2$ for any $t \in \mathbb{R}_+$ and all $i \in F$. Now, we have $\|x_0\| \leqslant 1$ and $\|x_0 + 2h\| \geqslant 2\|h\| - \|x_0\| \geqslant 1$. So we can find $t \in [0, 2]$ such that $\|x_0 + th\| = 1$; and then the vector $x := x_0 + th$ has the required properties. □

Example 2.7. — Let $1 \leqslant p < \infty$, and let $\mathbf{M}$ be the unit ball of $\ell_p(\mathbb{Z}_+ \times \mathbb{Z}_+)$. Consider the "row topology" τ and the "column topology" τ' on $\mathbf{M}$, defined as follows: τ is generated by the seminorms $q_i(x) := \left(\sum_{j=0}^{\infty} |x_{i,j}|^p\right)^{1/p}$, $i \in \mathbb{Z}_+$, and τ' is generated by the seminorms $q'_j(x) := \left(\sum_{i=0}^{\infty} |x_{i,j}|^p\right)^{1/p}$, $j \in \mathbb{Z}_+$. Then τ and τ' are (clearly incomparable) similar Polish topologies on $\mathbf{M}$.

Proof. — Each seminorm q_i is w^*-lower semicontinuous, being the supremum of the w^*-continuous seminorms $q_{i,N}(x) := \left(\sum_{j=0}^{N} |x_{i,j}|^p\right)^{1/p}$, $N \geqslant 0$; and we have $\bigcap_{i \in F} \ker(q_i) \neq \{0\}$ for any finite set $F \subseteq \mathbb{Z}_+$. The same is true for the seminorms q'_j, so the result follows from Corollary 2.6. □

Example 2.8. — Let $1 \leqslant p < \infty$ and let $\mathbf{M} := B_{\ell_p}$. For any bounded sequence of positive real numbers $\mathbf{w} = (w_n)_{n \geqslant 0}$ and any $p \leqslant q \leqslant \infty$, denote by $\|\cdot\|_{\mathbf{w},q}$ the associated weighted ℓ_q-norm on $X := \ell_p$, *i.e.*

$$\|x\|_{\mathbf{w},q} := \|\mathbf{w}x\|_q = \left(\sum_{n=0}^{\infty} w_n |x_n|^q\right)^{1/q}, \quad x \in \ell_p.$$

Then $(B_{\ell_p}, \|\cdot\|_{\mathbf{w},q})$ is Polish for any choice of $(\mathbf{w}, q)$. Moreover, if $\mathbf{w}$ and $\mathbf{w}'$ are two weight sequences such that $\inf_{n \in \mathbb{Z}_+} w_n = 0 = \inf_{n \in \mathbb{Z}_+} w'_n$, then the topologies generated by $\|\cdot\|_{\mathbf{w},q}$ and $\|\cdot\|_{\mathbf{w}',q'}$ are similar for any choice of $q, q' \geqslant p$. However, these topologies are incomparable as soon as $\inf_{n \in \mathbb{Z}_+} w_n/w'_n = 0 = \inf_{n \in \mathbb{Z}_+} w'_n/w_n$.

Proof. — Since $\inf_{\|h\|=1} \|h\|_{\mathbf{w},q} \leqslant \inf_{n \in \mathbb{Z}_+} \|e_n\|_{\mathbf{w},q} = \inf_{n \in \mathbb{Z}_+} w_n$, this follows again from Corollary 2.6. □

2.3. Operator topologies. — Let us now consider the case where $\mathbf{M} = \mathcal{B}_1(X)$, with two different topologies chosen within the set $\{\mathtt{WOT}, \mathtt{SOT}, \mathtt{SOT}_*, \mathtt{SOT}^*\}$. The following lemma is well known, but we give a proof for completeness' sake. Recall that a map $f : E \to E'$ between topological spaces E and E' is said to be *Borel* 1 if $f^{-1}(V)$ is F_σ in E for any open set $V \subseteq E'$. (This notion is really interesting only if all open sets in E are F_σ, e.g. if E is metrizable, since otherwise continuous maps need not be Borel 1; but it makes sense for arbitrary topological spaces.) It is well known that if E' is second-countable, then the set of continuity points of any Borel 1 map $f : E \to E'$ is G_δ and comeager in E (see e.g. [20, Theorem 24.14]).

Lemma 2.9. — *Let X be a Banach space, and let $\mathbf{M} := \mathcal{B}_1(X)$.*

(1) *if X is separable, then the identity map $\mathbf{i}_{\mathtt{WOT},\mathtt{SOT}}$ is Borel 1.*
(2) *if X^* is separable, then the map $\mathbf{i}_{\mathtt{WOT},\mathtt{SOT}^*}$ is Borel 1, and hence $\mathbf{i}_{\mathtt{WOT},\mathtt{SOT}}$, $\mathbf{i}_{\mathtt{WOT},\mathtt{SOT}_*}$ and $\mathbf{i}_{\mathtt{SOT},\mathtt{SOT}^*}$ are also Borel 1.*

Proof. — (1) We have to show that any $\mathtt{SOT}$-open set $\mathcal{U} \subseteq \mathcal{B}_1(X)$ is $\mathtt{WOT}$-F_σ. Since X is separable, $\mathcal{U}$ is a countable union of finite intersections of sets of the form

$$\mathcal{V}(A, x, \varepsilon) := \big\{T \in \mathcal{B}_1(X);\ \|(T - A)x\| < \varepsilon\big\},$$

where $A \in \mathcal{B}_1(X)$, $x \in X$ and $\varepsilon > 0$. Now, if $T \in \mathcal{B}_1(X)$, then

$$T \in \mathcal{V}(A, x, \varepsilon) \iff \exists n \in \mathbb{N} \Big(\forall x^* \in B_{X^*}\ :\ |\langle x^*, (T - A)x\rangle| \leqslant \varepsilon - \frac{1}{n}\Big).$$

This shows that each set $\mathcal{V}(A, x, \varepsilon)$ is $\mathtt{WOT}$-F_σ; and hence that $\mathcal{U}$ is $\mathtt{WOT}$-F_σ as well.

(2) Assume that X^* is separable, and let us show that $\mathbf{i}_{\mathtt{WOT},\mathtt{SOT}^*}$ is Borel 1. Let $\mathcal{U} \subseteq \mathcal{B}_1(X)$ be $\mathtt{SOT}^*$-open. Then $\mathcal{U}$ is a countable union of finite intersections of sets either of the form $\mathcal{V}(A, x, \varepsilon)$, or of the form $\mathcal{V}^*(A, x^*, \varepsilon) := \big\{T \in \mathcal{B}_1(X);\ \|(T - A)^*x^*\| < \varepsilon\big\}$, where $A \in \mathcal{B}_1(X)$, $x^* \in X^*$ and $\varepsilon > 0$. We already know that each set $\mathcal{V}(A, x, \varepsilon)$ is $\mathtt{WOT}$-F_σ. Moreover, if $T \in \mathcal{B}_1(X)$ then

$$T \in \mathcal{V}(A, x^*, \varepsilon) \iff \exists n \in \mathbb{N} \Big(\forall x \in B_X\ :\ |\langle x^*, (T - A)x\rangle| \leqslant \varepsilon - \frac{1}{n}\Big).$$

So each set $\mathcal{V}(A, x^*, \varepsilon)$ is $\mathtt{WOT}$-F_σ, and hence $\mathcal{U}$ is $\mathtt{WOT}$-F_σ. □

From Lemma 2.9 and since all the relevant topologies are Polish, we immediately deduce the following corollary:

Corollary 2.10. — *If X^* is separable, $\mathcal{C}(\mathtt{SOT}, \mathtt{SOT}^*)$ is a dense G_δ subset of $(\mathcal{B}_1(X), \mathtt{SOT})$. If X is reflexive and separable, then $\mathcal{C}(\mathtt{WOT}, \mathtt{SOT})$, $\mathcal{C}(\mathtt{WOT}, \mathtt{SOT}_*)$ and $\mathcal{C}(\mathtt{WOT}, \mathtt{SOT}^*)$ are dense G_δ subsets of $(\mathcal{B}_1(X), \mathtt{WOT})$.*

2.4. The Hilbertian case. — When X is the Hilbert space ℓ_2, it turns out that one can identify the points of continuity for each one of the five possible pairings (τ, τ') of the topologies $\mathtt{WOT}$, $\mathtt{SOT}$, $\mathtt{SOT}_*$, $\mathtt{SOT}^*$ such that $\tau \subseteq \tau'$. This will be used in Section 5.

Proposition 2.11. — *Let $\mathbf{M} := \mathcal{B}_1(\ell_2)$.*

(1) *$\mathcal{C}(\mathtt{WOT}, \mathtt{SOT})$ is the set of all isometries.*
(2) *$\mathcal{C}(\mathtt{WOT}, \mathtt{SOT}^*)$ is the set of all unitary operators.*
(3) *$\mathcal{C}(\mathtt{WOT}, \mathtt{SOT}_*)$ is the set of all co-isometries.*
(4) *$\mathcal{C}(\mathtt{SOT}, \mathtt{SOT}^*)$ is the set of all co-isometries.*
(5) *$\mathcal{C}(\mathtt{SOT}_*, \mathtt{SOT}^*)$ is the set of all isometries.*

Proof. — Since the topology $\mathtt{WOT}$ is self-adjoint, it is rather clear that an operator T belongs to $\mathcal{C}(\mathtt{WOT}, \mathtt{SOT}^*)$ if and only if T and T^* both belong to $\mathcal{C}(\mathtt{WOT}, \mathtt{SOT})$. So (2) follows from (1). Also, T belongs to $\mathcal{C}(\mathtt{WOT}, \mathtt{SOT}_*)$ if and only if T^* belongs to $\mathcal{C}(\mathtt{WOT}, \mathtt{SOT})$, so (3) follows from (1). Finally, since $\mathtt{SOT}^*$ is self-adjoint, T belongs to $\mathcal{C}(\mathtt{SOT}_*, \mathtt{SOT}^*)$ if and only if T^* belongs to $\mathcal{C}(\mathtt{SOT}, \mathtt{SOT}^*)$, so (5) follows from (4). Hence we concentrate on the proofs of assertions (1) and (4).

Since the norm of ℓ_2 has the Kadec-Klee Property, it is clear that if (T_n) is a sequence of operators in $\mathcal{B}_1(\ell_2)$ such that $T_n \xrightarrow{\mathtt{WOT}} J$ where J is an isometry, then in fact $T_n \xrightarrow{\mathtt{SOT}} J$. In other words, any isometry belongs to $\mathcal{C}(\mathtt{WOT}, \mathtt{SOT})$. Since $\mathtt{SOT}$-convergence implies $\mathtt{WOT}$-convergence for the adjoints, this implies that any co-isometry belongs to $\mathcal{C}(\mathtt{SOT}, \mathtt{SOT}^*)$.

Let us denote by $\mathcal{I}$ the set of all isometries, and by $\mathcal{I}_*$ the set of all co-isometries. By what has just been said, it remains to prove that $\mathcal{I} \supseteq \mathcal{C}(\mathtt{WOT}, \mathtt{SOT})$ and that $\mathcal{I}_* \supseteq \mathcal{C}(\mathtt{SOT}, \mathtt{SOT}^*)$. Moreover since $\mathcal{I}$ is obviously $\mathtt{SOT}$-closed in $\mathcal{B}_1(\ell_2)$ and since $\mathcal{I}_*$ is $\mathtt{SOT}^*$-closed, Lemma

2.3 (b) tells us that we just have to check the following facts: $\mathcal{I}$ is $\mathtt{WOT}$-dense in $\mathcal{B}_1(\ell_2)$, and $\mathcal{I}_*$ is $\mathtt{SOT}$-dense. Finally, since the topology $\mathtt{WOT}$ is self-adjoint and weaker than $\mathtt{SOT}$, it is in fact enough to show that $\mathcal{I}_*$ is $\mathtt{SOT}$-dense.

Let $T \in \mathcal{B}_1(\ell_2)$ be arbitrary. For any $N \in \mathbb{N}$, set $A_N := TP_N$ and

$$T_N := A_N + (I - A_N A_N^*)^{1/2} B^{N+1},$$

where B is the canonical backward shift on ℓ_2. A straightforward computation shows that T_N^* is an isometry, *i.e.* $T_N \in \mathcal{I}_*$. Moreover, since $A_N \xrightarrow{\mathtt{SOT}} T$ and $\|I - A_N A_N^*\| \leqslant 1$, it is also clear that $T_N \xrightarrow{\mathtt{SOT}} T$. This concludes the proof of Proposition 2.11. □

Remark 2.12. — Let X be a reflexive (separable) Banach space, and let $\mathbf{M} := \mathcal{B}_1(X)$. The beginning of the above proof shows the following: if the norm of X has the Kadec-Klee property, then any isometry $J \in \mathcal{B}_1(X)$ belongs to $\mathcal{C}(\mathtt{WOT}, \mathtt{SOT})$. It follows that if the norm of X^* has the Kadec-Klee property, then any co-isometry belongs to $\mathcal{C}(\mathtt{WOT}, \mathtt{SOT}_*)$ and hence to $\mathcal{C}(\mathtt{SOT}, \mathtt{SOT}^*)$.

2.5. Norming vectors and points of continuity. — In this sub-section, we elaborate on Remark 2.12 above. In particular, we are going to show that if X is a reflexive (separable) Banach space with the Kadec-Klee property, then any operator $T \in \mathcal{B}_1(X)$ with enough norming vectors belongs to $\mathcal{C}(\mathtt{WOT}, \mathtt{SOT})$. This will turn out to be crucial for the proof of Theorem 1.3 (see Section 4).

Given a (separable) Banach space X, let us denote by $\mathcal{C}(w, \|\cdot\|)$ the set of all points of continuity of the identity map $\mathbf{i}_{w,\|\cdot\|} : (B_X, w) \to (B_X, \|\cdot\|)$. Note that $\mathcal{C}(w, \|\cdot\|)$ is contained in the unit sphere S_X. Moreover, if X is reflexive then $\mathcal{C}(w, \|\cdot\|)$ is dense in (B_X, w) because (B_X, w) is compact metrizable (hence Polish) and the identity map $\mathbf{i}_{w,\|\cdot\|}$ is Borel 1.

Proposition 2.13. — *Let X be reflexive, and let $\mathbf{M} := \mathcal{B}_1(X)$. If $T \in \mathcal{B}_1(X)$ is such that* $\operatorname{span}\{z \in S_X;\ Tz \in \mathcal{C}(w, \|\cdot\|)\}$ *is dense in X, then $T \in \mathcal{C}(\mathtt{WOT}, \mathtt{SOT})$.*

Proof. — Let (T_n) be a sequence in $\mathcal{B}_1(X)$ such that $T_n \xrightarrow{\mathtt{WOT}} T$. We have to show that $T_n \xrightarrow{\mathtt{SOT}} T$, *i.e.* that $T_n x \to Tx$ for every $x \in X$. Now, if $z \in S_X$ is such that $Tz \in \mathcal{C}(w, \|\cdot\|)$, then obviously $T_n z \xrightarrow{\|\cdot\|} Tz$. Since the sequence (T_n) is bounded, the result follows. □

Recall that if $T \in \mathcal{B}(X)$, a vector $z \in X\backslash\{0\}$ is said to be *norming* for T if $\|Tz\| = \|T\|\|z\|$. We denote by $\mathcal{N}(T)$ the set of all norming vectors for T:

$$\mathcal{N}(T) := \{z \in X\backslash\{0\};\ \|Tz\| = \|T\|\|z\|\}.$$

Proposition 2.14. — *Let X be reflexive, and assume that X has the Kadec-Klee property. If $T \in \mathcal{B}_1(X)$ is such that $\|T\| = 1$ and* $\operatorname{span}(\mathcal{N}(T))$ *is dense in X, then $T \in \mathcal{C}(\mathtt{WOT}, \mathtt{SOT})$.*

Proof. — This follows immediately from Proposition 2.13. Indeed, by the Kadec-Klee property, we have $\mathcal{C}(w, \|\cdot\|) = S_X$; and this means that the set $\{z \in S_X;\ Tz \in \mathcal{C}(w, \|\cdot\|)\}$ is equal to $\mathcal{N}(T) \cap S_X$. □

Proposition 2.14 has the following consequence, which will be our main tool for proving Theorem 1.3 in Section 4.

Corollary 2.15. — *Let X be reflexive, and assume that X has the Kadec-Klee property. If the set $\{T \in \mathcal{B}_1(X);\ \overline{\mathrm{span}}(\mathcal{N}(T)) = X\}$ is* SOT *- dense in $\mathcal{B}_1(X)$, then the topologies* WOT *and* SOT *on $\mathcal{B}_1(X)$ are similar.*

Proof. — Under the above assumption, $\mathcal{C}(\mathtt{WOT}, \mathtt{SOT})$ is SOT - dense in $\mathcal{B}_1(X)$ by Proposition 2.14; so we may apply Lemma 2.3. □

Finally, here is another, perhaps unexpected, consequence of Proposition 2.14.

Corollary 2.16. — *If $T \in \mathcal{B}_1(\ell_2)$ is such that $\|T\| = 1$ and $\overline{\mathrm{span}}\,(\mathcal{N}(T)) = \ell_2$, then T is an isometry.*

Proof. — This follows from Proposition 2.14 and part (1) of Proposition 2.11. □

3. Real or complex ℓ_p - spaces with $p \neq 2$

Our aim in this section is to prove Theorem 1.2. The key point is the following proposition.

Proposition 3.1. — *Let $M = \mathcal{B}_1(\ell_p)$. If $p > 2$, then $\mathcal{C}(\mathtt{WOT}, \mathtt{SOT}_*)$ and $\mathcal{C}(\mathtt{SOT}, \mathtt{SOT}^*)$ are* SOT* *- dense in M; and if $1 < p < 2$, then $\mathcal{C}(\mathtt{WOT}, \mathtt{SOT})$ and $\mathcal{C}(\mathtt{SOT}_*, \mathtt{SOT}^*)$ are* SOT* *- dense in M.*

The proof of Proposition 3.1 relies on Propositions 5.15 and 5.16 from [14]. For the sake of readability, we restate here what is precisely needed. Recall that if $M \in \mathbb{Z}_+$ then $E_M = \mathrm{span}(e_0, \dots, e_M) \subseteq \ell_p$. Identifying an operator $B \in \mathcal{B}(E_M)$ with $P_M B P_M \in \mathcal{B}(\ell_p)$, we consider $\mathcal{B}(E_M)$ as a subspace of $\mathcal{B}(\ell_p)$.

Lemma 3.2. — *Let $1 < p < \infty$. For any* SOT* *- open set $\mathcal{U} \neq \varnothing$ in $\mathcal{B}_1(\ell_p)$ and any $n_0 \in \mathbb{Z}_+$, one can find $M \geqslant n_0$ and $B \in \mathcal{B}_1(E_M)$ such that $B \in \mathcal{U}$ and B has the following additional properties: $\|B\| = 1$, and B admits a norming vector $z \in E_M$ such that $\langle e_m^*, Bz\rangle \neq 0$ for all $m \in [\![0, M]\!]$.*

Proof. — This follows from [14, Proposition 5.15]. □

Lemma 3.3. — *Assume that $p > 2$. Let $M \in \mathbb{Z}_+$, and let $B \in \mathcal{B}_1(E_M)$. Assume that $\|B\| = 1$, and that B admits a norming vector $z \in E_M$ such that $\langle e_m^*, Bz\rangle \neq 0$ for all $m \in [\![0, M]\!]$. For any $\varepsilon > 0$, there exists $\delta > 0$ such that the following holds true: if $T \in \mathcal{B}_1(\ell_p)$ satisfies $\|P_M(T - B)P_M\| < \delta$, then $\|P_M(T - B)\| < \varepsilon$.*

Proof. — This is essentially [14, Proposition 5.16]. We outline here an argument, which is somewhat simpler than the one provided in [14].

Everything relies on the following elementary fact: since $p \geqslant 2$, there exists a constant $\alpha > 0$ such that $|1+w|^p + |1-w|^p \geqslant 2 + \alpha|w|^2$ for all $w \in \mathbb{C}$. More precisely, by [19, Lemma 2.1] one can take $\alpha := p$, so that (by homogeneity)

$$|u + v|^p + |u - v|^p \geqslant 2|u|^p + p|u|^{p-2}|v|^2 \qquad \text{for all } u, v \in \mathbb{C}. \tag{1}$$

Let us fix $\varepsilon > 0$. Let also $\delta > 0$ to be specified, and let $T \in \mathcal{B}_1(\ell_p)$ be such that $\|P_M(T - B)P_M\| < \delta$. Since $B = P_M B P_M$, it is enough to show that if δ is small enough, then $\|P_M T(I - P_M)\| < \varepsilon$.

Let $z \in S_{E_M}$ be a norming vector for B such that $\langle e_m^*, Bz\rangle \neq 0$ for all $m \in [\![0, M]\!]$. Since $\|Bz\| = 1$, $z \in E_M$ and $P_M B P_M = B$, we have $\|P_M T z\| > 1 - \delta$ and, if δ is small enough, $|\langle e_m^*, Tz\rangle| \geqslant \gamma/2$ for all $m \in [\![0, M]\!]$, where $\gamma := \min\big\{|\langle e_m^*, Bz\rangle|;\ m \in [\![0, M]\!]\big\} > 0$.

Now, let $x \in \ell_p$ with $\|x\| = 1$, and let $y := (I - P_M)x$. By (1), we have for every $s > 0$ and all $m \in [\![0, M]\!]$:

$$ps^2\,|\langle e_m^*, Tz\rangle|^{p-2}\,|\langle e_m^*, Ty\rangle|^2 \leqslant |\,\langle e_m^*, Tz + sTy\rangle\,|^p + |\,\langle e_m^*, Tz - sTy\rangle\,|^p - 2\,|\,\langle e_m^*, Tz\rangle\,|^p.$$

Summing over $m \in [\![0, M]\!]$ and assuming that δ is small enough, it follows that

$$\begin{aligned} ps^2\,(\gamma/2)^{p-2} \sum_{m=0}^{M} |\,\langle e_j^*, Ty\rangle\,|^2 &\leqslant \|P_M T(z + sy)\|^p + \|P_M T(z - sy)\|^p - 2\,\|P_M T z\|^p \\ &\leqslant \|z + sy\|^p + \|z - sy\|^p - 2(1 - \delta)^p. \end{aligned}$$

Therefore, since $\|z \pm sy\|^p = \|z\|^p + s^p\|y\|^p \leqslant 1 + s^p$ and $\sum_{m=0}^{M} |\,\langle e_m^*, Ty\rangle\,|^2 \geqslant \|P_M T y\|^2$ (because $p \geqslant 2$), we obtain, setting $c := p(\gamma/2)^{p-2}$ and $\eta := 1 - (1 - \delta)^p$, that

$$\|P_M T(I - P_M)x\|^2 \leqslant \frac{2}{c}\left(\frac{\eta}{s^2} + s^{p-2}\right),$$

for any $x \in S_{\ell_p}$ and all $s > 0$.

Optimising with respect to s, *i.e.* taking $s := \left(\frac{2\eta}{p-2}\right)^{1/p}$, it follows that

$$\|P_M T(I - P_M)\|^2 \leqslant K\,\eta^{1-\frac{2}{p}},$$

for some constant K independent of T and η. Since $\eta = 1 - (1 - \delta)^p$, this shows that $\|P_M T(I - P_M)\| < \varepsilon$ if δ is small enough. □

Proof of Proposition 3.1. — By duality, it is enough to consider the case $p > 2$. Moreover, since $\mathtt{SOT}$ is stronger than $\mathtt{WOT}$ and since $\mathtt{SOT}^* = \mathtt{SOT} \vee \mathtt{SOT}_*$, it is clear that $\mathcal{C}(\mathtt{WOT}, \mathtt{SOT}_*) \subseteq \mathcal{C}(\mathtt{SOT}, \mathtt{SOT}^*)$. So it is enough to show that $\mathcal{C}(\mathtt{WOT}, \mathtt{SOT}_*)$ is $\mathtt{SOT}^*$-dense in $\mathbf{M} = \mathcal{B}_1(\ell_p)$, $p > 2$.

For any $A, B \in \mathcal{B}_1(\ell_p)$, let us set

$$d_*(A, B) := \sum_{n=0}^{\infty} 2^{-n}\|P_n(B - A)\|.$$

Since $\|P_n(B - A)\| = \|(B^* - A^*)P_n^*\|$ for all $n \geqslant 0$, the metric d_* generates the topology $\mathtt{SOT}_*$. Therefore, if we define the sets

$$\mathcal{V}_\eta = \Big\{B \in \mathcal{B}_1(\ell_p);\ \exists \mathcal{W}\ \mathtt{WOT}\text{-neighbourhood of } B\ :\ d_*\text{-diam}\big(\mathcal{W}\big) < \eta\Big\},$$

then

$$\mathcal{C}(\mathtt{WOT}, \mathtt{SOT}_*) = \bigcap_{\eta > 0} \mathcal{V}_\eta = \bigcap_{k \in \mathbb{N}} \mathcal{V}_{1/k}.$$

Now, the sets $\mathcal{V}_\eta$ are $\mathtt{WOT}$-open in $\mathcal{B}_1(\ell_p)$, hence $\mathtt{SOT}^*$-open. So, by the Baire Category Theorem applied to $(\mathcal{B}_1(\ell_p), \mathtt{SOT}^*)$, it is enough to show that $\mathcal{V}_\eta$ is $\mathtt{SOT}^*$-dense in $\mathcal{B}_1(\ell_p)$ for every $\eta > 0$.

Let us fix $\eta > 0$, and let $\mathcal{U}$ be a non-empty open set in $(\mathcal{B}_1(\ell_p), \mathtt{SOT}^*)$. Choose $n_0 \in \mathbb{N}$ such that $\sum_{n > n_0} 2^{-n} < \eta/4$. By Lemmas 3.2 and 3.3, one can find an operator $B \in \mathcal{U}$ and an integer $M \geqslant n_0$ such that the following holds true: for any $\varepsilon > 0$, there exists $\delta > 0$ such that for any $T \in \mathcal{B}_1(\ell_p)$:

$$\|P_M(T - B)P_M\| < \delta \implies \|P_M(T - B)\| < \varepsilon.$$

Now, choose $\varepsilon := \eta/8$, let δ satisfy the above property, and define

$$\mathcal{W} := \big\{T \in \mathcal{B}_1(\ell_p);\ \|P_M(T-B)P_M\| < \delta.$$

The set $\mathcal{W}$ is a $\mathtt{WOT}$-neighbourhood of B; and by the choice of δ, we have

$$d_*\text{-}\operatorname{diam}(\mathcal{W}) \leqslant 2\sum_{n=0}^{M} 2^{-n}\,\eta/8 + 2\sum_{n>M} 2^{-n} < \eta.$$

This concludes the proof. □

We can now prove Theorem 1.2, which we restate here as Theorem 3.4.

Theorem 3.4. — *On* $\mathbf{M} = \mathcal{B}_1(\ell_p)$, $p > 2$, *the topologies* $\mathtt{SOT}$ *and* $\mathtt{SOT}^*$ *are similar and the topologies* $\mathtt{WOT}$ *and* $\mathtt{SOT}_*$ *are similar. On* $\mathbf{M} = \mathcal{B}_1(\ell_p)$, $1 < p < 2$, *the topologies* $\mathtt{SOT}_*$ *and* $\mathtt{SOT}^*$ *are similar and the topologies* $\mathtt{WOT}$ *and* $\mathtt{SOT}$ *are similar.*

Proof. — This follows immediately from Proposition 3.1 and Lemma 2.3. □

As a direct corollary, we retrieve the result from [14, Theorem 2.3] that in the complex case and for $p > 2$, a typical $T \in (\mathcal{B}_1(\ell_p), \mathtt{SOT})$ is such that the operators $2T$ and $2T^*$ are hypercyclic, and obtain an analogous statement for $1 < p < 2$:

Corollary 3.5. — *Let* $\mathbb{K} = \mathbb{C}$. *If* $p > 2$, *then an* $\mathtt{SOT}$*-typical* $T \in \mathcal{B}_1(\ell_p)$ *is such that* $2T$ *and* $2T^*$ *are hypercyclic, so that in particular* $T - \lambda I$ *is one-to-one with dense range for every* $\lambda \in \mathbb{C}$. *If* $1 < p < 2$, *then an* $\mathtt{SOT}_*$*-typical* $T \in \mathcal{B}_1(\ell_p)$ *is such that* $2T$ *and* $2T^*$ *are hypercyclic.*

Proof. — By [13, Proposition 2.3], we know that for any $1 < p < \infty$, an $\mathtt{SOT}^*$-typical $T \in \mathcal{B}_1(\ell_p)$ is such that $2T$ and $2T^*$ are hypercyclic. So this is also true for an $\mathtt{SOT}$-typical T if $p > 2$ and for an $\mathtt{SOT}_*$-typical T if $1 < p < 2$. □

We also derive the following result for $\mathtt{WOT}$-typical contractions on ℓ_p:

Corollary 3.6. — *Let* $\mathbb{K} = \mathbb{C}$. *If* $1 < p < 2$, *then a typical* $T \in (\mathcal{B}_1(\ell_p), \mathtt{WOT})$ *is such that* $T - \lambda I$ *has dense range for every* $\lambda \in \mathbb{C}$. *If* $p > 2$, *then a typical* $T \in (\mathcal{B}_1(\ell_p), \mathtt{WOT})$ *has no eigenvalue.*

Proof. — By [13, Proposition 2.3] again, an $\mathtt{SOT}$-typical $T \in \mathcal{B}_1(\ell_p)$ is such that $2T$ is hypercyclic, and hence such that $T - \lambda I$ has dense range for every $\lambda \in \mathbb{C}$. So this is also true for a $\mathtt{WOT}$-typical $T \in \mathcal{B}_1(\ell_p)$ if $1 < p < 2$. The second statement follows by duality. □

Finally, one can deduce from Theorem 3.4 combined with Proposition 7.13 below that a $\mathtt{WOT}$-typical contraction on ℓ_p, $p \neq 2$ has maximal essential spectrum (which is to be compared with the forthcoming Proposition 5.1):

Corollary 3.7. — *Let* $\mathbb{K} = \mathbb{C}$ *and* $X = \ell_p$, $1 < p \neq 2 < \infty$. *The essential spectrum of a typical* $T \in (\mathcal{B}_1(X), \mathtt{WOT})$ *is equal to* $\overline{\mathbb{D}}$.

Proof. — Since the property "to have an essential spectrum equal to $\overline{\mathbb{D}}$" is self-adjoint, this follows from Proposition 7.13 and Theorem 3.4. □

4. Real ℓ_p with $p = 3$ or $3/2$

In this section, we consider *real* ℓ_p-spaces, which we still denote by ℓ_p. Our aim is to prove Theorem 1.3, which we restate as follows.

Theorem 4.1. — *If $p = 3$ or $3/2$, then the topologies* WOT *and* SOT* *on $\mathcal{B}_1(\ell_p)$ are similar.*

Note that Theorem 4.1 is indeed equivalent to Theorem 1.3: if τ is any topology lying between WOT and SOT*, and if we know that WOT and SOT* have the same dense sets (which is the content of Theorem 4.1), then SOT* and τ clearly have the same dense sets.

Observe next that since 3 and 3/2 are conjugate exponents, the case $p = 3/2$ follows from the case $p = 3$ by duality. Also, by part (1a) of Theorem 1.2, we already know that for any $p > 2$, SOT* is similar to SOT on $\mathcal{B}_1(\ell_p)$. Hence, to prove Theorem 4.1, it is enough to show that if $p = 3$, then WOT and SOT are similar. Moreover, by Corollary 2.15, it is in fact enough to prove the following proposition.

Proposition 4.2. — *If $p = 3$, then the set of all $T \in \mathcal{B}_1(\ell_p)$ such that $\overline{\operatorname{span}}\big(\mathcal{N}(T)\big) = \ell_p$ is* SOT*-dense in $\mathcal{B}_1(\ell_p)$.*

Since the proof of Proposition 4.2 is rather convoluted, we divide this section into small sub-sections. We have tried to indicate very precisely where the assumptions that $p = 3$ and $\mathbb{K} = \mathbb{R}$ seem to be really crucial. However, we would be extremely surprised if $p = 3$ were the only value of p for which Proposition 4.2 holds true. We would rather be inclined to believe that in fact, Proposition 4.2 holds true for all $p > 2$, so that (by duality) the answer to Question 1.1 is "Yes", in the real case at least.

4.1. Notation and terminology. — Recall that $(e_n)_{n\in\mathbb{Z}_+}$ is the canonical basis of ℓ_p, and that (e_n^*) is the associated sequence of coordinate functionals. If $N \in \mathbb{Z}_+$, we set $E_N := \operatorname{span}(e_0, \dots, e_N)$, and we denote by P_N the canonical projection of ℓ_p onto E_N. If $N, M \in \mathbb{Z}_+$, we denote by $\mathcal{B}(E_N, E_M)$ the space of all (bounded) operators $S : E_N \to E_M$. Whenever this seems convenient, we consider $\mathcal{B}(E_N, E_M)$ as a subspace of $\mathcal{B}(\ell_p)$ or of $\mathcal{B}(E_N, \ell_p)$ *via* the obvious identifications. Finally, if $x \in \ell_p$, the *support* of x, denoted by $\operatorname{supp}(x)$, is the set $\{n \in \mathbb{Z}_+;\ \langle e_n^*, x\rangle \neq 0\}$.

4.2. Two useful facts. — The following lemma, which is part of [19, Theorem 2.2], will be used a number of times in the proof of Proposition 4.2. We give a proof for completeness' sake (which is not the same as the one in [19]).

Lemma 4.3. — *Let $p > 2$, let E be a subspace of ℓ_p, and let $S \in \mathcal{B}(E, \ell_p)$. Let also $x \in E$ be a norming vector for S. If $y \in E$ is such that $\operatorname{supp}(y) \cap \operatorname{supp}(x) = \varnothing$, then $\operatorname{supp}(Sy) \cap \operatorname{supp}(Sx) = \varnothing$.*

Proof. — We prove in fact the following more general result. *Let E be a real or complex Banach space, let (Ω, μ) be a measure space, and let $S : E \to L_p(\Omega, \mu)$ be a bounded operator. Let also $x \in \mathcal{N}(S)$. If $y \in E$ is such that (x, y) is a sub-ℓ_q-sequence for some $q > 2$,* i.e *$\|ax + by\|^q \leqslant \|ax\|^q + \|by\|^q$ for all $a, b \in \mathbb{K}$, then $(Sx)(Sy) = 0$.*

We may assume that $\|S\| = 1$ and $\|x\| = 1 = \|y\|$. Then

$$\|Sx + tSy\|_p^p \leqslant \|x + ty\|^p \leqslant \big(1 + |t|^q\big)^{p/q} =: \alpha(t) \qquad \text{for all } t \in \mathbb{R}.$$

Note that since $q > 2$, the function α is $\mathcal{C}^2$-smooth on $\mathbb{R}$, with $\alpha(0) = 1$ and $\alpha'(0) = \alpha''(0) = 0$.

Let $\phi : \Omega \to \mathbb{K}$ be a measurable function such that $|\phi| \equiv 1$ and $Sx = |Sx|\,\phi$. Then

$$|Sx + t\,Sy| = \Big||Sx| + t\,\bar{\phi}Sy\Big| \geqslant \big||Sx| + t\,\mathrm{Re}(\bar{\phi}Sy)\big|.$$

So, if we set $u := \mathrm{Re}(\bar{\phi}Sy)$ then

$$\big\||Sx| + t\,u\big\|_p^p \leqslant \alpha(t) \qquad \text{for all } t \in \mathbb{R}.$$

Now, consider the function $f : \mathbb{R} \to \mathbb{R}$ defined by

$$f(t) := \big\||Sx| + t\,u\big\|_p^p - \alpha(t) = \int_\Omega \big||Sx| + tu\big|^p d\mu - \alpha(t).$$

Since $p > 2$, the function f is $\mathcal{C}^2$-smooth on $\mathbb{R}$, with

$$f'(t) = p\int_\Omega \big||Sx| + tu\big|^{p-1} u d\mu - \alpha'(t) \quad \text{and} \quad f''(t) = p(p-1)\int_\Omega \big||Sx| + tu\big|^{p-2} u^2 d\mu - \alpha''(t).$$

Moreover, we have $f(t) \leqslant 0$ on $\mathbb{R}$ and $f(0) = \|Sx\|_p^p - 1 = 0$ since $x \in \mathcal{N}(S)$; so f has a maximum at $t = 0$. Hence $f''(0) \leqslant 0$, *i.e.*

$$\int_\Omega |Sx|^{p-2} u^2 d\mu \leqslant 0.$$

Since $|Sx|^{p-2}u^2 \geqslant 0$, this means that $(Sx)u = 0$. So we have shown that $(Sx)\,\mathrm{Re}(\bar{\phi}\,Sy) = 0$. In the case where $\mathbb{K} = \mathbb{R}$, this yields that $(Sx)(Sy)\bar{\phi} = 0$, and hence $(Sx)(Sy) = 0$ since $|\phi| \equiv 1$. In the case where $\mathbb{K} = \mathbb{C}$, starting from the inequality $\|Sx - itSy\|_p^p \leqslant \alpha(t)$ we obtain that $(Sx)\,\mathrm{Im}(\bar{\phi}\,Sy) = 0$. Again, $(Sx)(Sy)\bar{\phi} = 0$, and hence $(Sx)(Sy) = 0$. □

The following inequality (see [19, Lemma 2.1]) will also be needed. Note that this inequality has already been used in the proof of Proposition 3.1.

Lemma 4.4. — *Let $p > 2$. For any $u, v \in \mathbb{C}$,*

$$|u + v|^p + |u - v|^p \geqslant 2|u|^p + p\,|u|^{p-2}|v|^2.$$

4.3. The interval property. — The next definition will be essential for us.

Definition 4.5. — Let $M, N \in \mathbb{Z}_+$, and let $S \in \mathcal{B}(E_N, E_M)$. We say that S has the *interval property* if the support of any norming vector x for S is an interval of the form $[\![0, a]\!]$ for some $0 \leqslant a \leqslant N$.

The importance of this definition lies in the following lemma. We can prove it for $p = 3$ only, but it is quite plausible that it holds true for every $p > 2$.

Lemma 4.6. — *Assume that $p = 3$. If $S \in \mathcal{B}(E_N, E_M)$ has the interval property, then $\mathcal{N}(S) \cap S_{E_N}$ is finite.*

To prove Lemma 4.6, we need to introduce another definition (where the assumption that $\mathbb{K} = \mathbb{R}$ is essential).

Definition 4.7. — Let $M, N \in \mathbb{Z}_+$, and let $S \in \mathcal{B}(E_N, E_M)$. Let also $x, y \in E_N$. We say that *x and y are sign-compatible relative to S*, or simply that *x and y are sign-compatible*, if $\langle e_n^*, x\rangle\langle e_n^*, y\rangle \geqslant 0$ for all $n \in [\![0, N]\!]$ and $\langle e_m^*, Sx\rangle\langle e_m^*, Sy\rangle \geqslant 0$ for all $m \in [\![0, M]\!]$.

In other words, the nth coordinates of x and y have the same sign for every $n \in [\![0, N]\!]$, and the mth coordinates of Sx and Sy have the same sign for every $m \in [\![0, M]\!]$ – with the convention that 0 has both signs + and −. Note that in spite of what the terminology might suggest, we have to be a little bit careful: the relation of sign-compatibility is obviously reflexive and symmetric, but it is not transitive.

In the next lemma, the assumption that $p = 3$ seems to be crucial. At least, it *is* quite crucial for the proof we give.

Lemma 4.8. — *Assume that $p = 3$. Let $M, N \in \mathbb{Z}_+$, and let $S \in \mathcal{B}(E_N, E_M)$. Let also $x, y \in E_N$, and assume that x and y are sign-compatible with respect to S. Then the following facts hold true.*

(i) *There exists $c > 0$ such that x and $x + ty$ are sign-compatible for every real number $t \geqslant -c$.*
(ii) *If $x, y \in \mathcal{N}(S)$, then $x + ty \in \mathcal{N}(S)$ for all $t \geqslant 0$.*
(iii) *Assume that $x, y \in \mathcal{N}(S)$ and that $\operatorname{supp}(y) \subseteq \operatorname{supp}(x)$. Then $\operatorname{supp}(Sy) \subseteq \operatorname{supp}(Sx)$ and, if c is as in* (i)*, $x + ty \in \mathcal{N}(S) \cup \{0\}$ for every $t \geqslant -c$.*

Proof. — (i) It is clear that x and $x + ty$ are sign-compatible for every $t \geqslant 0$; so we only need to show that there exists $c > 0$ such that x and $x + ty$ are sign-compatible for every $t \in [-c, 0]$. But this is easy: just choose c small enough to ensure that $\langle e_n^*, x\rangle\langle e_n^*, x+ty\rangle > 0$ on $[-c, 0]$ for all $n \in [\![0, N]\!]$ such that $\langle e_n^*, x\rangle \neq 0$, and $\langle e_m^*, Sx\rangle\langle e_m^*, S(x + ty)\rangle > 0$ on $[-c, 0]$ for all $m \in [\![0, M]\!]$ such that $\langle e_m^*, Sx\rangle \neq 0$.

(ii) Assume that $x, y \in \mathcal{N}(S)$ and, without loss of generality, that $\|S\| = 1$.

Let us consider the function $\varphi_{x,y} : \mathbb{R} \to \mathbb{R}$ defined by

$$\varphi_{x,y}(t) := \|x + ty\|^3 - \|S(x + ty)\|^3.$$

Note that $\varphi_{x,y}(t) \geqslant 0$ on $\mathbb{R}$ because $\|S\| = 1$, and $\varphi_{x,y}(0) = 0$ because $x \in \mathcal{N}(S)$. Moreover, the function $\varphi_{x,y}$ is $\mathcal{C}^1$-smooth on $\mathbb{R}$. We have to show that $\varphi_{x,y}(t) = 0$ for all $t \geqslant 0$.

Let c be as in (i). For any $n \in [\![0, N]\!]$ such that $\langle e_n^*, x\rangle \neq 0$ and $t \geqslant -c$, we may write $|\langle e_n^*, x + ty\rangle| = \omega_n\langle e_n^*, x + ty\rangle$, where $\omega_n = \pm 1$ does not depend on t; and it follows that $|\langle e_n^*, x + ty\rangle|^3$ is a polynomial function of $t \in [-c, \infty)$. Similarly, if $m \in [\![0, M]\!]$ is such that $\langle e_m^*, Sx\rangle \neq 0$, then $|\langle e_m^*, S(x+ty)\rangle|^3$ is a polynomial function of $t \in [-c, \infty)$. Writing down the definition of $\varphi_{x,y}(t)$, it follows that for $t \geqslant -c$,

$$\varphi_{x,y}(t) = P(t) + d\,|t|^3,$$

where P is a polynomial of degree at most 3 and

$$d := \sum_{\langle e_n^*, x\rangle = 0} |\langle e_n^*, y\rangle|^3 - \sum_{\langle e_m^*, Sx\rangle = 0} |\langle e_m^*, Sy\rangle|^3.$$

Let us write P as $P(t) = c_0 + c_1 t + c_2 t^2 + c_3 t^3$, so that

$$\varphi_{x,y}(t) = c_0 + c_1 t + c_2 t^2 + c_3 t^3 + d\,|t|^3 \qquad \text{for all } t \geqslant -c.$$

We have $c_0 = \varphi_{x,y}(0) = 0$, and $c_1 = \varphi'_{x,y}(0) = 0$ because $\varphi_{x,y}$ attains its minimum at $t = 0$; so

$$\varphi_{x,y}(t) = c_2 t^2 + c_3 t^3 + d\,|t|^3.$$

Next, we use a symmetry argument. Let $\varphi_{y,x}$ be the function defined by

$$\varphi_{y,x}(t) := \|y + tx\|^3 - \|S(y + tx)\|^3.$$

If $t > 0$ then $\varphi_{x,y}(t) = t^3\varphi_{y,x}(1/t)$; hence,

$$\forall u > 0 \ : \ \varphi_{y,x}(u) = u^3\varphi_{x,y}(1/u) = d + c_3 + c_2 u.$$

Since $\varphi_{y,x}(0) = 0 = \varphi'_{y,x}(0)$ because $y \in \mathcal{N}(S)$, it follows that $d + c_3 = 0 = c_2$. Thus, we obtain

$$\varphi_{x,y}(t) = d\,(|t|^3 - t^3) \qquad \text{for all } t \geqslant -c.$$

In particular, $\varphi_{x,y}(t) = 0$ for all $t \geqslant 0$, which gives (ii).

(iii) With the notation of (ii), we have $\varphi_{x,y}(t) = 2d\,|t|^3$ for all $t \in [-c, 0]$. Since $\varphi_{x,y} \geqslant 0$, it follows that

$$d = \sum_{\langle e_n^*, x\rangle = 0} |\langle e_n^*, y\rangle|^3 - \sum_{\langle e_m^*, Sx\rangle = 0} |\langle e_m^*, Sy\rangle|^3 \geqslant 0.$$

Since $\mathrm{supp}(y) \subseteq \mathrm{supp}(x)$, the first sum in the right-hand side of the expression above is equal to 0; so the second sum must also be equal to 0, *i.e.* $\mathrm{supp}(Sy) \subseteq \mathrm{supp}(Sx)$. Altogether $d = 0$; so $\varphi_{x,y}(t) = 0$ for all $t \geqslant -c$, which means that $x + ty \in \mathcal{N}(S) \cup \{0\}$. □

We can now prove Lemma 4.6.

Proof of Lemma 4.6. — Let us fix $S \in \mathcal{B}(E_N, E_M)$ with the interval property. We start with the following fact.

Fact 4.9. — If $x, y \in \mathcal{N}(S)$ are sign-compatible, then x and y are proportional.

Proof of Fact 4.9. — Towards a contradiction, assume that there exist two sign-compatible and linearly independent vectors x, y in $\mathcal{N}(S)$. By the interval property, we may also assume that $\mathrm{supp}(y) \subseteq \mathrm{supp}(x)$.

Claim 4.10. — There exists a largest $c > 0$ such that x and $x + ty$ are sign-compatible for all $t \geqslant -c$.

Proof of Claim 4.10. — By Lemma 4.8, there exists at least one $c > 0$ such that x and $x + ty$ are sign compatible for all $t \geqslant -c$, and for any such c we have $x + ty \in \mathcal{N}(S)$ for all $t \geqslant -c$ (we cannot have $x + ty = 0$ because x and y are linearly independent). Moreover, since $\langle e_0^*, u\rangle \neq 0$ for any $u \in \mathcal{N}(S)$ by the interval property, it cannot be that $x + ty \in \mathcal{N}(S)$ for all $t \in \mathbb{R}$. This proves our claim. □

Claim 4.11. — Let c be as in Claim 4.10 and let $z := x - cy$. Then $z \in \mathcal{N}(S)$ and $\mathrm{supp}(z)$ is strictly contained in $\mathrm{supp}(y)$.

Proof of Claim 4.11. — That $z \in \mathcal{N}(S)$ follows from Lemma 4.8. Moreover, since we have that $\mathrm{supp}(z) \subseteq \mathrm{supp}(x)$ and x, z are sign-compatible, it also follows from Lemma 4.8 that $\mathrm{supp}(Sz) \subseteq \mathrm{supp}(Sx)$.

We first note that z and y are sign-compatible: indeed, since x, y and x, z are sign-compatible and $\mathrm{supp}(z) \subseteq \mathrm{supp}(x)$, we have $\langle e_n^*, z\rangle\langle e_n^*, x\rangle > 0$ and $\langle e_n^*, x\rangle\langle e_n^*, y\rangle \geqslant 0$ for all $n \in \mathrm{supp}(z)$, so that $\langle e_n^*, z\rangle\langle e_n^*, y\rangle \geqslant 0$; and similarly, $\langle e_m^*, Sz\rangle\langle e_m^*, Sy\rangle \geqslant 0$ for all $m \in \mathrm{supp}(Sz)$.

Now, assume that $\mathrm{supp}(z)$ is not strictly contained in $\mathrm{supp}(y)$. Then $\mathrm{supp}(y) \subseteq \mathrm{supp}(z)$ by the interval property. By Lemma 4.8 and since z, y are sign-compatible, it follows that one can find $\varepsilon > 0$ such that z and $z + sy$ are sign-compatible for all $s \geqslant -\varepsilon$, and it also follows that $\mathrm{supp}(Sy) \subseteq \mathrm{supp}(Sz)$. Let us show that x and $x + ty$ are sign-compatible for all $t \geqslant -c - \varepsilon$, which will contradict the maximality of c. Write $t = -c + s$ with $s \geqslant -\varepsilon$. Then

$x+ty = z+sy$, so that $z, x+ty$ are sign-compatible and $\mathrm{supp}(x+ty) \subseteq \mathrm{supp}(z)$. So we have $\langle e_n^*, z\rangle\langle e_n^*, x+ty\rangle > 0$ for all $n \in \mathrm{supp}(x+ty)$, and also $\langle e_n^*, z\rangle\langle e_n^*, x\rangle \geqslant 0$ since z, x are sign-compatible. Hence, we see that $\langle e_n^*, x\rangle\langle e_n^*, x+ty\rangle \geqslant 0$ for all $n \in \mathrm{supp}(x+ty)$. Similarly, using the sign-compatibility of $z, x+ty$ and the fact that $\mathrm{supp}(S(x+ty)) \subseteq \mathrm{supp}(Sz)$, we get $\langle e_m^*, Sx\rangle\langle e_m^*, S(x+ty)\rangle \geqslant 0$ for all $m \in \mathrm{supp}(S(x+ty))$. □

Claim 4.12. — We have $\mathrm{supp}(x) = \mathrm{supp}(y)$.

Proof of Claim 4.12. — Towards a contradiction, assume that $\mathrm{supp}(x) = [\![0, L]\!]$ and that $\mathrm{supp}(y) = [\![0, L']\!]$ with $L' < L$ (recall that S has the interval property and that $\mathrm{supp}(y) \subseteq \mathrm{supp}(x)$). By Claim 4.11, one can find $n \in [\![0, N]\!]$ such that $\langle e_n^*, x\rangle\langle e_n^*, y\rangle \neq 0$ and $\langle e_n^*, z\rangle = 0$. Then $n \leqslant L' < L$, and $n \notin \mathrm{supp}(z)$. However, since $L \in \mathrm{supp}(x)\backslash\mathrm{supp}(y)$ and $z = x - cy$, we have $L \in \mathrm{supp}(z)$. Since $z \in \mathcal{N}(S)$, this contradicts the interval property. □

So far, we have proved the following: whenever $x, y \in \mathcal{N}(S)$ are linearly independent and sign-compatible, it holds that $\mathrm{supp}(x) = \mathrm{supp}(y)$. Now, if we start with two linearly independent sign-compatible vectors $x, y \in \mathcal{N}(S)$ with $\mathrm{supp}(y) \subseteq \mathrm{supp}(x)$, and if we apply this property to x and to the vector $z = x - cy$ of Claim 4.11, we obtain a contradiction. This shows that in fact, there cannot exist linearly independent sign-compatible vectors $x, y \in \mathcal{N}(S)$. Hence, we have proved Fact 4.9. □

Now we can prove Lemma 4.6. Towards a contradiction, assume that $\mathcal{N}(S) \cap S_{E_N}$ is infinite. Consider the map $\Phi : \mathcal{N}(S) \cap S_{E_N} \to \{-1, 0, 1\}^{N+M+2}$ defined by

$$\Phi(u) := \big(\mathrm{sgn}(\langle e_0^*, u\rangle), \ldots, \mathrm{sgn}(\langle e_N^*, u\rangle), \mathrm{sgn}(\langle e_0^*, Su\rangle), \ldots, \mathrm{sgn}(\langle e_M^*, Su\rangle)\big),$$

where $\mathrm{sgn}(0) = 0$. Since Φ has finite range and $\mathcal{N}(S) \cap S_{E_N}$ is assumed to be infinite, Φ cannot be one-to-one. So one can find $x, y \in \mathcal{N}(S) \cap S_{E_N}$ such that $\Phi(x) = \Phi(y)$. Then x and y are sign compatible; and they are also linearly independent because $\Phi(-u) \neq \Phi(u)$ for every $u \in \mathcal{N}(S) \cap S_{E_N}$. This contradicts Fact 4.9. □

4.4. Special operators. — The next definition introduces a somewhat strange-looking class of operators, which will nonetheless be quite useful to us.

Definition 4.13. — Let $N \in \mathbb{Z}_+$, and let $S \in \mathcal{B}(E_N, \ell_p)$. We say that S is a *special operator* if there exists $R \in \mathbb{Z}_+$ such that

$$\forall n \in [\![0, N]\!] \ : \ (I - P_R)Se_n = \alpha_n \sum_{j \in \Lambda_n} e_j,$$

where $\alpha_0, \ldots, \alpha_N$ are non-zero scalars and $\Lambda_0, \ldots, \Lambda_N$ are finite subsets of $\mathbb{Z}_+$ with $\Lambda_n \cap [\![0, R]\!] = \varnothing$ which are "pairwise intersecting but not 3 - by - 3 intersecting", *i.e.* $\Lambda_n \cap \Lambda_{n'} \neq \varnothing$ for any n, n' but $\Lambda_n \cap \Lambda_{n'} \cap \Lambda_{n''} = \varnothing$ for any pairwise distinct n, n', n''.

Lemma 4.14. — *Let $A \in \mathcal{B}_1(\ell_p)$, and let $N_0 \in \mathbb{Z}_+$. For any $\varepsilon > 0$, one can find a special operator $S \in \mathcal{B}(E_{N_0+1}, \ell_p)$ such that $\|S\| = 1$ and $\|Se_n - Ae_n\| < \varepsilon$ for all $n \in [\![0, N_0]\!]$.*

Proof. — Choose $R \in \mathbb{Z}_+$ such that $\|P_R Ae_n - Ae_n\| < \varepsilon/2$ for all $n \in [\![0, N_0]\!]$. Choose also finite sets $\Lambda_0, \ldots, \Lambda_{N_0+1} \subseteq \mathbb{Z}_+$ with $\Lambda_n \cap [\![0, R]\!] = \varnothing$, pairwise but not 3 - by - 3 intersecting. Finally, let $\alpha, \beta > 0$, and define $S = S_{\alpha,\beta} \in \mathcal{B}(E_{N_0+1}, \ell_p)$ as follows:

$$\begin{cases} Se_n := (1-\varepsilon/2)P_R Ae_n + \alpha \sum_{j\in\Lambda_n} e_j & \text{for } n = 0,\dots,N_0, \\ Se_{N_0+1} := \beta \sum_{j\in\Lambda_{N_0+1}} e_j. \end{cases}$$

Then S is a special operator by definition, and if α is small enough we have $\|SP_{N_0}\| < 1$ and $\|Se_n - Ae_n\| < \varepsilon$ for all $n \in [\![0, N_0]\!]$. Having fixed α small enough, we may then choose β in such a way that $\|S\| = 1$. (Take the largest β such that $\|S_{\alpha,\beta}\| \leqslant 1$, which exists since $\|S_{\alpha,\beta}\| \to \infty$ as $\beta \to \infty$.) □

Remark. — In order to construct the pairwise intersecting but not 3-by-3 intersecting sets $\Lambda_0, \dots, \Lambda_{N_0+1}$, one may proceed as follows. Denote by $\mathcal{F}$ the family of all 2-elements subsets of $[\![0, N_0+1]\!]$; choose a family $(k_{\{n,n'\}})_{\{n,n'\}\in\mathcal{F}}$ of pairwise distinct integers greater than R; and set $\Lambda_n := \{k_{\{n,n'\}};\ n' \neq n\}$ for $n = 0,\dots,N_0+1$.

Lemma 4.15. — *Let $p > 2$. If $S \in \mathcal{B}(E_N, \ell_p)$ is a special operator, then every norming vector x for S has full support, i.e.* $\operatorname{supp}(x) = [\![0, N]\!]$. *In particular, S has the interval property.*

Proof. — Let $S \in \mathcal{B}(E_N, \ell_p)$ be a special operator with witnesses $\Lambda_n, \alpha_n, \beta_n$, and let $x \in \mathcal{N}(S)$. Towards a contradiction, assume that $\langle e_n^*, x\rangle = 0$ for some $n \in [\![0, N]\!]$. Then e_n and x have disjoint supports. Since $p > 2$ and $x \in \mathcal{N}(S)$, it follows that Se_n and Sx have disjoint supports as well, by Lemma 4.3.

Since $x \neq 0$ we may fix $n' \in [\![0, N]\!]$ such that $\langle e_{n'}^*, x\rangle \neq 0$; and by assumption on the sets $\Lambda_0, \dots, \Lambda_N$, we may choose $j \in \Lambda_n \cap \Lambda_{n'}$. Then $j \notin \Lambda_{n''}$ for any $n'' \neq n, n'$; hence

$$\langle e_j^*, Sx\rangle = \alpha_n \langle e_n^*, x\rangle + \alpha_{n'} \langle e_{n'}^*, x\rangle.$$

However, we have $\langle e_n^*, x\rangle = 0$, and also $\langle e_j^*, Sx\rangle = 0$ because $j \in \Lambda_n$ (so that $j \in \operatorname{supp}(Se_n)$). So we get $\langle e_{n'}^*, x\rangle = 0$, which is the required contradiction. □

4.5. Modifications. — The next definition describes a way of "modifying" a finite-dimensional operator without losing any norming vector and with in mind the goal of creating new ones.

Definition 4.16. — Le $N, M \in \mathbb{Z}_+$, and let $S \in \mathcal{B}(E_N, E_M)$ be such that $\|S\| = 1$ and $\operatorname{span}(\mathcal{N}(S)) \neq E_N$. Denote by $\mathcal{N}(S)^\perp \subseteq E_N^*$ the annihilator of $\mathcal{N}(S)$, and let $\mathbf{y} = (y_1^*, \dots, y_L^*)$ be a finite sequence of non-zero vectors in $\mathcal{N}(S)^\perp$. Let also $\eta \in (0, (2L)^{-1/p})$ and $\delta > 0$. The $(\mathbf{y}, \eta, \delta)$-*modification of S* is the operator $S_{\mathbf{y},\eta,\delta} \in \mathcal{B}(E_{N+1}, E_{M+2L})$ defined as follows:

$$\begin{cases} S_{\mathbf{y},\eta,\delta} x := Sx + \delta \sum_{l=1}^{L} \langle y_l^*, x\rangle \left(e_{M+2l-1} + e_{M+2l}\right) & \text{for all } x \in E_N, \\ S_{\mathbf{y},\eta,\delta} e_{N+1} := \eta \sum_{l=1}^{L} \left(e_{M+2l-1} - e_{M+2l}\right). \end{cases}$$

Thus, in matrix form,

$$S_{\mathbf{y},\eta,\delta} = \begin{bmatrix} \boxed{\quad S \quad} & \begin{matrix} 0 \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ 0 \end{matrix} \\ \begin{matrix} \delta\langle y_1^*, e_0\rangle & \cdots & \delta\langle y_1^*, e_N\rangle \\ \delta\langle y_1^*, e_0\rangle & \cdots & \delta\langle y_1^*, e_N\rangle \\ \vdots & & \vdots \\ \vdots & & \vdots \\ \vdots & & \vdots \\ \delta\langle y_L^*, e_0\rangle & \cdots & \delta\langle y_L^*, e_N\rangle \\ \delta\langle y_L^*, e_0\rangle & \cdots & \delta\langle y_L^*, e_N\rangle \end{matrix} & \begin{matrix} \eta \\ -\eta \\ \vdots \\ \vdots \\ \vdots \\ \eta \\ -\eta \end{matrix} \end{bmatrix}$$

The following lemma is obvious.

Lemma 4.17. — *With the notation of Definition 4.16, we have $\|S_{\mathbf{y},\eta,\delta}x\| \geqslant \|Sx\|$ for all $x \in E_N$ and $S_{\mathbf{y},\eta,\delta}x = Sx$ for all $x \in \mathcal{N}(S)$. In particular $\|S_{\mathbf{y},\eta,\delta}\| \geqslant \|S\| = 1$, and if $\|S_{\mathbf{y},\eta,\delta}\| = 1$ then $\mathcal{N}(S) \subseteq \mathcal{N}(S_{\mathbf{y},\eta,\delta})$.*

Note that, given S, $\mathbf{y}$ and η, it is not at all clear that one can find $\delta > 0$ such that $\|S_{\mathbf{y},\eta,\delta}\| = 1$. This motivates the following definition.

Definition 4.18. — We will say that p is *good exponent* if the following holds true: for any $S \in \mathcal{B}(E_N, E_M)$ such that $\|S\| = 1$ and $\mathcal{N}(S) \cap S_{E_N}$ is a finite set, for any $\mathbf{y} = (y_1^*, \dots, y_L^*) \subseteq \mathcal{N}(S)^\perp$ and for any $\eta \in (0, (2L)^{-1/p})$, one can find $\delta > 0$ such that $\|S_{\mathbf{y},\eta,\delta}\| = 1$.

The next lemma is the most technical part in the proof of Proposition 4.2. We can prove it for $p = 3$ only, but it seems hard to believe that it does not hold true for all p.

Lemma 4.19. — *$p := 3$ is a good exponent.*

Proof. — Let us fix $S \in \mathcal{B}(E_N, E_M)$ with $\|S\| = 1$ such that $\mathcal{N}(S) \cap S_{E_N}$ is a finite set. For notational simplicity and since $\mathbf{y}$ and η will remain fixed, we write S_δ instead of $S_{\mathbf{y},\eta,\delta}$. For any $x \in \mathcal{N}(S)$ and $\varepsilon > 0$, we set

$$\Sigma(x, \varepsilon) := \big\{u \in E_N;\ \|u\| = \varepsilon \text{ and } \|x + u\| = 1\big\}.$$

Finally, recall the definition of sign-compatibility: two vectors $x, y \in E_N$ are said to be sign-compatible with respect to S if $\langle e_n^*, x\rangle\langle e_n^*, y\rangle \geqslant 0$ for all $n \in [\![0, N]\!]$ and $\langle e_m^*, Sx\rangle\langle e_m^*, Sy\rangle \geqslant 0$ for all $m \in [\![0, M]\!]$.

Claim 4.20. — One can find $\varepsilon \in (0, 1]$ such that, for any $x \in \mathcal{N}(S) \cap S_{E_N}$ and all $u \in \Sigma(x, \varepsilon)$, the following holds true: $x + u$ and x are sign-compatible with respect to S, and $\|S(x + u)\| < \|x + u\|$.

Proof of Claim 4.20. — Since $\mathcal{N}(S) \cap S_{E_N}$ is a finite set, one can find $\varepsilon > 0$ such that, for every $x \in \mathcal{N}(S) \cap S_{E_N}$, the set $\overline{B}(x, \varepsilon) \cap S_{E_N}$ contains no norming vector for S except x. Then $\|S(x+u)\| < \|x+u\|$ for every $x \in \mathcal{N}(S) \cap S_{E_N}$ and all $u \in \Sigma(x, \varepsilon)$. Moreover, taking ε small enough, we may also assume that the sign-compatibility condition is satisfied (and that $\varepsilon \leqslant 1$); see the beginning of the proof of Lemma 4.8. □

Claim 4.21. — Let ε be as in Claim 4.20. One can find $\delta_0 > 0$ such that, for any $0 \leqslant \delta \leqslant \delta_0$, every $x \in \mathcal{N}(S) \cap S_{E_N}$ and all $z \in E_{N+1}$ such that $P_N z \in \Sigma(x, \varepsilon)$, we have $\|S_\delta(x+z)\| \leqslant \|x+z\|$.

Proof of Claim 4.21. — Let

$$\kappa := \inf\big\{\|x+u\|^p - \|S(x+u)\|^p;\ x \in \mathcal{N}(S) \cap S_{E_N}\ ,\ u \in \Sigma(x, \varepsilon)\big\}$$

and note that, since $\mathcal{N}(S) \cap S_{E_N}$ is a finite set, $\kappa > 0$ by compactness of the sets $\Sigma(x, \varepsilon)$.

Now, let us fix $x \in \mathcal{N}(S) \cap S_{E_N}$ and $z \in E_{N+1}$ such that $P_N z \in \Sigma(x, \varepsilon)$. Write $z = u + \mu e_{N+1}$, where $u := P_N z$ and $\mu \in \mathbb{R}$. For any $\delta \geqslant 0$, the definition of $S_\delta = S_{\mathbf{y},\eta,\delta}$ gives

$$\|S_\delta(x+z)\|^p = \|S(x+u)\|^p + \sum_{l=1}^{L}\Big(|\delta\,\langle y_l^*, u\rangle + \eta\mu|^p + |\delta\,\langle y_l^*, u\rangle - \eta\mu|^p\Big).$$

(We have used the fact that $S_\delta x = Sx$.) Hence, setting $C := \max(\|y_1^*\|, \dots, \|y_L^*\|)$, we get

$$\begin{aligned}\|S_\delta(x+z)\|^p &\leqslant \|S(x+u)\|^p + 2L\,\big(C\delta\|u\| + \eta|\mu|\big)^p \\ &\leqslant \|S(x+u)\|^p + 2L\,\big(C\delta + \eta|\mu|\big)^p \qquad \text{because } \|u\| = \varepsilon \leqslant 1.\end{aligned}$$

By the definition of κ, it follows that

$$\|x+z\|^p - \|S_\delta(x+z)\|^p \geqslant \kappa + |\mu|^p - 2L\,\big(C\delta + \eta|\mu|\big)^p.$$

So, if we introduce the function $f_\delta : \mathbb{R}_+ \to \mathbb{R}$ defined by

$$f_\delta(s) := s^p - 2L\big(C\delta + \eta s)^p,$$

it is enough to show that if $\delta_0 > 0$ is small enough, then $f_\delta \geqslant -\kappa$ on $\mathbb{R}_+$ for every $0 \leqslant \delta \leqslant \delta_0$.

Let $\alpha > 0$ to be determined in a few lines. We choose δ_0 such that

$$C\delta_0 \leqslant \alpha\eta \qquad \text{and} \qquad (C\delta_0 + \eta s)^p \leqslant \frac{\kappa}{2L} + (\eta s)^p \quad \text{for all } s \in [0,1].$$

If $0 \leqslant \delta \leqslant \delta_0$ then, on the one hand,

$$f_\delta(s) \geqslant s^p - \kappa - 2L\eta^p s^p \geqslant -\kappa \qquad \text{for all } s \in [0,1],$$

since $2L\eta^p < 1$. On the other hand, if $s \geqslant 1$ then $C\delta \leqslant \alpha\eta s$, and hence

$$f_\delta(s) \geqslant s^p - 2L\,(1+\alpha)^p\eta^p\, s^p.$$

So, if we take α small enough to ensure that $2L(1+\alpha)^p\eta^p \leqslant 1$, which is possible since $2L\eta^p < 1$, then δ_0 has the required property. □

Claim 4.22. — Let ε be as in Claim 4.20, and let δ_0 be as in Claim 4.21. If $0 \leqslant \delta \leqslant \delta_0$ then, for every $x \in \mathcal{N}(S) \cap S_{E_N}$ and all $z \in E_{N+1}$ such that $P_N z \in \Sigma(x, \varepsilon)$, it holds that

$$\forall t \in [0,1]\ :\ \|S_\delta(x+tz)\| \leqslant \|x+tz\|.$$

Proof of Claim 4.22. — Let us fix $0 \leqslant \delta \leqslant \delta_0$, $x \in \mathcal{N}(S) \cap S_{E_N}$ and $z \in E_{N+1}$ such that $u := P_N z \in \Sigma(x, \varepsilon)$. Define $\varphi : \mathbb{R} \to \mathbb{R}$ by

$$\varphi(t) := \|x + tz\|^3 - \|S_\delta(x + tz)\|^3.$$

We are going to show that $\varphi(t) \geqslant 0$ for all $t \in [0, 1]$.

Since $x \in \mathcal{N}(S)$, we have $S_\delta x = Sx$. Looking at the definition of S_δ, it follows that there exists some real number α such that

$$\begin{aligned} \forall t \in \mathbb{R} \ : \ \varphi(t) &= \|x + tu\|^3 - \|S(x + tu)\|^3 + \alpha\, |t|^3 \\ &=: \psi(t) + \alpha\, |t|^3. \end{aligned}$$

Now, since x and $x + u$ are sign-compatible (with respect to S) by the choice of ε, x and $x + tu$ are sign-compatible for every $t \in [0, 1]$ by convexity. Writing down the definition of $\psi(t)$, it follows (as in the proof of Lemma 4.8) that the function ψ is polynomial (of degree at most 3) on $[0, 1]$; and since $\psi(t) \geqslant 0$ on $\mathbb{R}$ with $\psi(0) = 0 = \psi'(0)$ (because x is a norming vector for S and $\|S\| = 1$), it must have the form $\psi(t) = at^2 + bt^3$, with $a \geqslant 0$. So we see that

$$\forall t \in [0, 1] \ : \ \varphi(t) = at^2 + ct^3,$$

for some constants $a \geqslant 0$ and $c \in \mathbb{R}$.

Moreover, by Claim 4.21 we know that $\varphi(1) = \|x + z\|^3 - \|S_\delta(x + z)\|^3 \geqslant 0$, *i.e.* $a + c \geqslant 0$. So we get $\varphi(t) = at^2 + ct^3 \geqslant (a + c)t^3 \geqslant 0$ for all $t \in [0, 1]$, as required. □

Claim 4.23. — Let ε and δ_0 be as in Claim 4.21, and let $0 \leqslant \delta \leqslant \delta_0$. If $x \in \mathcal{N}(S) \cap S_{E_N}$, then one can find $r > 0$ such that $\|S_\delta v\| \leqslant \|v\|$ for all $v \in E_{N+1}$ satisfying $\|P_N v - x\| < r$.

Proof of Claim 4.23. — Let us fix $x \in \mathcal{N}(S) \cap S_{E_N}$. Let also $r > 0$, and let $v \in E_{N+1}$ be such that $\|P_N v - x\| < r$. Write $v = u + \mu e_{N+1}$ with $u := P_N v$ and $\mu \in \mathbb{R}$.

If $u \in \mathbb{R}x$, then $S_\delta u = Su$ and hence $\|S_\delta v\|^p = \|Su\|^p + 2L\eta^p\, |\mu|^p \leqslant \|u\|^p + |\mu|^p = \|v\|^p$ because $2L\eta^p \leqslant 1$. So we assume that $u \notin \mathbb{R}x$. By Claim 4.22, it is enough to show that one can find $\lambda \in \mathbb{R}$, $z \in P_N^{-1}\big(\Sigma(x, \varepsilon)\big)$ and $t \in [0, 1]$ such that $v = \lambda(x + tz)$.

Consider the 2-dimensional space $F := \operatorname{span}(x, u) \subseteq E_N$, its unit sphere S_F and the sphere $S_F(x, \varepsilon) = \{w \in F\,;\, \|w - x\| = \varepsilon\} \subseteq F$. Since $x \in S_F$ and $\|u - x\| < r$, it is geometrically clear that if r is small enough, then one can find $w \in S_F(x, \varepsilon) \cap S_F$ such that the half-line $\mathbb{R}_+ u$ intersects the half-open segment $(x, w]$. This means that one can find $\alpha > 0$, $\widetilde{u} \in \Sigma(x, \varepsilon)$ and $t \in (0, 1]$ such that $\alpha u = x + t\widetilde{u}$. Since $v = u + \mu e_{N+1}$, it follows that one can find $\lambda \in \mathbb{R}$ and $z \in P_N^{-1}(\Sigma(x, \varepsilon))$ such that $v = \lambda(x + tz)$; explicitly, $v = (1/\alpha)\big(x + t\,(\widetilde{u} + (\alpha\mu/t)e_{N+1})\big)$. □

We can now prove Lemma 4.19. Let δ_0 be as in Claim 4.21. By Claim 4.23, one can find an open set $V \subseteq E_N$ containing $\mathcal{N}(S) \cap S_{E_N}$ such that

$$\forall \delta \in [0, \delta_0] \ \ \forall v \in P_N^{-1}(V) \ : \ \|S_\delta v\| \leqslant \|v\|.$$

Note that, by homogeneity, the inequality $\|S_\delta v\| \leqslant \|v\|$ holds in fact for every $v \in P_N^{-1}\big(\widehat{V}\big)$ where $\widehat{V} := \bigcup_{\lambda \in \mathbb{R}} \lambda V$.

Let $K := S_{E_N} \backslash \widehat{V}$ and $\widehat{K} := \bigcup_{\lambda \in \mathbb{R}} \lambda K$. Since $V \supseteq \mathcal{N}(S) \cap S_{E_N}$, we have $\|Su\| < 1$ for all $u \in K$. Since K is compact, one can find $c < 1$ such that $\|Su\| \leqslant c$ for all $u \in K$. Then $\|Su\| \leqslant c\,\|u\|$ for all $u \in \widehat{K}$ by homogeneity. We claim that if δ is small enough, then

$$\forall v \in P_N^{-1}(\widehat{K}) \ : \ \|S_\delta v\| \leqslant \|v\|.$$

Indeed, let $v \in P_N^{-1}(\widehat{K})$ and write $v = u + \mu e_{N+1}$ where $u := P_N v \in \widehat{K}$. Then

$$\begin{aligned}\|S_\delta(v)\| &\leqslant \|Su + S_\delta(\mu e_{N+1})\| + \|(I - P_M)S_\delta u\| \\ &\leqslant \|Su + S_\delta(\mu e_{N+1})\| + C\delta\|u\|,\end{aligned}$$

where $C := (2L)^{1/p}\max(\|y_1^*\|, \dots, \|y_L^*\|)$. Moreover,

$$\begin{aligned}\|Su + S_\delta(\mu e_{N+1})\|^p &= \|Su\|^p + 2L\eta^p|\mu|^p \\ &\leqslant \theta^p(\|u\|^p + |\mu|^p) = \theta^p\|v\|^p,\end{aligned}$$

where $\theta := \max(c, (2L)^{1/p}\eta) < 1$. So, if we choose δ such that $\theta + C\delta \leqslant 1$, we get that $\|S_\delta v\| \leqslant \|v\|$ for all $v \in P_N^{-1}(\widehat{K})$.

Altogether, we have shown that if $\delta > 0$ is small enough, then $\|S_\delta v\| \leqslant \|v\|$ for all $v \in P_N^{-1}(\widehat{V} \cup \widehat{K})$. Since $\widehat{K} \cup \widehat{V}$ contains B_{E_N}, this implies that $\|S_\delta\| \leqslant 1$, which concludes the proof. □

Lemma 4.24. — *Let p be a good exponent. Let $S \in \mathcal{B}(E_N, E_M)$ and $\mathbf{y} = (y_1^*, \dots, y_L^*)$ be as in Definition 4.16, and assume that $\mathcal{N}(S) \cap S_{E_N}$ is a finite set. Let also $\eta \in (0, (2L)^{-1/p})$.*

(i) *There exists a largest $\delta > 0$ such that $\|S_{\mathbf{y},\eta,\delta}\| = 1$.*

(ii) *If δ is as in* (i), *then* $\dim \operatorname{span}\big(\mathcal{N}(S_{\mathbf{y},\eta,\delta})\big) \geqslant 2 + \dim \operatorname{span}\big(\mathcal{N}(S)\big)$.

Proof. — (i) Since p is a good exponent, there exists at least one $\delta > 0$ such that $\|S_{\mathbf{y},\eta,\delta}\| = 1$. Moreover, since the linear functionals y_l^* are non-zero, it is clear that $\|S_{\mathbf{y},\eta,\delta}\| \to \infty$ as $\delta \to \infty$. This proves (i).

(ii) This will follow from the next three claims, which will also be used in the proof of Lemma 4.31 below. Let δ be as in (i) and, for notational simplicity, let $\widetilde{S} := S_{\mathbf{y},\eta,\delta}$.

Claim 4.25. — $\mathcal{N}(S)$ is strictly contained in $\mathcal{N}(\widetilde{S})$.

Proof of Claim 4.25. — By Lemma 4.17, we know that $\mathcal{N}(\widetilde{S}) \supseteq \mathcal{N}(S)$. Towards a contradiction, assume that $\mathcal{N}(\widetilde{S}) = \mathcal{N}(S)$. Then, extending the linear functionals $y_1^*, \dots, y_L^*$ to E_{N+1} by setting $\langle y_l^*, e_{N+1}\rangle := 0$, we have $y_l^* \in \mathcal{N}(\widetilde{S})^\perp$ for $l = 1, \dots, L$; so we may consider $(\mathbf{y}, \eta, \delta')$-modifications of $\widetilde{S} \in \mathcal{B}(E_{N+1}, E_{M+2L})$ for any $\delta' > 0$.

Since p is a good exponent and $\mathcal{N}(\widetilde{S}) \cap S_{E_{N+1}} = \mathcal{N}(S) \cap S_{E_N}$ is a finite set, one can find $\varepsilon > 0$ such that the operator $T := \widetilde{S}_{\mathbf{y},\eta,\varepsilon\delta} \in \mathcal{B}(E_{N+2}, E_{M+4L})$ satisfies $\|T\| = 1$. Thus, for any $(x, \lambda, \mu) \in E_N \times \mathbb{R} \times \mathbb{R}$ we have

$$\|T(x + \lambda e_{N+1} + \mu e_{N+2})\|^p \leqslant \|x\|^p + |\lambda|^p + |\mu|^p.$$

By the definition of T and since $\langle y_l^*, e_{N+1}\rangle = 0$ for $l = 1, \dots, L$, this reads as follows:

$$\|\widetilde{S}(x + \lambda e_{N+1})\|^p + \sum_{l=1}^{L}\Big(|\varepsilon\delta\langle y_l^*, x\rangle + \eta\mu|^p + |\varepsilon\delta\langle y_l^*, x\rangle - \eta\mu|^p\Big) \leqslant \|x\|^p + |\lambda|^p + |\mu|^p.$$

Taking $\mu := \varepsilon\lambda$, we obtain that for all $(x, \lambda) \in E_N \times \mathbb{R}$,

$$\|\widetilde{S}(x + \lambda e_{N+1})\|^p + \varepsilon^p\sum_{l=1}^{L}\Big(\big|\delta\langle y_l^*, x\rangle + \eta\lambda\big|^p + \big|\delta\langle y_l^*, x\rangle - \eta\lambda\big|^p\Big) \leqslant \|x\|^p + (1 + \varepsilon^p)|\lambda|^p.$$

However, the definition of $\widetilde{S} = S_{\mathbf{y},\eta,\delta}$ gives

$$\|\widetilde{S}(x+\lambda e_{N+1})\|^p = \|Sx\|^p + \sum_{l=1}^{L}\Big(\big|\delta\langle y_l^*, x\rangle + \eta\,\lambda\big|^p + \big|\delta\langle y_l^*, x\rangle - \eta\,\lambda\big|^p\Big).$$

So we see that for any $(x,\lambda) \in E_N \times \mathbb{R}$,

$$\|Sx\|^p + (1+\varepsilon^p)\sum_{l=1}^{L}\Big(\big|\delta\langle y_l^*, x\rangle + \eta\,\lambda\big|^p + \big|\delta\langle y_l^*, x\rangle - \eta\,\lambda\big|^p\Big) \leqslant \|x\|^p + (1+\varepsilon^p)|\lambda|^p$$

Setting $\kappa := (1+\varepsilon^p)^{1/p}$ and replacing λ by λ/κ, this may be re-written as follows:

$$\forall (x,\lambda) \in E_N \times \mathbb{R} \ :\ \|Sx\|^p + \sum_{l=1}^{L}\Big(\big|\kappa\delta\langle y_l^*, x\rangle + \eta\,\lambda\big|^p + \big|\kappa\delta\langle y_l^*, x\rangle - \eta\,\lambda\big|^p\Big) \leqslant \|x\|^p + |\lambda|^p.$$

This means exactly that the operator $S_{\mathbf{y},\eta,\kappa\delta}$ satisfies $\|S_{\mathbf{y},\eta,\kappa\delta}\| \leqslant 1$, which contradicts the maximality of δ since $\kappa > 1$. □

Claim 4.26. — Let $z \in E_{N+1}$, and assume that $z \in \mathcal{N}(\widetilde{S})\backslash\mathcal{N}(S)$. Then $\langle e_{N+1}^*, z\rangle \neq 0$ and $P_N z \notin \operatorname{span}\big(\mathcal{N}(S)\big)$.

Proof of Claim 4.26. — Assume that $\langle e_{N+1}^*, z\rangle = 0$. Then z and e_{N+1} have disjoint supports. Since $z \in \mathcal{N}(\widetilde{S})$ and $p > 2$, it follows that $\widetilde{S}z$ and $\widetilde{S}e_{N+1}$ also have disjoint supports, by Lemma 4.3. By the definition of $\widetilde{S} = S_{\mathbf{y},\eta,\delta}$, we thus see that $\langle y_l^*, z\rangle = 0$ for $l = 1,\dots,L$. Hence $\widetilde{S}z = Sz$, which contradicts our assumption that $z \in \mathcal{N}(\widetilde{S})\backslash\mathcal{N}(S)$ since $\|\widetilde{S}\| = 1 = \|S\|$.

Assume now that $P_N z \in \operatorname{span}\big(\mathcal{N}(S)\big)$. Then by the definition of $\widetilde{S}$ we have

$$\widetilde{S}z = S(P_N z) + \eta\langle e_{N+1}^*, z\rangle\sum_{l=1}^{L}(e_{M+2l-1} - e_{M+2l}),$$

so that $\|\widetilde{S}z\|^p = \|SP_N z\|^p + 2L\eta^p|\langle e_{N+1}^*, z\rangle|^p$. Since $\langle e_{N+1}^*, z\rangle \neq 0$ and $2L\eta^p < 1$, it follows that

$$\|\widetilde{S}z\|^p < \|SP_N z\|^p + |\langle e_{N+1}^*, z\rangle|^p \leqslant \|P_N z\|^p + |\langle e_{N+1}^*, z\rangle|^p = \|z\|^p,$$

which contradicts the fact that $z \in \mathcal{N}(\widetilde{S})$. □

Claim 4.27. — Let $z \in \mathcal{N}(\widetilde{S})$, and write z as $z = x + \lambda e_{N+1}$ with $x \in E_N$ and $\lambda \in \mathbb{R}$. Then the vector $z' := x - \lambda e_{N+1}$ also belongs to $\mathcal{N}(\widetilde{S})$.

Proof of Claim 4.27. — We have $\|z'\| = \|z\|$, and it is clear from the definition of $\widetilde{S}$ that $\|\widetilde{S}z'\| = \|\widetilde{S}z\|$. □

Remark 4.28. — With the notation of Claim 4.27, if $\lambda \neq 0$ then $e_{N+1} \in \operatorname{span}(z, z')$. So, from Claims 4.25 and 4.26, we see that $e_{N+1} \in \operatorname{span}\big(\mathcal{N}(\widetilde{S})\big)$. This remark will be used in the proof of Lemma 4.31 below.

We can now prove (ii), *i.e.* that $\dim \operatorname{span}\big(\mathcal{N}(S_{\mathbf{y},\eta,\delta})\big) \geqslant 2 + \dim \operatorname{span}\big(\mathcal{N}(S)\big)$. Let us fix $z \in \mathcal{N}(\widetilde{S})\backslash\mathcal{N}(S)$ given by Claim 4.25. Write $z = x + \lambda e_{N+1}$ where $x \in E_N$, and let $z' := x - \lambda e_{N+1}$, so that $z' \in \mathcal{N}(\widetilde{S})$ by Claim 4.27. Since $x \neq 0$ and $\lambda \neq 0$ by Claim 4.26, z and z' are linearly independent. Hence, to conclude the proof of (ii), it is enough to show that $\operatorname{span}(z, z') \cap \operatorname{span}\big(\mathcal{N}(S)\big) = \{0\}$.

Let $u \in \operatorname{span}(z, z') \cap \operatorname{span}\big(\mathcal{N}(S)\big)$. There exist $a, b \in \mathbb{R}$ such that $u = az + bz' = (a+b)x + \lambda(a-b)e_{N+1}$. Since $\lambda \neq 0$ and $u \in \operatorname{span}\big(\mathcal{N}(S)\big) \subseteq E_N$, we must have $a - b = 0$. Hence $u = 2ax$; and since $u \in \operatorname{span}\big(\mathcal{N}(S)\big)$ and $x \notin \operatorname{span}\big(\mathcal{N}(S)\big)$, it follows that $a = 0$. So $u = 0$, as required. □

Lemma 4.29. — *Assume that $p > 2$. Let $N_0 \in \mathbb{Z}_+$ and let $\varepsilon > 0$. Let also $N, M \in \mathbb{Z}_+$ with $N \geqslant N_0$, and let $S \in \mathcal{B}(E_N, E_M)$ and $\mathbf{y} = (y_1^*, \dots, y_L^*)$ be as in Definition 4.16. If $\eta \in (0, (2L)^{-1/p})$ is close enough to $(2L)^{-1/p}$ (depending only on ε and $\mathbf{y}$) then, for any $\delta > 0$ such that $\|S_{\mathbf{y},\eta,\delta}\| = 1$, it holds that $\|S_{\mathbf{y},\eta,\delta} e_n - S e_n\| < \varepsilon$ for all $n \in [\![0, N_0]\!]$.*

Proof. — For any $\delta > 0$ and any $n \in [\![0, N_0]\!]$, we have

$$\|S_{\mathbf{y},\eta,\delta} e_n - S e_n\|^p = 2 \left(\sum_{l=1}^{L} |\langle y_l^*, e_n \rangle|^p \right) \delta.$$

So it is enough to show that if we choose η sufficiently close to $(2L)^{-1/p}$, then any δ such that $\|S_{\mathbf{y},\eta,\delta}\| = 1$ must be very small.

Let $c_l := \langle y_l^*, e_0 \rangle$ for $l = 1, \dots, L$. For any $\delta > 0$ such that $\|S_{\mathbf{y},\eta,\delta}\| = 1$ and for every $t > 0$, we have

$$\begin{aligned} 1 + t^p = \|te_0 + e_{N+1}\|^p &\geqslant \|S_{\mathbf{y},\eta,\delta}(te_0 + e_{N+1})\|^p \\ &\geqslant \sum_{l=1}^{L} \Big(|\delta c_l t + \eta|^p + |\delta c_l t - \eta|^p \Big). \end{aligned}$$

Now, since $p > 2$, we can make use of Lemma 4.4: for any $u, v \in \mathbb{C}$,

$$|u + v|^p + |u - v|^p \geqslant 2|u|^p + p\,|u|^{p-2}|v|^2.$$

Taking $u := \eta$ and $v := \delta c_l t$ for $l = 1, \dots, L$, this gives

$$2L\eta^p + C\eta^{p-2}\delta^2 t^2 \leqslant 1 + t^p\,, \qquad \text{where} \quad C := p \sum_{l=1}^{L} |c_l|^2.$$

Thus, writing $2L\eta^p = 1 - \alpha$ and assuming, as we may, that $\alpha \leqslant 1/2$, we obtain

$$\forall t > 0 \ : \ K\delta^2 \leqslant \frac{\alpha}{t^2} + +t^{p-2},$$

where $K := C \times (4L)^{-1+\frac{2}{p}}$. Optimising with respect to t, *i.e.* taking $t := \left(\frac{2\alpha}{p-2}\right)^{1/p}$, this gives

$$K\delta^2 \leqslant C_p\, \alpha^{1-\frac{2}{p}}$$

for some constant C_p depending only on p. Hence we see that δ is indeed very small if η is close enough to $(2L)^{-1/p}$. □

4.6. Maximal modifications. — The following definition looks rather natural.

Definition 4.30. — Let $N, M \in \mathbb{Z}_+$ and let $S \in \mathcal{B}(E_N, E_M)$ with $\|S\| = 1$. We say that an operator $\widetilde{S} \in \mathcal{B}(E_{N+1}, \ell_p)$ is a *maximal modification of S* if $\widetilde{S}$ is a $(\mathbf{y}, \eta, \delta)$-modification of S for some triple $(\mathbf{y}, \eta, \delta)$ such that $\mathbf{y}$ is a basis of $\mathcal{N}(S)^\perp$.

The interest of this definition lies in the fact that, roughly speaking, maximal modifications "preserve the interval property". This is the content of the next lemma.

Lemma 4.31. — *Assume that $p > 2$. Let $N, M \in \mathbb{Z}_+$, and let $(S_0, \dots, S_K)$ be a finite sequence of operators, with $S_k \in \mathcal{B}(E_{N+k}, E_{M+L_k})$ for some $L_k \in \mathbb{Z}_+$ (with $L_0 = 0$) and $\|S_k\| = 1$, such that S_{k+1} is a maximal modification of S_k for every $k < K$. If S_0 is a special operator, then each S_k has the interval property.*

Proof. — Since S_0 is a special operator, any $z \in \mathcal{N}(S_0)$ has support equal to $[\![0, N]\!]$ by Lemma 4.15. In particular, S_0 has the interval property.

Let us first show that S_1 has the interval property. Let $z \in E_{N+1}$ be a norming vector for S_1. We have to show that $\operatorname{supp}(x) = [\![0, a]\!]$ for some $a \leqslant N+1$. If $z \in \mathcal{N}(S_0)$, then $\operatorname{supp}(z) = [\![0, N]\!]$ because S_0 is a special operator. So we assume that $z \notin \mathcal{N}(S_0)$. Then $\langle e^*_{N+1}, z\rangle \neq 0$ and $P_N z \neq 0$ by Claim 4.26. We show that $\operatorname{supp}(z) = [\![0, N+1]\!]$.

Choose $n \in [\![0, N]\!]$ such that $\langle e^*_n, z\rangle \neq 0$ and, towards a contradiction, assume that there exists $n' \in [\![0, N]\!]$ such that $\langle e^*_{n'}, z\rangle = 0$. Since $S_0 \in \mathcal{B}(E_N, E_M)$ is a special operator, one can find $m \in [\![0, M]\!]$ such that $m \in \operatorname{supp}(S_0 e_n) \cap \operatorname{supp}(S_0 e_{n'})$ but $m \notin \operatorname{supp}(S_0 e_{n''})$ for any $n'' \in [\![0, N]\!] \backslash \{n, n'\}$. Since S_1, being a modification of S_0, satisfies $P_M S_1 P_N = S_0$ and $P_M S_1 e_{N+1} = 0$, we see that $m \in \operatorname{supp}(S_1 e_n) \cap \operatorname{supp}(S_1 e_{n'})$ and $m \notin \operatorname{supp}(S_1 e_{n''})$ for any $n'' \in [\![0, N+1]\!] \backslash \{n, n'\}$. Since $\langle e^*_n, z\rangle \neq 0$ and $\langle e^*_{n'}, z\rangle = 0$, it follows that $\langle e^*_m, S_1 z\rangle \neq 0$, *i.e.* $m \in \operatorname{supp}(S_1 z)$. However, since z and $e_{n'}$ have disjoint supports and $z \in \mathcal{N}(S_1)$, the vectors $S_1 e_{n'}$ and $S_1 z$ must have disjoint supports by Lemma 4.3. This is the required contradiction.

Now, assume that we have been able to prove that $S_0, \dots, S_k$ have the interval property for some $1 \leqslant k < K$, and let us show that S_{k+1} has the interval property. Let $z \in E_{N+k+1}$ be a norming vector for S_{k+1}. We have to show that $\operatorname{supp}(z)$ is an interval $[\![0, L]\!]$. If $z \in \mathcal{N}(S_k)$, then we are done by our induction hypothesis; so we assume that $z \notin \mathcal{N}(S_k)$. Then $\langle e^*_{N+k+1}, z\rangle \neq 0$ by Claim 4.26. Hence, we have to show that $\langle e^*_n, z\rangle \neq 0$ for all $n \in [\![0, N+k]\!]$.

Let us first consider n such that $N < n \leqslant N+k$, and write $n = N + k'$ with $1 \leqslant k' \leqslant k$. Towards a contradiction, assume that $\langle e^*_{N+k'}, z\rangle = 0$. Then $\operatorname{supp}(S_{k+1} e_{N+k'}) \cap \operatorname{supp}(S_{k+1} z) = \varnothing$, by Lemma 4.3. In particular, since $\operatorname{supp}(S_{k'} e_{N+k'}) \subseteq \operatorname{supp}(S_{k+1} e_{N+k'})$ (because S_{k+1} is obtained from $S_{k'}$ by successive modifications), we have $\operatorname{supp}(S_{k+1} z) \cap \operatorname{supp}(S_{k'} e_{N+k'}) = \varnothing$. Denoting by $\mathbf{y} = (y_1, \dots, y^*_L) \subseteq \mathcal{N}(S_{k'-1})^\perp$ the sequence of linear functionals used to construct $S_{k'}$ from $S_{k'-1}$, this means that $\langle y^*_l, P_{N+k'-1} z\rangle = 0$ for $l = 1, \dots, L$; and since $S_{k'}$ is a *maximal* modification of $S_{k'-1}$, *i.e.* $\mathbf{y}$ is a basis of $\mathcal{N}(S_{k'-1})^\perp$, it follows that $P_{N+k'-1} z \in \operatorname{span}\big(\mathcal{N}(S_{k'-1})\big)$. In particular, $P_{N+k'-1} z \in \operatorname{span}\big(\mathcal{N}(S_k)\big)$. Moreover, by successive applications of Remark 4.28, we see that $e_{N+r} \in \operatorname{span}\big(\mathcal{N}(S_r)\big)$ for all $1 \leqslant r \leqslant K$. In particular, $e_{N+r} \in \operatorname{span}\big(\mathcal{N}(S_k)\big)$ for all $k' \leqslant r \leqslant k$. So we get that $P_{N+k} z \in \operatorname{span}\big(\mathcal{N}(S_k)\big)$, which contradicts Claim 4.26 since $z \in \mathcal{N}(S_{k+1}) \backslash \mathcal{N}(S_k)$.

It remains to consider the case where $n \in [\![0, N]\!]$. Using the fact that S_0 is a special operator, one shows that $\langle e_n, z\rangle \neq 0$ exactly as in the above proof for S_1. □

We are finally ready to prove Proposition 4.2.

4.7. Proof of Proposition 4.2. — Assume that $p = 3$. We have to show that for any $A \in \mathcal{B}_1(\ell_p)$, for any $N_0 \in \mathbb{Z}_+$ and any $\varepsilon > 0$, one can find $T \in \mathcal{B}_1(\ell_p)$ such that $\overline{\operatorname{span}}\big(\mathcal{N}(T)\big) = \ell_p$ and $\|Te_n - Ae_n\| < \varepsilon$ for all $n \in [\![0, N_0]\!]$. Let us fix A, N_0 and ε.

We only need to show that there exist $N', M' \in \mathbb{Z}_+$ with $N' \geqslant N_0$ and $S \in \mathcal{B}(E_{N'}, E_{M'})$ with $\|S\| = 1$ such that $\operatorname{span}\big(\mathcal{N}(S)\big) = E_{N'}$ and $\|Se_n - Ae_n\| < \varepsilon$ for all $n \in [\![0, N_0]\!]$. Indeed,

if this is has been done then, denoting by $\mathbf{S}$ the canonical forward shift on ℓ_p, the operator $T := S + \mathbf{S}^{M'}(I - P_{N'})$ does the job because $\operatorname{span}\big(\mathcal{N}(T)\big)$ contains e_n for all $n \in \mathbb{Z}_+$.

Let $N := N_0 + 1$. By Lemma 4.14, one can find $M \in \mathbb{Z}_+$ and a special operator $S_0 \in \mathcal{B}(E_N, E_M)$ such that $\|S_0\| = 1$ and $\|S_0 e_n - Ae_n\| < \varepsilon/2$ for all $n \in [\![0, N_0]\!]$. Let $d_0 := \dim \operatorname{span}\big(\mathcal{N}(S_0)\big)$. If $d_0 = N+1$, *i.e.* $\operatorname{span}\big(\mathcal{N}(S_0)\big) = E_N$, we are done. Otherwise, by Lemmas 4.15, 4.6, 4.19 and 4.24, one can find $S_1 \in \mathcal{B}(E_{N+1}, M_1)$, a maximal modification of S_0, such that $\|S_1 e_n - S_0 e_n\| < \varepsilon/2^2$ for $n = 0, \dots, N_0$ and $\dim \operatorname{span}\big(\mathcal{N}(S_1)\big) \geqslant d_0 + 2$. If $d_0 + 2 = N + 2 = \dim E_{N+1}$, we are done. Otherwise, we can repeat the process (since S_1 has the interval property by Lemma 4.31 and hence $\mathcal{N}(S_1) \cap S_{E_{N+1}}$ is finite by Lemma 4.6), and continue as long as possible. In other words, as long as this is possible, we construct operators $S_k \in \mathcal{B}(E_{N+k}, M_k)$, $k \geqslant 0$ such that S_{k+1} is a maximal modification of S_k with $\|S_{k+1} e_n - S_k e_n\| < \varepsilon/2^{k+1}$ for $n = 0, \dots, N_0$ and $\dim \operatorname{span}\big(\mathcal{N}(S_k)\big) \geqslant d_0 + 2k$. Since at each step the dimension of $\operatorname{span}\big(\mathcal{N}(S_k)\big)$ is increased at least by 2 but the dimension of E_{N+k} is only increased by 1, the process must end after finitely many steps: there exists K such that $\dim \operatorname{span}\big(\mathcal{N}(S_K)\big) \geqslant \dim E_{N+K}$, and hence $\operatorname{span}\big(\mathcal{N}(S_K)\big) = E_{N+K}$. Setting $N' := N + K$, $M' := M_K$ and $S := S_K$, we have $\|Se_n - Ae_n\| < \varepsilon \sum_{k=0}^{K} 2^{-k-1} < \varepsilon$ for $n = 0, \dots, N_0$, which concludes the proof. □

4.8. A remark concerning the case $p = 4$. — In this last sub-section, we would like to point out the following fact, which is in the spirit of Lemma 4.8.

Fact 4.32. — Assume that $p = 4$. Let E be a subspace of ℓ_p, and let $S \in \mathcal{B}(E, \ell_p)$. If $x, y \in \mathcal{N}(S)$ and if $\operatorname{span}(x, y)$ contains a vector $z \in \mathcal{N}(S)$ which is not a multiple of x or y, then $\operatorname{span}(x, y) \subseteq \mathcal{N}(S) \cup \{0\}$. In particular, if $\dim(E) = 2$ and if S admits 3 pairwise linearly independent norming vectors, then S is a scalar multiple of an isometry.

Proof. — We may assume that $\|S\| = 1$. Let $x, y \in \mathcal{N}(S)$, and let us consider the function $\varphi : \mathbb{R}^2 \to \mathbb{R}$ defined by

$$\varphi(s, t) := \|sx + ty\|^4 - \|S(sx + ty)\|^4.$$

Since $p = 4$, this a polynomial function of degree at most 4. Moreover, we have $\varphi(s, t) \geqslant 0$ on $\mathbb{R}^2$, and $\varphi(s, 0) \equiv 0 \equiv \varphi(0, t)$. We have to show that if there exists (s_0, t_0) such that $s_0 t_0 \neq 0$ and $\varphi(s_0, t_0) = 0$, then $\varphi(s, t) = 0$ for all $(s, t) \in \mathbb{R}^2$.

Since $\varphi(s, 0) = 0 = \varphi(0, t)$, we may write

$$\varphi(s, t) = st\, \psi(s, t),$$

where ψ is a polynomial function of degree at most 2. Moreover since $\varphi(s, t) \geqslant 0$ on $\mathbb{R}^2$, the function φ has a minimum at $(s, 0)$ and at $(0, t)$ for any $s, t \in \mathbb{R}$. So we must have

$$\frac{\partial \varphi}{\partial t}(s, 0) \equiv 0 \equiv \frac{\partial \varphi}{\partial s}(0, t),$$

i.e.

$$s\psi(s, 0) \equiv 0 \equiv t\psi(0, t).$$

This means that $\psi(s, 0) \equiv 0 \equiv \psi(0, t)$, and hence that $\psi(s, t) = c\, st$ for some constant c. So we obtain

$$\varphi(s, t) = c\, s^2 t^2,$$

and the result follows. □

Remark 4.33. — The proof of Fact 4.32 has shown the following. Assume that $p = 4$, let E be a subspace of ℓ_p and let $S \in \mathcal{B}(E, \ell_p)$ with $\|S\| = 1$. If $x, y \in \mathcal{N}(S)$ then, for all $s, t \in \mathbb{R}$:

$$\|sx + ty\|^4 - \|S(sx + ty)\|^4 = \left(\|x + y\|^4 - \|S(x + y)\|^4\right) s^2 t^2.$$

In particular, we see that $\|sx + ty\|^4 - \|S(sx + ty)\|^4 = \|sy + tx\|^4 - \|S(sy + tx)\|^4$; this is a "symmetry" property that was not immediately apparent. It seems natural to wonder whether this symmetry property holds true for any even integer $p = 2n$. Note that this is indeed the case for $p = 2$, since a direct computation using the scalar product reveals that $\|sx + ty\|^2 - \|S(sx + ty)\|^2 = \left(\|x + y\|^2 - \|S(x + y)\|^2\right) st$.

Remark 4.34. — It is shown in [24] that if $E = \ell_p^2$ and $p > 1$ is arbitrary, then any $S \in \mathcal{B}(E)$ having 3 pairwise linearly independent norming vectors is a scalar multiple of an isometry. This might be true for any 2 - dimensional subspace of ℓ_p. In fact, it is tempting to "conjecture" the following: if E is a d - dimensional subspace of ℓ_p and if $S \in \mathcal{B}(E, \ell_p)$ admits $d + 1$ norming vectors such that any d of them are linearly independent, then S is a scalar multiple of an isometry.

5. Typical properties of Hilbert space contractions

In this section, we illustrate the point of continuity approach by giving short and streamlined proofs of two nice results due to respectively to Eisner [9] and Eisner-Mátrai [10] concerning Hilbert space contractions. We start with the following proposition, which is the main result of [9].

Proposition 5.1. — *A typical $T \in (B_1(\ell_2), \mathtt{WOT})$ is unitary.*

Proof. — This follows immediately from Corollary 2.10 and part (2) of Proposition 2.11. □

Now we turn to a rather surprising result from [10]. Let us denote by B_∞ the backward shift with infinite multiplicity, *i.e.* the canonical backward shift acting on $\ell_2(\mathbb{Z}_+, \ell_2)$.

Theorem 5.2. — *A typical $T \in \left(\mathcal{B}_1(\ell_2), \mathtt{SOT}\right)$ is unitarily equivalent to B_∞.*

The proof relies on the next two lemmas, which we state in greater generality than what is needed for the proof of Theorem 5.2.

Lemma 5.3. — *If X is any separable Banach space, then a typical $T \in (\mathcal{B}_1(X), \mathtt{SOT})$ is strongly stable, i.e. such that $T^n \xrightarrow{\mathtt{SOT}} 0$ as $n \to \infty$.*

Proof. — This is proved in [10, Lemma 5.9] or [14, Proposition 3.7]. However, the proof is so short that we can reproduce it. Let D be a countable dense subset of X. Then since $\|T^n x\|$ is non-increasing for any $T \in \mathcal{B}_1(X)$ and any $x \in X$, an operator $T \in \mathcal{B}_1(X)$ is strongly stable if and only if $\forall z \in D \ : \ \inf_{n \in \mathbb{N}} \|T^n z\| = 0$. It follows that the set of strongly stable operators is G_δ in $(\mathcal{B}_1(X), \mathtt{SOT})$; and this set is also dense since it contains all $T \in \mathcal{B}_1(X)$ such that $\|T\| < 1$. □

Recall that an operator $T \in \mathcal{B}(X)$ is said to be *upper semi-Fredholm* if T has closed range and $\dim \ker(T) < \infty$. Recall also that the Banach space X has the *Metric Approximation Property* if the identity operator I_X can be uniformly approximated on compact sets by finite rank operators in $\mathcal{B}_1(X)$.

Lemma 5.4. — *Assume that the Banach space X has the* MAP. *Then a typical $T \in (\mathcal{B}_1(X), \mathtt{SOT})$ is not upper semi-Fredholm.*

Proof. — We need the following fact, which is proved in [10, Lemma 5.13] when X is a Hilbert space.

Fact 5.5. — A typical $T \in (\mathcal{B}_1(X), \mathtt{SOT})$ has the following property: for any $\varepsilon > 0$ and any $n \in \mathbb{N}$, there exists a subspace $E \subseteq X$ with $\dim(E) > n$ such that $\|T_{|E}\| < \varepsilon$.

Proof of Fact 5.5. — For any fixed finite-dimensional subspace $E \subseteq X$ and any $\varepsilon > 0$, the set $\{T \in \mathcal{B}_1(X);\ \|T_{|E}\| < \varepsilon\}$ is easily seen to be $\mathtt{SOT}$-open in $\mathcal{B}_1(X)$. So the set of all $T \in \mathcal{B}_1(X)$ with the above property is $\mathtt{SOT}$-G_δ. Moreover, this set is also $\mathtt{SOT}$-dense in $\mathcal{B}_1(X)$ since it contains all finite rank contraction operators (and the latter are dense in $\mathcal{B}_1(X)$ thanks to the MAP). □

For any $n \in \mathbb{Z}_+$, let us set

$$\mathcal{M}_n := \big\{T \in \mathcal{B}_1(X);\ T \text{ has closed range and } \dim\ker(T) = n\big\}.$$

If $T \in \mathcal{M}_n$, then there is a subspace $F \subseteq X$ such that $\operatorname{codim}(F) = n$ and $T_{|F}$ is an embedding. By Fact 5.5, it follows that each set $\mathcal{M}_n$ is $\mathtt{SOT}$-meager in $\mathcal{B}_1(X)$; which proves the lemma. □

Remark 5.6. — Lemma 5.4 has the following rather curious consequence: if a typical $T \in (\mathcal{B}_1(X), \mathtt{SOT})$ has closed range, then a typical $T \in \mathcal{B}_1(X)$ has an infinite-dimensional kernel.

Proof of Theorem 5.2. — In this proof, the word "typical" refers to the topology $\mathtt{SOT}$. By the Wold Decomposition Theorem, an operator $T \in \mathcal{B}_1(\ell_2)$ is unitarily equivalent to B_∞ if and only if T is a co-isometry with infinite-dimensional kernel and $T^n \xrightarrow{\mathtt{SOT}} 0$ as $n \to \infty$. So, by the Baire Category Theorem, it is enough to prove separately the following three facts: a typical $T \in \mathcal{B}_1(\ell_2)$ is a co-isometry; a typical $T \in \mathcal{B}_1(\ell_2)$ is such that $T^n \xrightarrow{\mathtt{SOT}} 0$ as $n \to \infty$; and a typical $T \in \mathcal{B}_1(\ell_2)$ is such that $\dim\ker(T) = \infty$. The first fact follows from part (3) of Proposition 2.11 and Corollary 2.10. The second fact is Lemma 5.3. As to the third fact, note that a typical $T \in \mathcal{B}_1(\ell_2)$ is surjective since T^* is an isometry, and hence a typical $T \in \mathcal{B}_1(\ell_2)$ has closed range. By Remark 5.6, it follows that a typical $T \in \mathcal{B}_1(\ell_2)$ is such that $\dim\ker(T) = \infty$. □

Remark 5.7. — As shown in [10, Proposition 2.7 and Corollary 6.2], the situation is quite the opposite if one considers the topologies $\mathtt{WOT}$ and $\mathtt{SOT}^*$ rather than $\mathtt{SOT}$; namely, all orbits of unitary equivalence are $\mathtt{WOT}$-meager and $\mathtt{SOT}^*$-meager.

To conclude this section, we show that Theorem 5.2 fails on ℓ_p for any $p \neq 2$.

Proposition 5.8. — *Let $X = \ell_p$, $1 \leqslant p \neq 2 < \infty$. Denote by* $\operatorname{Iso}(X)$ *the group of all surjective linear isometries of X. For any $A \in \mathcal{B}_1(X)$, the set $\big\{J^{-1}AJ;\ J \in \operatorname{Iso}(X)\big\}$ is meager in $\big(\mathcal{B}_1(X), \mathtt{SOT}\big)$.*

The proof of Proposition 5.8 relies on the following lemma.

Lemma 5.9. — *Let X be a (separable) Banach space, and let $\mathcal{A} \subseteq \mathcal{B}_1(X)$. Assume that there exists some vector $e \neq 0$ in X such that the set $\{Te;\ T \in \mathcal{A}\}$ is meager in X. Then $\mathcal{A}$ is meager in $\big(\mathcal{B}_1(X), \tau\big)$ for any topology τ on $\mathcal{B}_1(X)$ finer than* $\mathtt{SOT}$ *and weaker than the operator norm topology.*

Proof. — We may assume that $\|e\| = 1$. Let us denote by B_X the closed unit ball of X and by $\delta_e : \mathcal{B}_1(X) \to B_X$ the evaluation map defined by $\delta_e(T) := Te$.

Claim 5.10. — The map $\delta_e : \mathcal{B}_1(X) \to B_X$ is is continuous on $(\mathcal{B}_1(X), \tau)$ and, for any non-empty τ-open set $\mathcal{O} \subseteq \mathcal{B}_1(X)$, the set $\delta_e(\mathcal{O})$ has non-empty interior in B_X.

Proof of Claim 5.10. — That δ_e is continuous on $(\mathcal{B}_1(X), \tau)$ is clear since τ is finer than SOT. Let $\mathcal{O} \subseteq \mathcal{B}_1(X)$ be non-empty and τ-open. Since τ is weaker than the operator norm topology, we can choose $T_0 \in \mathcal{O}$ such that $\|T_0\| < 1$, and then $\delta > 0$ such that the open ball $B(T_0, \delta)$ is contained in $\mathcal{O}$. Let $x_0 := T_0 e$. For any $x \in B(x_0, \delta)$, the operator $T := T_0 + e^* \otimes (x - x_0)$ is in $B(T_0, \delta)$, hence in $\mathcal{O}$, and satisfies $Te = x$. This shows that $\delta_e(\mathcal{O})$ has non-empty interior. □

It follows from Claim 5.10 that for any meager set $M \subseteq B_X$, the set $\delta_e^{-1}(M)$ is meager in $(\mathcal{B}_1(X), \tau)$: indeed, if $F \subseteq B_X$ is a closed set with empty interior, then $\delta_e^{-1}(F)$ is closed in $(\mathcal{B}_1(X), \tau)$, and it must have empty interior because $\delta_e\big(\delta_e^{-1}(F)\big) \subseteq F$. Taking $M := \{Te;\ T \in \mathcal{A}\} = \delta_e(\mathcal{A})$, we immediately get that $\mathcal{A}$ is meager in $(\mathcal{B}_1(X), \tau)$. □

Remark 5.11. — Let B_∞ be the backward shift with infinite multiplicity acting on $H = \ell_2(\mathbb{Z}_+, \ell_2)$. By Theorem 5.2 and Lemma 5.9, we see that for any $u \in H \backslash \{0\}$, the set $\Omega_u := \{J^{-1} B_\infty J u;\ J \in \mathrm{Iso}(H)\}$ is non-meager in H. In fact, one can check directly that Ω_u contains every vector $v \in H$ such that $\|v\| \leqslant \|u\|$ and $\sqrt{\|u\|^2 - \|v\|^2}\, \|v\| \geqslant |\langle u, v\rangle|$, and hence that Ω_u has non-empty interior. Indeed, since $\dim \ker(B_\infty) \geqslant 2$, these conditions allow one to find $v', z \in \ker(B_\infty)$ such that $\|v'\| = \|v\|$, $\|z\| = \sqrt{\|u\|^2 - \|v\|^2}$ and $\langle z, v'\rangle = \langle u, v\rangle$. Then $u' := z + B_\infty^* v'$ satisfies $\|u'\| = \|u\|$ and $B_\infty u' = v'$ because B_∞^* is an isometry, and $\langle u', v'\rangle = \langle u, v\rangle$. Since $\|u'\| = \|u\|$, $\|v'\| = \|v\|$ and $\langle u', v'\rangle = \langle u, v\rangle$, one can find $J \in \mathrm{Iso}(H)$ such that $Ju = u'$ and $Jv = v'$, and we have $B_\infty J u = J v$.

Proof of Proposition 5.8. — Let $A \in \mathcal{B}_1(\ell_p)$. By Lemma 5.10, it is enough to find some vector $e \neq 0$ in ℓ_p such that the set $\big\{J^{-1} A J e;\ J \in \mathrm{Iso}(\ell_p)\big\}$ is meager in ℓ_p. We take $e := e_0$, the first vector of the canonical basis of ℓ_p.

It is well known (see e.g. [23, Proposition 2.f.14] that if $J \in \mathrm{Iso}(\ell_p)$, then there exist a permutation $\sigma : \mathbb{Z}_+ \to \mathbb{Z}_+$ and scalars $\lambda_j \in \mathbb{T}$, $j \in \mathbb{Z}_+$, such that

$$\forall j \in \mathbb{Z}_+ \ :\ J e_j = \lambda_j e_{\sigma(j)}.$$

We then have

$$(J^*)^{-1} e_k^* = \lambda_k^{-1} e_{\sigma(k)}^* \qquad \text{for all } k \in \mathbb{Z}_+,$$

and hence

$$\langle e_0^*, J^{-1} A J e_0\rangle = \langle e_{\sigma(0)}^*, A e_{\sigma(0)}\rangle.$$

So, we see that the set $\big\{\langle e_0^*, J^{-1} A J e_0\rangle;\ J \in \mathrm{Iso}(\ell_p)\big\}$ is countable, and hence meager in $\mathbb{K}$. Since the non-zero linear functional $e_0^* : \ell_p \to \mathbb{K}$ is continuous and open, it follows that $\{J^{-1} A J e_0;\ J \in \mathrm{Iso}(\ell_p)\}$ is meager in ℓ_p. □

6. Operators which do not attain their norm

This section is motivated by Corollary 2.15 and also by the theory of *norm-attaining* operators. We recall that an operator $T \in \mathcal{B}_1(X)$ on a (real or complex) separable Banach space X is said to be norm-attaining if $\mathcal{N}(T) \neq \varnothing$, *i.e.* there exists $z \neq 0$ such that $\|Tz\| = \|z\|$; equivalently, if there exists $x \in B_X$ such that $\|Tx\| = \|T\|$.

Let $\mathbf{M} := \mathcal{B}_1(X)$. Assuming that X has the Kadec-Klee property, Corollary 2.15 says that if one wants to prove that $(\mathcal{B}_1(X), \mathtt{WOT})$ and $(\mathcal{B}_1(X), \mathtt{SOT})$ have the same comeager sets, it is enough to show that there is an $\mathtt{SOT}$-dense set of $T \in \mathcal{B}_1(X)$ having many norming vectors. So it may look natural to ask whether a *typical* $T \in \big(\mathcal{B}_1(X), \mathtt{SOT}\big)$ has many norming vectors.

Note that if one considers the operator norm topology rather than the topology $\mathtt{SOT}$, then it is well known that the answer is negative. Indeed, it follows from a classical result of Bourgain [5] that if X is a Banach space with the Radon-Nikodým property, then a typical $T \in (\mathcal{B}_1(X), \|\cdot\|)$ is *absolutely strongly exposing*, which means the following: there exists $x \in S_X$ such that, whenever $(x_n) \subseteq B_X$ is a maximising sequence for T, *i.e.* $\|Tx_n\| \to \|T\|$, one can find a sequence $(\omega_n) \subseteq \mathbb{T}$ such that $\omega_n x_n \to x$. This implies in particular that T is norm-attaining and attains its norm at a unique point of S_X (up to rotation). We refer to [18] and the references therein for much more on this topic.

Somewhat surprisingly, our next result will show that for $X = \ell_p$, $1 < p < \infty$, and *with respect to the topology* $\mathtt{SOT}^*$, the typical behaviour is in fact to have no norming vector at all. This, of course, does not exclude the possibility of finding an $\mathtt{SOT}$-dense (or even $\mathtt{SOT}^*$-dense) set of operators T with many norming vectors.

Before stating the result, let us introduce some terminology.

The Banach space X is said to be *smooth* if its norm is Gateaux differentiable outside the origin; equivalently, if for every $x \in X\backslash\{0\}$, there is only one linear functional $x^* \in X^*$ such that $\|x^*\| = 1$ and $\langle x^*, x\rangle = \|x\|$.

By a *duality mapping* for X we mean any map $\mathbf{J} : X \to X^*$ with the following properties:

- $\mathbf{J}(0) = 0$ and $\mathbf{J}(x) \neq 0$ for all $x \neq 0$,
- $\langle \mathbf{J}(x), x\rangle = \|\mathbf{J}(x)\|\|x\|$ for all $x \in X$,
- $\|\mathbf{J}(x)\|$ depends only on $\|x\|$ and is a non-decreasing function of $\|x\|$.

Note that if X is smooth, then there is a "canonical" duality mapping $\mathbf{J}_0 : X \to X^*$; namely, $\mathbf{J}_0(0) = 0$ and $\mathbf{J}_0(x)$ is the Gateaux differential of the norm at x if $x \neq 0$, *i.e.* the unique linear functional such that $\|\mathbf{J}_0(x)\| = 1$ and $\langle \mathbf{J}_0(x), x\rangle = \|x\|$. In this case, any duality mapping has the form $\mathbf{J}(x) = c(\|x\|)\mathbf{J}_0(x)$ for some non-decreasing function $c : [0, \infty) \to [0, \infty)$.

We will say that a projection operator $P \in \mathcal{B}(X)$ is *M-like* if for every $T \in \mathcal{B}(X)$, it holds that

$$\|PTP + (I - P)T(I - P)\| = \max\big(\|PTP\|, \|(I - P)T(I - P)\|\big).$$

(Note that this forces $\|P\| = 1 = \|I - P\|$.) For example, if $X = \ell_p$ or c_0 and $\Lambda \subseteq \mathbb{Z}_+$, the canonical projection P_Λ onto $\overline{\mathrm{span}}\,\{e_j;\ j \in \Lambda\}$ is M-like. More generally, any L_p-projection, $1 \leqslant p \leqslant \infty$ is M-like. The terminology is justified by the following fact: denoting by $\mathcal{D}_P(X)$ the subspace of $\mathcal{B}(X)$ consisting of all operators which are block-diagonal with respect to the decomposition $X = \mathrm{ran}(P) \oplus \ker(P)$, the projection P is M-like if and only if the canonical projection $\pi : A \oplus B \mapsto A$ is an M-projection on $\mathcal{D}_P(X)$ in the usual sense, *i.e.* $\|T\| = \max\big(\|\pi(T)\|, \|(I - \pi)(T)\|\big)$ for every $T \in \mathcal{D}_P(X)$.

Theorem 6.1. — *Let X be a reflexive (separable) Banach space. Assume that X is smooth, that there exists a duality mapping $\mathbf{J} : X \to X^*$ which is (w, w^*)-continuous on bounded sets, and that there exists a sequence $(P_N)_{N\geqslant 0}$ of finite rank M-like projections on X such that $P_N \xrightarrow{\mathtt{SOT}^*} I$. Then a typical $T \in (\mathcal{B}_1(X), \mathtt{SOT}^*)$ does not attain its norm.*

From this result, we deduce

Corollary 6.2. — *For any $1 < p < \infty$, a typical $T \in \big(\mathcal{B}_1(\ell_p), \mathtt{SOT}^*\big)$ does not attain its norm.*

Proof. — We only have to check that there exists a duality mapping $\mathbf{J} : \ell_p \to \ell_q$ which is (w, w^*) - continuous on bounded sets. This is quite well known. For $x = (x_j)_{j\geqslant 0} \in \ell_p$, define $\mathbf{J}(x)$ as follows:

$$\mathbf{J}(x)_j := \frac{|x_j|^p}{x_j}, \qquad j \in \mathbb{Z}_+,$$

with the natural convention $\frac{|0|^p}{0} = 0$. Then $\mathbf{J}(x) \in \ell_q$ with $\|\mathbf{J}(x)\|_q = \|x\|_p^{p/q} = \|x\|_p^{p-1}$ and $\langle \mathbf{J}(x), x\rangle = \|x\|_p^p$, so $\mathbf{J}$ is a duality mapping. Moreover, $\mathbf{J}$ is (w, w^*) - continuous on bounded sets because it is bounded on bounded sets and w^*- convergence is equivalent to coordinate-wise convergence on bounded subsets of ℓ_q. □

Remark 6.3. — On can replace the space ℓ_p in the statement of Corollary 6.2 by any Banach space X which is an ℓ_p - direct sum of a sequence $(E_j)_{j\geqslant 0}$ of smooth finite-dimensional spaces. The duality mapping $\mathbf{J} : X \to X^*$ with the required continuity property is defined as follows: if $x = (x_j)_{j\geqslant 0} \in X$, then $\mathbf{J}(x) = (\mathbf{J}_j(x_j))_{j\geqslant 0}$, where $\mathbf{J}_j(x_j) \in E_j^*$ is the unique linear functional satisfying $\langle \mathbf{J}_j(x_j), x_j\rangle = \|x_j\|^p$ and $\|\mathbf{J}_j(x_j)\| = \|x_j\|^{p-1}$.

The proof of Theorem 6.1 relies on the next two lemmas. In what follows, we denote by $\mathcal{S}(X)$ the unit sphere of $\mathcal{B}(X)$. Our first lemma is well known.

Lemma 6.4. — *If X is any infinite-dimensional Banach space, then $\mathcal{S}(X)$ is $\mathtt{SOT}$ - G_δ and $\mathtt{SOT}^*$ - dense in $\mathcal{B}_1(X)$.*

Proof. — It is clear that $\mathcal{S}(X)$ is $\mathtt{SOT}$ - G_δ since if $T \in \mathcal{B}_1(X)$, then

$$T \in \mathcal{S}(X) \iff \forall n \in \mathbb{N}\ \exists x \in B_X \ :\ \|Tx\| > 1 - \frac{1}{n}\cdot$$

In order to prove that $\mathcal{S}(X)$ is $\mathtt{SOT}^*$- dense in $\mathcal{B}_1(X)$, it is enough to show the following: for any $A \in \mathcal{B}_1(X)$ and for any $x_1, \dots, x_N \in X$ and $x_1^*, \dots, x_N^* \in X^*$, one can find $T \in \mathcal{S}(X)$ such that $Tx_i = Ax_i$ and $T^*x_i^* = A^*x_i^*$ for $i = 1, \dots, N$. Choose $x^* \in X^*\backslash\{0\}$ such that $\langle x^*, x_i\rangle = 0$ for $i = 1, \dots, N$ and $x \in X\backslash\{0\}$ such that $\langle x_i^*, x\rangle = 0$ for $i = 1, \dots, N$. Since $\|A\| \leqslant 1$, one can find $\varepsilon \geqslant 0$ such that $T := A + \varepsilon\, x^* \otimes x$ satisfies $\|T\| = 1$; and T has the required properties. □

Our second lemma is essentially a rephrasing of [29, Theorem 2.2].

Lemma 6.5. — *Assume that X is smooth, and let $\mathbf{J} : X \to X^*$ be any duality mapping. If $T \in \mathcal{S}(X)$ and $x \in X\backslash\{0\}$, then*

$$x \in \mathcal{N}(T) \iff T^*\big(\mathbf{J}(Tx)\big) = \mathbf{J}(x).$$

Proof. — Assume that $x \in \mathcal{N}(T)$, *i.e.* $\|Tx\| = \|x\|$. Then

$$\langle T^*\big(\mathbf{J}(Tx)\big), x\rangle = \langle \mathbf{J}(Tx), Tx\rangle = \|\mathbf{J}(Tx)\|\|Tx\| = \|\mathbf{J}(Tx)\|\|x\|.$$

Since $\|T^*\big(\mathbf{J}(Tx)\big)\| \leqslant \|\mathbf{J}(Tx)\|$, it follows that in fact $\|T^*\big(\mathbf{J}(Tx)\big)\| = \|\mathbf{J}(Tx)\|$ and that $\langle T^*\big(\mathbf{J}(Tx)\big), x\rangle = \|T^*\big(\mathbf{J}(Tx)\big)\|\|x\|$. So $T^*\big(\mathbf{J}(Tx)\big)$ is a non-negative multiple of $\mathbf{J}_0(x)$, where $\mathbf{J}_0$ is the canonical duality mapping of X, and hence $T^*\big(\mathbf{J}(Tx)\big) = c\mathbf{J}(x)$ for some $c \geqslant 0$. Finally, we have $c = \|T^*\big(\mathbf{J}(Tx)\big)\|/\|\mathbf{J}(x)\| = \|\mathbf{J}(Tx)\|/\|\mathbf{J}(x)\|$, hence $c = 1$ since $\|\mathbf{J}(u)\|$ depends only on $\|u\|$ and $\|Tx\| = \|x\|$.

Conversely, assume that $T^*\big(\mathbf{J}(Tx)\big) = \mathbf{J}(x)$. Then

$$\|\mathbf{J}(x)\|\|x\| = \langle \mathbf{J}(x), x\rangle = \langle \mathbf{J}(Tx), Tx\rangle = \|\mathbf{J}(Tx)\|\|Tx\|;$$

and since $\|Tx\| \leqslant \|x\|$ and $\|\mathbf{J}(u)\|$ is a non-decreasing function of $\|u\|$, it follows that we must have $\|Tx\| = \|x\|$. □

Proof of Theorem 6.1. — Let us denote by $\mathrm{NA}(X)$ the set of all $T \in \mathcal{B}(X)$ which attain their norm. By Lemma 6.4, it is enough to show that $\mathrm{NA}(X) \cap \mathcal{S}(X)$ is meager in $(\mathcal{S}(X), \mathtt{SOT}^*)$. We are going to show that $\mathrm{NA}(X) \cap \mathcal{S}(X)$ is $\mathtt{SOT}^*$-F_σ in $\mathcal{S}(X)$, and that $\mathcal{S}(X)\backslash\mathrm{NA}(X)$ is $\mathtt{SOT}^*$-dense in $\mathcal{S}(X)$.

To show that $\mathrm{NA}(X)\cap\mathcal{S}(X)$ is $\mathtt{SOT}^*$-F_σ in $\mathcal{S}(X)$, we start with the following observation.

Claim 6.6. — For any bounded set $B \subseteq X$, the map $(T, x) \mapsto T^*\big(\mathbf{J}(Tx)\big)$ is continuous from $(\mathcal{B}_1(X), \mathtt{SOT}^*) \times (B, w)$ into (X^*, w^*).

Proof of Claim 6.6. — Let (T_n, x_n) be a sequence of elements of $\mathcal{B}_1(X) \times B$ such that $T_n \xrightarrow{\mathtt{SOT}^*} T \in \mathcal{B}_1(X)$ and $x_n \xrightarrow{w} x \in B$. Then $T_n x_n \xrightarrow{w} Tx$: indeed, if $x^* \in X^*$ then $\langle x^*, T_n x_n\rangle = \langle T_n^* x^*, x_n\rangle \to \langle T^* x^*, x\rangle = \langle x^*, Tx\rangle$ because $x_n \xrightarrow{w} x$ and $T_n^* x^* \xrightarrow{\|\cdot\|} T^* x^*$. Since $\mathbf{J}$ is (w, w^*)-continuous on bounded sets, it follows that $\mathbf{J}(T_n x_n) \xrightarrow{w^*} \mathbf{J}(Tx)$; and since $T_n z \xrightarrow{\|\cdot\|} Tz$ for all $z \in X$, this implies (as above) that $T_n^*\big(\mathbf{J}(T_n x_n)\big) \xrightarrow{w^*} T^*\big(\mathbf{J}(Tx)\big)$. □

Let $(B_q)_{q\in\mathbb{N}}$ be a family of closed balls in X such that $\bigcup_{q\in\mathbb{N}} B_q = X\backslash\{0\}$. By Lemma 6.5, if $T \in \mathcal{S}(X)$ then

$$T \in \mathrm{NA}(X) \iff \exists q \in \mathbb{N}\ \exists x \in B_q \ :\ T^*\big(\mathbf{J}(Tx)\big) = \mathbf{J}(x).$$

By Claim 6.6 and since $\mathbf{J}$ is (w, w^*)-continuous on bounded sets, we see that for each $q \in \mathbb{N}$, the set $\{(T, x) \in \mathcal{S}(X);\ T^*\big(\mathbf{J}(Tx)\big) = \mathbf{J}(x)\}$ is closed in $(\mathcal{S}(X), \mathtt{SOT}^*) \times (B_q, w)$; and since B_q is weakly compact, it follows that the set $\{T \in \mathcal{S}(X);\ \exists x \in B_q \ :\ T^*\big(\mathbf{J}(Tx)\big) = \mathbf{J}(x)\}$ is closed in $(\mathcal{S}(X), \mathtt{SOT}^*)$. Hence, $\mathrm{NA}(X) \cap \mathcal{S}(X)$ is F_σ in $(\mathcal{S}(X), \mathtt{SOT}^*)$.

Let us now show that $\mathcal{S}(X)\backslash\mathrm{NA}(X)$ is $\mathtt{SOT}^*$-dense in $\mathcal{S}(X)$.

First, note that for any $N \geqslant 0$, the space $F_N := \mathrm{ran}(I - P_N)$ is reflexive, infinite-dimensional, and has the Approximation Property (even the Metric Approximation Property) since the sequence of finite rank operators $\big((I - P_N)(P_k)_{|F_N}\big)_{k\geqslant 0}$ converges $\mathtt{SOT}$ to I_{F_N}. By a classical result of Holub [16], it follows that there is an operator $B_N \in \mathcal{B}(F_N)$ which does not attain its norm; and of course we may assume that $\|B_N\| = 1$.

Now, let $A \in \mathcal{S}(X)$ be arbitrary. Let $(\varepsilon_N)_{N\geqslant 0}$ be any sequence of positive numbers tending to 0. For each $N \geqslant 0$, considering B_N as an operator from F_N to X, define

$$T_N := (1 - \varepsilon_N)\, P_N A P_N + (I - P_N) B_N (I - P_N) \in \mathcal{B}(X).$$

Since P_N is M-like, we have $\|T_N\| = 1$, *i.e.* $T_N \in \mathcal{S}(X)$, and $T_N \xrightarrow{\mathtt{SOT}^*} A$ because $P_N \xrightarrow{\mathtt{SOT}^*} I$ and the sequence (B_N) is bounded. So it is enough to show that $T_N \notin \mathrm{NA}(X)$. For this, we will use the following fact.

Fact 6.7. — Let $P \in \mathcal{B}(X)$ be a M-like projection. If $u, v \in X$ are such that $\|Pu\| < \|Pv\|$ and $\|(I - P)u\| < \|(I - P)v\|$, then $\|u\| < \|v\|$.

Proof of Fact 6.7. — We may assume that $Pu \neq 0$ and $(I-P)u \neq 0$. Since $\|Pu\| < \|Pv\|$, one can find $x^* \in X^*$ such that $\|x^*\| < 1$ and $\langle x^*, Pv\rangle = \|Pu\|$. Then the operator $A := x^* \otimes \frac{Pu}{\|Pu\|}$ satisfies $\|A\| < 1$ and $APv = Pu$. Similarly, one can find $B \in \mathcal{B}(X)$ such that $\|B\| < 1$ and $B(I-P)v = (I-P)u$. Then $T := PAP + (I-P)B(I-P)$ satisfies $\|T\| < 1$ because P is M-like, and $Tv = u$. Hence $\|u\| < \|v\|$. □

Going back to T_N, let us show that $T_N \notin \mathrm{NA}(X)$. Let $x \in X\backslash\{0\}$: we have to show that $\|Tx\| < \|x\|$. If $P_N x = 0$ then $x \in F_N$ and $T_N x = B_N x$, so $\|T_N x\| < \|x\|$ because B_N does not attain its norm. If $(I-P_N)x = 0$, then $T_N x = (1-\varepsilon_N)\,Ax$, so we have $\|T_N x\| \leqslant (1-\varepsilon_N)\,\|x\| < \|x\|$. Finally, assume that $P_N x \neq 0$ and $(I-P_N)x \neq 0$. Then $\|P_N T_N x\| = \|\,(1-\varepsilon_N)\,P_N A P_N x\| < \|P_N x\|$, and $\|(I-P_N)T_N x\| = \|(I-P_N)B_N(I-P_N)x\| < \|(I-P_N)x\|$ because $B_N \notin \mathrm{NA}(F_N)$. Hence $\|T_N x\| < \|x\|$ by Fact 6.7. □

Remark 6.8. — Let $X = \ell_p$, $1 < p < \infty$. The proof of Theorem 6.1 has shown that $\mathrm{NA}(X) \cap \mathcal{S}(X)$ is F_σ in $(\mathcal{S}(X), \mathtt{SOT}^*)$, hence in $(\mathcal{S}(X), \|\cdot\|)$. On the other hand, by Bourgain's result [5] mentioned above, $\mathrm{NA}(X) \cap \mathcal{S}(X)$ is comeager in $(\mathcal{S}(X), \|\cdot\|)$. So, we obtain that $\mathcal{S}(X)\backslash\mathrm{NA}(X)$ is in fact *nowhere dense* in $(\mathcal{S}(X), \|\cdot\|)$. In other words, $\mathrm{NA}(\ell_p) \cap \mathcal{S}(\ell_p)$ contains a norm-dense open subset of $\mathcal{S}(\ell_p)$; from which it follows that $\mathrm{NA}(\ell_p)$ contains a norm-dense open subset of $\mathcal{B}(\ell_p)$.

It follows from Theorems 6.1 and 3.4 that if $2 < p < \infty$, a typical $T \in \big(\mathcal{B}_1(\ell_p), \mathtt{SOT}\big)$ does not attain its norm. This leads to the following question:

Question 6.9. — Let $1 < p < 2$. Is it true that a typical $T \in \big(\mathcal{B}_1(\ell_p), \mathtt{SOT}\big)$ does not attain its norm?

7. Additional results

7.1. More on similarity of topologies. — Let $\mathbf{M}$ be an abstract set. We have observed that two topologies on $\mathbf{M}$ sharing the same comeager sets may not be similar. In the next proposition, we show that if some extra assumptions on the topologies are added, then the two properties become equivalent.

Proposition 7.1. — *Let τ and τ' be two topologies on $\mathbf{M}$. Assume that $\tau \subseteq \tau'$, that τ' is Baire and second-countable, and that the identity map $\mathbf{i}_{\tau,\tau'} : (\mathbf{M}, \tau) \to (\mathbf{M}, \tau')$ is Borel 1, i.e. any τ'-open subset of $\mathbf{M}$ is τ-F_σ. Then τ and τ' are similar if and only if they have the same comeager sets.*

Proof. — By Lemma 2.1, we only need to prove the "if" part. Assume that τ and τ' have the same comeager sets. Since $\mathbf{i}_{\tau,\tau'}$ is Borel 1 and τ' is second-countable, we know that $\mathcal{C}(\tau, \tau')$ is τ-comeager. So $\mathcal{C}(\tau, \tau')$ is τ'-comeager by our assumption, and hence τ'-dense in $\mathbf{M}$ since τ' is Baire. By Lemma 2.3, it follows that τ and τ' are similar. (Note that it is not really necessary to assume that $\tau \subseteq \tau'$: it is enough to know that $\mathcal{C}(\tau', \tau)$ is τ-dense.) □

One gets the same conclusion with the following "symmetric" assumptions: τ and τ' are both Baire and second-countable, and the two identity maps are Borel 1.

We now prove a result that allows to give a few other characterisations of similarity.

Proposition 7.2. — *Let τ and τ' be similar topologies on $\mathbf{M}$, and assume that τ and τ' are second-countable. Then $\mathcal{C}(\tau, \tau') \cap \mathcal{C}(\tau', \tau)$ is comeager (for both τ and τ').*

Proof. — It is enough to show that $\mathcal{A} := \mathcal{C}(\tau,\tau') \cap \mathcal{C}(\tau',\tau)$ is τ-comeager; and to this end we use the *Banach-Mazur game* $\mathbf{G}(\mathcal{A})$ associated with $\mathcal{A}$ (see e.g. [20, p. 51]), in the topological space $(\mathbf{M},\tau)$. Recall that $\mathbf{G}(\mathcal{A})$ is an infinite game with two players, denoted by I and II. The two players play alternatively non-empty τ-open sets $U_0 \supseteq U_1 \supseteq U_2 \supseteq \cdots$, and player II wins the run if $\bigcap_{n\geqslant 0} U_n \subseteq \mathcal{A}$. To prove that $\mathcal{A}$ is τ-comeager, it is enough to show that player II has a winning strategy in $\mathbf{G}(\mathcal{A})$.

Let $(O_k)_{k\geqslant 0}$ and $(O'_k)_{k\geqslant 0}$ be countable bases for $(\mathbf{M},\tau)$ and $(\mathbf{M},\tau')$. Assume that player I has just played a τ-open set $U_{2k} \neq \varnothing$. Then II answers as follows. First, II chooses a τ'-open set $W'_k \neq \varnothing$ such that $W'_k \subseteq U_{2k}$ and, moreover, $W'_k \subseteq U_{2k} \cap O_k$ if $U_{2k} \cap O_k \neq \varnothing$. This is possible since τ and τ' are similar. Then II plays a τ-open set $U_{2k+1} \neq \varnothing$ such that $U_{2k+1} \subseteq W'_k$ and, moreover, $U_{2k+1} \subseteq W'_k \cap O'_k$ if $W'_k \cap O'_k \neq \varnothing$. Again, this is possible since τ and τ' are similar. Let us check that this strategy is winning for II.

Let $(U_n)_{n\geqslant 0}$ be a run of the game $\mathbf{G}(\mathcal{A})$ where II has followed the above strategy, and let $x \in \bigcap_{n\geqslant 0} U_n$. We have to show that $x \in \mathcal{A} = \mathcal{C}(\tau,\tau') \cap \mathcal{C}(\tau',\tau)$ *i.e.* that any τ-neighbourhood of x is a τ'-neighbourhood and vice-versa. Let V be a τ-neighbourhood of x. Choose $k \geqslant 0$ such that $x \in O_k \subseteq V$. Then $x \in O_k \cap U_{2k}$, so that in particular $O_k \cap U_{2k} \neq \varnothing$. With the above notation, it follows that $W'_k \subseteq O_k \cap U_{2k}$ and hence $W'_k \subseteq V$. Since $x \in U_{2k+1} \subseteq W'_k$, this shows that V is a τ'-neighbourhood of x. Now, let V' be a τ'-neighbourhood of x. Choose $k \geqslant 0$ such that $x \in O'_k \subseteq V'$. Then $x \in U_{2k+1} \cap O'_k \subseteq W'_k \cap O'_k$, so $W'_k \cap O'_k \neq \varnothing$. Hence $U_{2k+1} \subseteq W'_k \cap O'_k \subseteq V'$; and since $x \in U_{2k+1}$, it follows that V' is a τ-neighbourhood of x. □

Corollary 7.3. — *Assume that τ and τ' are Baire and second-countable. Then the following assertions are equivalent.*

(i) *τ and τ' are similar.*
(ii) *$\mathcal{C}(\tau,\tau') \cap \mathcal{C}(\tau,\tau')$ is comeager for both τ and τ'.*
(iii) *$\mathcal{C}(\tau,\tau')$ and $\mathcal{C}(\tau,\tau')$ are both τ-dense and τ'-dense.*
(iv) *$\mathcal{C}(\tau,\tau')$ is τ'-dense and $\mathcal{C}(\tau,\tau')$ is τ-dense.*

Proof. — This follows from Proposition 7.2 and Lemma 2.3. □

In order to state our second corollary, we need to recall a perhaps not so well known notion. A subset A of a topological space E is said to be *semi-open* if A is contained in the closure of its interior; equivalently, if there exists an open set O such that $O \subseteq A \subseteq \overline{O}$. This notion was introduced in [21], and independently in [25] (where semi-open sets are called *β-sets*). The somewhat dual property of being contained in the interior of its closure defines the class of *locally dense* sets introduced in [7]. There is actually a rather abundant literature concerning various classes of "generalised open sets"; see e.g. [8].

Corollary 7.4. — *Assume that $\tau \subseteq \tau'$, that τ is second-countable, and that τ' is Baire and second-countable. Then the following assertions are equivalent.*

(a) *τ and τ' are similar;*
(b) *$\mathcal{C}(\tau,\tau')$ is τ'-dense in $\mathbf{M}$;*
(c) *every τ'-semi-open set is τ-semi-open;*
(d) *every τ'-open set is τ-semi-open.*

Proof. — By Proposition 7.2 and since τ' is Baire, we know that (a) $\Longrightarrow$ (b).

(b) $\Longrightarrow$ (c) Assume that $\mathcal{C}(\tau,\tau')$ is τ'-dense in $\mathbf{M}$. Let $A \subseteq \mathbf{M}$ be τ'-semi-open, and choose a τ'-open set O' such that $O' \subseteq A \subseteq \overline{O'}^{\tau'}$. Denote by O the τ-interior of O', so that $O \subseteq O' \subseteq A$. Since $\mathcal{C}(\tau,\tau')$ is τ'-dense in $\mathbf{M}$ and O' is τ'-open, $\mathcal{C}(\tau,\tau') \cap O'$ is τ'-

dense in O'; and since $\tau \subseteq \tau'$, it follows that $\overline{O'}^{\tau'} \subseteq \overline{\mathcal{C}(\tau,\tau') \cap O'}^{\tau}$ Moreover, $\mathcal{C}(\tau,\tau') \cap O'$ is contained in O by the definition of $\mathcal{C}(\tau,\tau')$. So we see that $O \subseteq A \subseteq \overline{O}^{\tau}$, and hence A is τ - semi-open.

(c) $\Longrightarrow$ (d) is obvious.

(d) $\Longrightarrow$ (a) Assume (d). Since $\tau \subseteq \tau'$, it is enough to show that any set $A \subseteq \mathbf{M}$ with non-empty τ' - interior has non-empty τ - interior; which is obvious by (d). □

Remark 7.5. — The implications (b) $\Longrightarrow$ (c) $\Longrightarrow$ (d) $\Longrightarrow$ (a) do not require τ and τ' to be second-countable or Baire. Moreover, without assuming that $\tau \subseteq \tau'$, the proof of (b) $\Longrightarrow$ (c) shows that if (b) holds true then, for any τ' - semi-open set $A \subseteq \mathbf{M}$, one can find a τ - open set O such that $O \subseteq A \subseteq \overline{O}^{\tau'}$; and the proof of (d) $\Longrightarrow$ (a) shows that if (d) holds true then any τ - dense set is τ' - dense.

Remark 7.6. — One cannot replace condition (c) in Corollary 7.4 above by the condition "τ and τ' have the same semi-open sets". In fact, having the same semi-open sets should be considered as a much stronger property than similarity. For example, it follows from [25, Propositions 1 and 8] that if τ and τ' are two *regular* topologies with the same semi-open sets, then $\tau = \tau'$. (Recall that a topology is regular if every point of the space has a neighbourhood basis consisting of closed sets.)

7.2. More on points of `WOT-SOT` continuity. — In this sub-section, we add some examples of situations where one can conclude that an operator $T \in \mathcal{B}_1(X)$ belongs to $\mathcal{C}(\mathtt{WOT}, \mathtt{SOT})$.

Our first example concerns isometries. Let us say that a projection operator $P \in \mathcal{B}(X)$ is *uniformly strictly contractive* if $\|P\| = 1$ and, whenever (x_n) is a sequence in B_X such that $\|Px_n\| \to 1$, it follows that $\|(I-P)x_n\| \to 0$. For example, any L_p - projection, $1 \leqslant p < \infty$ is uniformly strictly contractive. (A projection $P \in \mathcal{B}(X)$ is an L_p - projection if $X = \operatorname{ran}(P) \oplus_{\ell_p} \ker(P)$; see [4].)

Example 7.7. — Let X be reflexive. If $T \in \mathcal{B}_1(X)$ admits a left inverse $L \in \mathcal{B}_1(X)$ such that the projection $P := TL$ is uniformly strictly contractive, then $T \in \mathcal{C}(\mathtt{WOT}, \mathtt{SOT})$. In particular:

(i) any surjective isometry of X belongs to $\mathcal{C}(\mathtt{WOT}, \mathtt{SOT}^*)$;
(ii) if E is any reflexive space and $1 < p < \infty$, the forward shift S on $X := \ell_p(\mathbb{Z}_+, E)$ belongs to $\mathcal{C}(\mathtt{WOT}, \mathtt{SOT})$.

Proof. — Note that the assumption made on T implies that T is an isometry since $\|T\| \leqslant 1$ and $\|L\| \leqslant 1$.

Since X is reflexive and separable, the set $\mathcal{C}(w, \|\cdot\|)$ is weakly dense in B_X; in particular, we have $\overline{\operatorname{span}}\ \mathcal{C}(w, \|\cdot\|) = X$. Hence, by Proposition 2.13, it is enough to show that $T\big(\mathcal{C}(w, \|\cdot\|)\big) \subseteq \mathcal{C}(w, \|\cdot\|)$.

Let $z \in \mathcal{C}(w, \|\cdot\|)$, and let (x_n) be a sequence in B_X such that $x_n \xrightarrow{w} Tz$. Since $LTz = z$ and $z \in \mathcal{C}(w, \|\cdot\|)$, we see that $Lx_n \xrightarrow{\|\cdot\|} z$. Hence $\|TLx_n\| \to \|Tz\| = 1$. Since $P = TL$ is uniformly strictly contractive, it follows that $\|(I - TL)x_n\| \to 0$; and since $TLx_n \xrightarrow{\|\cdot\|} Tz$, we conclude that $x_n \xrightarrow{\|\cdot\|} Tz$, which shows that $Tz \in \mathcal{C}(w, \|\cdot\|)$.

Since the identity operator is a uniformly strictly contractive projection, we see that any surjective isometry belongs to $\mathcal{C}(\mathtt{WOT}, \mathtt{SOT})$, and (i) follows by duality. As to (ii), take $L := B$, the backward shift on $X = \ell_p(\mathbb{Z}_+, E)$. □

Our second example follows from Proposition 2.14. Recall that a point $x \in S_X$ is said to be a *strongly exposed point of* B_X if there exists a linear functional $x^* \in S_{X^*}$ such that $\langle x^*, x\rangle = 1$ and $x^*_{|B_X}$ has a strong maximum at x, *i.e.* every maximising sequence $(x_n) \subseteq B_X$ for x^* converges in norm to x.

Example 7.8. — Let X be reflexive with the Kadec-Klee property. If $T \in \mathcal{B}_1(X)$ is such that $\|T\| = 1$ and the weak closure of $\mathcal{N}(T)$ contains the strongly exposed points of B_X, then $T \in \mathcal{C}(\mathtt{WOT}, \mathtt{SOT})$.

Proof. — Since X is reflexive, B_X is the closed convex hull of its strongly exposed points, by [22, Theorem 4]. So the result is clear by Proposition 2.14. □

Remark. — It is quite possible that $\mathcal{N}(T)$ contains all extreme points of B_X and yet T is not an isometry. Here is a simple example. Let $1 < p < \infty$ and let $X := X_1 \oplus_{\ell_p} X_2$, where $X_1 := (\mathbb{K}^2, \|\cdot\|_1)$ and X_2 is any reflexive Banach space with the Kadec-Klee property. Then X is reflexive with the Kadec-Klee property, and the extreme points of B_X are the points $x = x_1 \oplus x_2$, where $\|x_1\|^p + \|x_2\|^p = 1$ and $x_j \in \mathrm{Ext}\big(\|x_j\| B_{X_j}\big)$. Let $T := A \oplus I_{X_2} \in \mathcal{B}_1(X)$, where $A \in \mathcal{B}_1(X_1)$ is any non-isometric operator such that $\|Ae\| = \|Ae'\| = 1$, where (e, e') is the canonical basis of X_1. Then T is not an isometry but $\mathcal{N}(T) \supseteq \mathrm{Ext}(B_X)$.

We now prove a variant of Proposition 2.14 involving only "approximately norming" vectors. Recall that the Banach space X is said to have the *Uniform Kadec-Klee property* if the following holds true: for any $\varepsilon > 0$, there exists $\delta > 0$ such that, whenever (x_n) is an ε-separated sequence in B_X such that $x_n \xrightarrow{w} x \in X$, it follows that $\|x\| < 1 - \delta$. (This property was introduced by Huff [17].) For example, any uniformly convex space is UKK. Also, if $T \in \mathcal{B}(X)$, let us say that a vector $z \in X\backslash\{0\}$ is $(1-\delta)$-*norming* for T is $\|Tz\| > (1-\delta)\|T\|\|z\|$. We denote by $\mathcal{N}_\delta(T)$ the set of all $(1-\delta)$-norming vectors for T.

Proposition 7.9. — *Assume that X is reflexive with the Uniform Kadec-Klee property, and let $T \in \mathcal{B}_1(X)$. Assume that $\|T\| = 1$ and that there exists some constant $c > 0$ such that, for every $\delta > 0$, the set $\overline{\mathrm{conv}}\,\big(\mathcal{N}_\delta(T) \cap S_X\big)$ contains the ball cB_X. Then, $T \in \mathcal{C}(\mathtt{WOT}, \mathtt{SOT})$.*

Proof. — Let (T_n) be a sequence in $\mathcal{B}_1(X)$ such that $T_n \xrightarrow{\mathtt{WOT}} T$, and let $x \in X$. We have to show that $T_n x \xrightarrow{\|\cdot\|} Tx$. So we fix $\alpha > 0$, and we intend to show that $\|T_n x - Tx\| \leqslant \alpha$ for all large enough n. Moreover, we may assume that $\|x\| \leqslant 1$.

Claim 7.10. — Let $\varepsilon > 0$, and let $\delta = \delta(\varepsilon)$ be given by the UKK property. For any $z \in \mathcal{N}_\delta(T) \cap S_X$, we have $\overline{\lim}\, \|T_n z - Tz\| \leqslant \varepsilon$.

Proof of Claim 7.10. — Since $z \in \mathcal{N}_\delta(T) \cap S_X$, we have $\|Tz\| > 1-\delta$; and since $T_n z \xrightarrow{w} Tz$, it follows that the sequence $(T_n z)$ has no ε-separated subsequence. From that, it is not hard to deduce that one can find an increasing sequence of integers $(n_k)_{k\geqslant 0}$ such that $\|T_{n_k} z - T_{n_{k'}} z\| < \varepsilon$ for every $k, k' \geqslant 0$. Fixing k and letting $k' \to \infty$, we get $\|T_{n_k} z - Tz\| \leqslant \varepsilon$ for all $k \geqslant 0$. So we have $\underline{\lim}\, \|T_n - Tz\| \leqslant \varepsilon$; and since this can be applied to any subsequence of (T_n), the result follows. □

Now, let us choose $\varepsilon := \alpha c/3$, and let δ be given by the UKK property. By assumption on T, one can find $u \in X$ which is a convex combination of vectors $z_1, \ldots, z_N \in \mathcal{N}_\delta(T) \cap S_X$ such that $\|u - cx\| < \alpha c/3$. By Claim 7.10, we have $\|T_n z_i - Tz_i\| \leqslant \varepsilon = \alpha c/3$ for $i = 1, \ldots, N$ if n is large enough; and then $\|T_n u - Tu\| \leqslant \alpha c/3$ by convexity. By the triangle inequality

and since $\|T\|, \|T_n\| \leqslant 1$, it follows that $\|T_n(cx) - T(cx)\| \leqslant \alpha c$ for all large enough n; so $\|T_n x - Tx\| \leqslant \alpha$, as required. □

Remark. — By the Hahn-Banach theorem and since $\mathcal{N}_\delta(T)$ is balanced for any $\delta > 0$, the assumption in Proposition 7.9 can be written as follows:

$$\inf_{x^* \in S_{X^*}} \inf_{\delta > 0} \sup_{z \in \mathcal{N}_\delta(T) \cap S_X} |\langle x^*, z \rangle| > 0.$$

We conclude this sub-section with a rather natural question. As observed in [19, Proposition 2.6], if the Banach space X is strictly convex then any operator $T \in \mathcal{B}_1(X)$ such that $\|T\| = 1$ and $\overline{\operatorname{span}}\big(\mathcal{N}(T)\big) = X$ (in particular, any isometry) has to be an *extreme point* of $\mathcal{B}_1(X)$. In fact, without any assumption on X, any $T \in \mathcal{B}_1(X)$ such that $\|T\| = 1$, $\mathcal{N}(T) \cap S_X \subseteq \operatorname{Ext}(B_X)$ and $\overline{\operatorname{span}}\big(\mathcal{N}(T)\big) = X$ is an extreme point of $\mathcal{B}_1(X)$. By Proposition 2.11, it follows that for $X = \ell_2$, any $T \in \mathcal{C}(\mathtt{WOT}, \mathtt{SOT})$ is an extreme point of $\mathcal{B}_1(X)$. Also, it is easy to see that $\mathcal{C}(\mathtt{WOT}, \mathtt{SOT})$ is an extreme subset of $\mathcal{B}_1(X)$, for any X: if $A, B \in \mathcal{B}_1(X)$ and $\frac{A+B}{2} \in \mathcal{C}(\mathtt{WOT}, \mathtt{SOT})$, then $A \in \mathcal{C}(\mathtt{WOT}, \mathtt{SOT})$ and $B \in \mathcal{C}(\mathtt{WOT}, \mathtt{SOT})$. This motivates the following question.

Question 7.11. — For which reflexive spaces X is it true that every $T \in \mathcal{C}(\mathtt{WOT}, \mathtt{SOT})$ is an extreme point of $\mathcal{B}_1(X)$?

7.3. Typical properties related to Fredholm theory. — In this final sub-section, we gather a few remarks and some questions related to Lemma 5.4. Since we consider spectral properties, we assume that $\mathbb{K} = \mathbb{C}$. We begin by observing that the $\mathtt{SOT}^*$ version of Lemma 5.4 holds true if X is reflexive and has the MAP. In other words:

Fact 7.12. — If X is reflexive and has the MAP then a typical $T \in (\mathcal{B}_1(X), \mathtt{SOT}^*)$ is not upper semi-Fredholm.

Proof. — By the proof of Lemma 5.4, it is in fact enough to show that the finite rank contraction operators are $\mathtt{SOT}^*$- dense in $\mathcal{B}_1(X)$; and this will follow if we can show that there exists a sequence of finite rank operators $(R_n) \subseteq \mathcal{B}_1(X)$ such that $R_n \xrightarrow{\mathtt{SOT}^*} I_X$.

Since X has the MAP, there is a sequence of finite rank operators $(A_k) \subseteq \mathcal{B}_1(X)$ such that $A_k \xrightarrow{\mathtt{SOT}} I_X$. Then $A_k \xrightarrow{\mathtt{WOT}} I_X$, hence $A_k^* \xrightarrow{\mathtt{WOT}} I_{X^*}$ since X is reflexive. By Mazur's Theorem, it follows that I_{X^*} belongs to the $\mathtt{SOT}$- closure of the set $\big\{A_k^*;\ k \geqslant n\big\}$, for any $n \in \mathbb{N}$. So, by metrizability of $\big(\mathcal{B}_1(X^*), \mathtt{SOT}\big)$, one can find a sequence (S_n) of convex combinations of the A_k^* such that $S_n \xrightarrow{\mathtt{SOT}} I_{X^*}$ and $S_n \in \operatorname{conv}\big\{A_k^*;\ k \geqslant n\big\}$ for all $n \in \mathbb{N}$. The sequence of finite-rank operators (R_n) defined by $R_n := S_n^*$ is then such that $R_n \xrightarrow{\mathtt{SOT}} I_X$ (because $A_k \xrightarrow{\mathtt{SOT}} I_X$) and $R_n^* \xrightarrow{\mathtt{SOT}} I_{X^*}$. □

The next proposition shows in particular that Lemma 5.4 can be improved when X is an ℓ_p - direct sum or a c_0 - direct sum of finite-dimensional spaces..

Proposition 7.13. — *Assume that there exists a sequence $(P_N)_{N \geqslant 0}$ of finite rank M - like projections on X such that $P_N \xrightarrow{\mathtt{SOT}} I$. Then a typical $T \in (\mathcal{B}_1(X), \mathtt{SOT})$ is such that $T - \lambda I$ is not upper semi-Fredholm for any $\lambda \in \overline{\mathbb{D}}$. In particular, the essential spectrum of a typical $T \in \mathcal{B}_1(X)$ is equal to $\overline{\mathbb{D}}$.*

Proof. — We use an improved version of Fact 5.5.

Fact 7.14. — Let $\lambda \in \overline{\mathbb{D}}$. A typical $T \in \mathcal{B}_1(X)$ has the following property: for any $\varepsilon > 0$ and any $n \in \mathbb{N}$, there exists a subspace $E \subseteq X$ with $\dim(E) > n$ such that $\|(T-\lambda I)_{|E}\| < \varepsilon$.

Proof of Fact 7.14. — Let us denote by $\mathcal{G}$ the set of all $T \in \mathcal{B}_1(X)$ with the above property. The set $\mathcal{G}$ is $\mathtt{SOT}$-G_δ; so we just have to check that $\mathcal{G}$ is $\mathtt{SOT}$-dense in $\mathcal{B}_1(X)$. Let $A \in \mathcal{B}_1(X)$ be arbitrary, and for any $N \geqslant 0$, set $T_N := P_N A P_N + \lambda(I - P_N)$. Since the projection P_N is M-like, we see that $T_N \in \mathcal{B}_1(X)$; and $T_N \xrightarrow{\mathtt{SOT}} A$. Moreover, we have $T_N \equiv \lambda I$ on $E := \ker(P_N)$, so $T \in \mathcal{G}$. □

It follows from Fact 7.14 that for any fixed $\lambda \in \overline{\mathbb{D}}$, a typical $T \in \mathcal{B}_1(X)$ is such that $T-\lambda I$ is not upper semi-Fredholm. Now, let Λ be a countable dense subset of $\overline{\mathbb{D}}$. By the Baire Category Theorem, a typical $T \in \mathcal{B}_1(X)$ is such that $T - \lambda I$ is not upper semi-Fredholm for any $\lambda \in \Lambda$. Since the set of all $\lambda \in \mathbb{C}$ such that $T - \lambda I$ is not upper semi-Fredholm is closed in $\mathbb{C}$, the result follows. □

Remark 7.15. — The $\mathtt{SOT}^*$ version of Proposition 7.13 holds true (with exactly the same proof) if one assumes that $P_N \xrightarrow{\mathtt{SOT}^*} I$. This applies in particular to $X = c_0$ or ℓ_p, $1 < p < \infty$.

Remark 7.16. — Let $X = \ell_1$. It is proved in [14] that a typical $T \in (\mathcal{B}_1(X), \mathtt{SOT})$ has the following two properties: T^* is an isometry, and every $\lambda \in \mathbb{D}$ is an eigenvalue of T of infinite multiplicity. The proof of the first property is rather simple, but the proof of the second property given in [14] is a bit technical. By Proposition 7.13, we now see that the typicality of the second property actually *follows* from the typicality of the first one. Indeed, if T^* is an isometry, then $T - \lambda I$ is surjective for every $\lambda \in \mathbb{D}$ since $(T - \lambda I)^*$ is an embedding, and hence $\dim\ker(T - \lambda I) = \infty$ if $T - \lambda I$ is not Fredholm.

Assume that the Banach space X has the MAP. As already pointed out, it follows from Lemma 5.4 that if a typical $T \in \mathcal{B}_1(X)$ has closed range, then a typical $T \in \mathcal{B}_1(X)$ has an infinite-dimensional kernel. As the next remark shows, saying that a typical $T \in \mathcal{B}_1(X)$ has closed range is in fact the same as saying that a typical $T \in \mathcal{B}_1(X)$ is surjective.

Remark 7.17. — If X has the MAP, then a typical $T \in \big(\mathcal{B}_1(X), \mathtt{SOT}\big)$ has dense range. Consequently, a typical $T \in \mathcal{B}_1(X)$ has closed range if and only if a typical $T \in \mathcal{B}_1(X)$ is surjective.

Proof. — For any $z \in X$, set $\mathcal{G}_z := \big\{T \in \mathcal{B}_1(X);\ z \in \overline{\operatorname{ran}(T)}\big\}$. Let also D be a countable dense subset of X. Then an operator $T \in \mathcal{B}_1(X)$ has dense range if and only if $T \in \bigcap_{z\in D} \mathcal{G}_z$. Moreover, each set $\mathcal{G}_z$ is easily seen to be $\mathtt{SOT}$-G_δ in $\mathcal{B}_1(X)$. So it is enough to show that $\mathcal{G}_z$ is $\mathtt{SOT}$-dense in $\mathcal{B}_1(X)$, for any $z \in X$.

Let $\mathcal{U}$ be a non-empty open set in $(\mathcal{B}_1(X), \mathtt{SOT})$. Since X has the MAP, one can find a finite rank operator A such that $A \in \mathcal{U}$ and $\|A\| < 1$. Since $\dim(X) = \infty$, we can choose $x \in \ker(A)$ with $x \neq 0$, and then $x^* \in X^*$ such that $\langle x^*, x\rangle = 1$. Let $\varepsilon > 0$, and set $T := A + \varepsilon\, x^* \otimes z$. If ε is small enough, then $\|T\| \leqslant 1$ and $T \in \mathcal{U}$. Moreover, since $Tx = \varepsilon z$, we have $z \in \operatorname{ran}(T)$ and hence $T_N \in \mathcal{G}_z$. □

We finish the paper with the following two questions:

Question 7.18. — Is a typical $T \in (\mathcal{B}_1(X), \mathtt{SOT})$ surjective when $X = \ell_p$, $1 < p < 2$?

We have stated Question 7.18 for $1 < p < 2$ only because the answer is already known for all other values of p. Indeed, the answer is "Yes" for $X = \ell_2$ by [10], and also "Yes" for $X = \ell_1$ by [14, Theorem 4.1]. On the other hand, since a typical $T \in (\mathcal{B}_1(X), \mathtt{SOT})$ is not invertible (by [14, Proposition 2.1], see also Proposition 7.13), the answer is "No" if $X = \ell_p$, $p > 2$ or c_0, because a typical $T \in (\mathcal{B}_1(X), \mathtt{SOT})$ is one-to-one by [14, Theorem 2.3]. The answer is also "No" for the topology $\mathtt{SOT}^*$ and any $1 < p < \infty$, for the same reason. It follows from Theorem 4.1 that the answer is "No" as well for the topology $\mathtt{SOT}$ if $\mathbb{K} = \mathbb{R}$ and $p = 3/2$.

Question 7.19. — Assume that X has the MAP. Is it true that a typical $T \in \big(\mathcal{B}_1(X), \mathtt{SOT}\big)$ is such that $T - \lambda I$ has dense range for every $\lambda \in \mathbb{C}$?

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