

## Small sets and hypercyclic vectors

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*Abstract.* We study the “smallness” of the set of non-hypercyclic vectors for some classical hypercyclic operators.

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### 1. Introduction

Let  $X$  be a separable Fréchet space. A continuous linear operator  $T$  on  $X$  is said to be *hypercyclic* if there exists some vector  $x \in X$  whose  $T$ -orbit  $O_T(x) := \{T^n(x); n \in \mathbb{N}\}$  is dense in  $X$ . Such a vector is said to be hypercyclic for  $T$ , and the set of hypercyclic vectors is denoted by  $HC(T)$ . We refer to [6] for background on hypercyclicity.

It is easy to see that if  $T \in \mathcal{L}(X)$  is hypercyclic, then  $HC(T)$  is a dense  $G_\delta$  subset of  $X$ . Thus,  $X \setminus HC(T)$  is always “small” in the sense of Baire category, provided  $HC(T) \neq \emptyset$ . However, there exist many other natural notions of smallness in infinite-dimensional analysis. In this note, we consider two of them:  $\sigma$ -porosity, and Haar-negligibility.

In a metric space  $(E, d)$ , a set  $A$  is said to be *porous* if the following property holds true: for each point  $a \in A$ , there exists some positive constant  $\lambda$  and a sequence of positive numbers  $(r_n)$  tending to 0 such that, for each  $n \in \mathbb{N}$ , one can find  $x \in B(a, r_n)$  with  $B(x, \lambda r_n) \cap A = \emptyset$ . The set  $A$  is said to be  *$\sigma$ -porous* if it can be covered by countably many porous sets. Porosity was introduced by E.P. Dolženko in 1967 ([4]), and extensively studied since then; see [9] and [10] for more details.

In a Polish abelian group  $G$ , a universally measurable set  $A$  is said to be *Haar-null* if there exists some Borel probability measure  $\mu$  on  $X$  such that  $\mu(A+x) = 0$  for all  $x \in G$ . This notion was discovered by J.P.R. Christensen in 1972 ([3]), and it has received much attention in the last few years; see e.g. [7].

In this note, we study the smallness of the set of non-hypercyclic vectors for weighted backward shifts on  $c_0(\mathbb{N})$  or  $\ell^p(\mathbb{N})$  ( $1 \leq p < \infty$ ) and operators commuting with translations on the space of entire functions  $\mathcal{H}(\mathbb{C})$ . We first recall the definitions.

Let  $X = c_0(\mathbb{N})$  or  $\ell^p(\mathbb{N})$  ( $1 \leq p < \infty$ ), and let us denote by  $(e_i)_{i \in \mathbb{N}}$  the canonical basis of  $X$ . If  $\mathbf{w} = (w_i)_{i \geq 1}$  is any bounded sequence of positive numbers, then the weighted backward shift on  $X$  associated to  $\mathbf{w}$  is the operator  $B_{\mathbf{w}} : X \rightarrow X$  defined by  $B_{\mathbf{w}}(e_0) = 0$  and  $B_{\mathbf{w}}(e_i) = w_i e_{i-1}$ ,  $i \geq 1$ . By a result of H. Salas ([8]),  $B_{\mathbf{w}}$  is hypercyclic if and only if

$$\limsup_{n \rightarrow \infty} \prod_{i=1}^n w_i = \infty.$$

For example, the operator  $T = 2B$  is hypercyclic, where  $B$  is the usual, unweighted backward shift on  $X$ . This is a classical result of S. Rolewicz.

If  $\lambda \in \mathbb{C}$ , then the translation-by- $\lambda$  operator on  $\mathcal{H}(\mathbb{C})$  is the operator  $\tau_\lambda$  defined by  $\tau_\lambda f(z) = f(z + \lambda)$ . By a classical result of G.D. Birkhoff,  $\tau_\lambda$  is hypercyclic whenever  $\lambda \neq 0$ . More generally, it was proved by G. Godefroy and J.H. Shapiro ([5]) that if  $T \in \mathcal{L}(H(\mathbb{C}))$  is not a scalar multiple of the identity and commutes with all translation operators, then  $T$  is hypercyclic. A typical example is the derivation operator  $D$ : this is another classical result, due to S. McLane.

The first two sections of the paper concern  $\sigma$ -porosity, which had already been considered in [1]. We show that if a weighted backward shift  $T$  has at least one orbit staying away from 0, then the set of non-hypercyclic vectors for  $T$  is not  $\sigma$ -porous; this improves the first main result in [1]. On the other hand, we give a criterion for an operator to be “ $\sigma$ -porous hypercyclic”, which can be applied to an interesting shift constructed by D. Preiss (unpublished), and to translation operators on  $\mathcal{H}(\mathbb{C})$  for a certain class of metrics; here, of course, an operator  $T$  is said to be  $\sigma$ -porous hypercyclic if the set of non-hypercyclic vectors for  $T$  is  $\sigma$ -porous. The final section concerns Haar-negligibility. We show that weighted backward shifts with large weights are not “Haar-null hypercyclic”, and we get the same conclusion for a class of operators on  $\mathcal{H}(\mathbb{C})$  commuting with translations.

### 2. Weighted shifts which are not $\sigma$ -porous hypercyclic

In this section, we exhibit a class of non  $\sigma$ -porous subsets in Banach spaces, and we apply the result to show that quite a lot of weighted backward shifts on  $c_0(\mathbb{N})$  or  $\ell^p(\mathbb{N})$  are not  $\sigma$ -porous hypercyclic.

In what follows,  $X$  is a Banach space over  $\mathbb{K} = \mathbb{R}$  or  $\mathbb{C}$ , and we assume that  $X$  has an unconditional basis  $(e_i)_{i \in \mathbb{N}}$ . We denote by  $(e_i^*)$  the associated sequence of coordinate functionals. We will consider subsets of  $X$  of the form

$$F_{[L]}^{\mathcal{A}} = \{x \in X; \forall n \in \mathbb{N} L_n(x) \in \mathcal{A}\},$$

where  $\mathcal{A}$  is a subset of  $\mathbb{K}^{\mathbb{N}}$  and  $[L] = (L_n)$  is a sequence of continuous linear maps from  $X$  into  $\mathbb{K}^{\mathbb{N}}$ . We view a sequence  $[L] = (L_n) \subset \mathcal{L}(X, \mathbb{K}^{\mathbb{N}})$  as an infinite matrix  $(L_{nj})$  with entries in  $X^*$ , and we denote by  $\mathcal{L}$  the family of all such matrices.

We will say that a matrix  $[L]' = (L'_n) \in \mathfrak{L}$  is an *admissible modification* of a matrix  $[L]$  if  $L'_{nj} = \alpha_{nj} L_{nj}$  for all  $(n, j) \in \mathbb{N}^2$ , where  $(\alpha_{nj})$  is a bounded sequence of scalars.

Finally, we say that a set  $\mathcal{A} \subset \mathbb{K}^{\mathbb{N}}$  is *monotone* if, whenever  $(u_j) \in \mathcal{A}$  and  $|v_j| \geq |u_j|$  for all  $j \in \mathbb{N}$ , it follows that  $(v_j) \in \mathcal{A}$ .

**Theorem 2.1.** *Let  $\mathcal{A}$  be a monotone subset of  $\mathbb{K}^{\mathbb{N}}$ , and let  $[L] \in \mathfrak{L}$ . Assume that  $F_{[L]}^{\mathcal{A}} \neq \emptyset$ , and that the following properties hold:*

- $L_{nj} \in \bigcup_i \mathbb{K}e_i^*$  for all pairs  $(n, j) \in \mathbb{N}^2$ ;
- $F_{[L]}'^{\mathcal{A}}$  is closed in  $X$  for each admissible modification  $[L]'$  of  $[L]$ .

Then  $F_{[L]}^{\mathcal{A}}$  is not  $\sigma$ -porous.

From this, we get the following improvement of the first main result of [1].

**Corollary 2.2.** *Let  $T$  be a weighted backward shift on  $X = c_0(\mathbb{N})$  or  $\ell^p(\mathbb{N})$ ,  $1 \leq p < \infty$ . If there exists some point  $x \in X$  such that  $\inf_n \|T^n(x)\| > 0$ , then  $T$  is not  $\sigma$ -porous hypercyclic.*

PROOF: It is enough to show that the set

$$F = \{x \in X; \forall n \in \mathbb{N} \|T^n(x)\| \geq 1\}$$

is not  $\sigma$ -porous. Now, the set  $F$  is nonempty and has the form  $F_{[L]}^{\mathcal{A}}$ , with  $L_{nj}(x) = \langle e_j^*, T^n(x) \rangle$  and  $\mathcal{A} = \{(u_j) \in \mathbb{K}^{\mathbb{N}}; \|\sum_j u_j e_j\| \geq 1\}$ , where we have put  $\|\sum_j u_j e_j\| = \infty$  if  $(u_j)$  does not define an element of  $X$ .

Since  $L_{nj} = T^{*n}(e_j^*)$  and  $T$  is a shift, we have  $L_{nj} \in \mathbb{K}e_{n+j}^*$  for each  $(n, j) \in \mathbb{N}^2$ . Moreover, if  $[L]' = (L'_n)$  is an admissible modification of  $[L]$ , so that  $L'_{nj} = \alpha_{nj} L_{nj}$  for some bounded sequence of scalars  $(\alpha_{nj})$ , then each  $L'_n : X \rightarrow \mathbb{K}^{\mathbb{N}}$  defines a bounded operator  $T'_n : X \rightarrow X$ , namely  $T'_n(\sum_j x_j e_j) = T^n(\sum_j \alpha_{nj} x_j e_j)$ . Therefore,  $F_{[L]}'^{\mathcal{A}} = \{x \in X; \forall n \in \mathbb{N} \|T'_n(x)\| \geq 1\}$  is closed in  $X$ . Thus, we may apply 2.1.  $\square$

To prove 2.1, we will use a version of *Foran's Lemma* (see [9, Lemma 4.3]), which is essentially the only known tool to check that some sets are not  $\sigma$ -porous.

A subset  $A$  of a metric space  $(E, d)$  is said to be  $\lambda$ -porous ( $\lambda > 0$ ) at some point  $a \in E$  if there exists a sequence of positive numbers  $(r_n)$  tending to 0 such that, for each  $n \in \mathbb{N}$ , one can find  $x \in B(a, r_n)$  with  $B(x, \lambda r_n) \cap A = \emptyset$ . The set  $A$  is said to be  $\lambda$ -porous if it is  $\lambda$ -porous at each point  $a \in A$ . It is proved in [10] that given  $\lambda \in (0, \frac{1}{2})$ , any  $\sigma$ -porous set  $A \subset E$  can in fact be covered by countably many  $\lambda$ -porous sets. From this and Lemma 4.3 in [9] applied to the porosity relation  $V$  defined by  $V(x, A) \Leftrightarrow A$  is  $\lambda$ -porous at  $x$ , one gets the following result.

**Lemma 2.3.** *Let  $(E, d)$  be a complete metric space, and let  $\lambda \in (0, \frac{1}{2})$ . Let also  $\mathcal{F}$  be a family of nonempty closed subsets of  $E$ . Assume  $\mathcal{F}$  has the following property: for each set  $F \in \mathcal{F}$  and each open set  $V$  such that  $V \cap F \neq \emptyset$ , one can find  $F' \in \mathcal{F}$  such that  $F' \subset F$ ,  $F' \cap V \neq \emptyset$  and  $F$  is  $\lambda$ -porous at no point of  $F'$ . Then no set  $F \in \mathcal{F}$  is  $\sigma$ -porous.*

PROOF OF THEOREM 2.1: Let us denote by  $[L]_0$  the matrix given in the hypotheses of Theorem 2.1, and by  $\mathfrak{L}_0$  the family of all matrices  $[L] \in \mathfrak{L}$  which are admissible modifications of  $[L]_0$  and satisfy  $F_{[L]}^{\mathcal{A}} \neq \emptyset$ . For notational simplicity, we will drop the superscript  $\mathcal{A}$  in  $F_{[L]}^{\mathcal{A}}$  and simply write  $F_{[L]}$ .

If  $L \in \mathfrak{L}_0$ , we can choose a map  $(n, j) \mapsto \langle n, j \rangle$  from  $\mathbb{N}^2$  into  $\mathbb{N}$  such that  $L_{nj} \in \mathbb{K}e_{\langle n, j \rangle}^*$ . We do not indicate explicitly that the map  $\langle \cdot, \cdot \rangle$  depends on the matrix  $[L]$ , but this will cause no confusion.

Finally, for each set  $J \subset \mathbb{N}$ , we denote by  $\pi_J$  the canonical projection from  $X$  onto  $\overline{\text{span}}\{e_i; i \in J\}$ . This projection is well-defined by unconditionality of the basis  $(e_i)$ .

Let  $L \in \mathfrak{L}_0$ . For each triple  $\mathbf{p} = (\varepsilon, K, I)$ , where  $\varepsilon > 0$ ,  $K > 1$  and  $I$  is a finite subset of  $\mathbb{N}$ , we define a new matrix  $[L^{\mathbf{p}}]$  in the following way:

$$L_{nj}^{\mathbf{p}} = \begin{cases} (1 + \varepsilon)^{-1} L_{nj} & \text{if } \langle n, j \rangle \in I; \\ K^{-1} L_{nj} & \text{if } \langle n, j \rangle \notin I. \end{cases}$$

Then  $[L^{\mathbf{p}}]$  is an admissible modification of  $[L]$ , and hence an admissible modification of the matrix  $[L]_0$  we started with. Moreover, if  $x \in F_{[L]}$ , then  $y := (1 + \varepsilon)\pi_I(x) + K\pi_{\mathbb{N} \setminus I}(x)$  satisfies  $L_{nj}^{\mathbf{p}}(y) = L_{nj}(x)$  for all  $(n, j) \in \mathbb{N}^2$ , whence  $y \in F_{[L^{\mathbf{p}}]}$ . Thus  $[L^{\mathbf{p}}] \neq \emptyset$ , so that  $[L^{\mathbf{p}}] \in \mathfrak{L}_0$ . Notice also that  $F_{[L^{\mathbf{p}}]} \subset F_{[L]}$  by the monotonicity property of  $\mathcal{A}$ .

**Claim 1.** *Let  $[L] \in \mathfrak{L}_0$ , and let  $K > 1$  be given. If  $V \subset X$  is an open set such that  $F_{[L]} \cap V \neq \emptyset$ , then one can find  $\varepsilon, I$  such that  $F_{[L^{\mathbf{p}}]} \cap V \neq \emptyset$ , where  $\mathbf{p} = (\varepsilon, K, I)$ .*

PROOF: Here, the basis  $(e_i)$  needs not be unconditional because we consider projections on finite or co-finite sets only. Choose a point  $x \in V \cap F_{[L]}$  and  $r > 0$  such that  $B(x, r) \subset V$ . Since  $(e_i)$  is a basis for  $X$ , one can choose a finite set  $I \subset \mathbb{N}$  such that  $\|\pi_{\mathbb{N} \setminus I}(x)\|$  is very small, and then  $\varepsilon > 0$  such that  $y := (1 + \varepsilon)\pi_I(x) + K\pi_{\mathbb{N} \setminus I}(x)$  satisfies  $\|y - x\| < r$ . This point  $y$  shows that  $F_{[L^{\mathbf{p}}]} \cap V \neq \emptyset$ .  $\square$

**Claim 2.** *Let  $[L] \in \mathfrak{L}_0$  and  $\lambda \in (0, 1)$ . If  $K > 0$  is large enough, then, for each  $\mathbf{p} = (\varepsilon, K, I)$ , the set  $F_{[L]}$  is  $\lambda$ -porous at no point of  $F_{[L^{\mathbf{p}}]}$ .*

PROOF: Let  $\alpha \in (0, 1)$  to be chosen later. If  $x \in X$ ,  $\varepsilon > 0$  and a finite set  $I \subset \mathbb{N}$  are given, one can find  $\delta = \delta(x, \varepsilon, I) > 0$  such that if  $y \in X$  satisfies  $\|y - x\| < \delta$ , then

$$|\langle e_i^*, y \rangle| \geq (1 + \varepsilon)^{-1} |\langle e_i^*, x \rangle| \quad \text{for all } i \in I.$$

Since  $(e_i)$  is unconditional, one may associate to each such point  $y$  another point  $\tilde{y} \in X$  such that

$$\langle e_i^*, \tilde{y} \rangle = \begin{cases} \langle e_i^*, y \rangle & \text{if } i \in I \text{ or } |\langle e_i^*, y \rangle| > \frac{\alpha}{2} |\langle e_i^*, x \rangle|; \\ \alpha \langle e_i^*, x \rangle + (1 - \alpha) \langle e_i^*, y \rangle & \text{otherwise.} \end{cases}$$

Then  $|\langle e_i^*, \tilde{y} \rangle| \geq (1 + \varepsilon)^{-1} |\langle e_i^*, x \rangle|$  if  $i \in I$ , and  $|\langle e_i^*, \tilde{y} \rangle| \geq \frac{\alpha}{2} |\langle e_i^*, x \rangle|$  if  $i \notin I$ , by the triangle inequality. Thus, if  $K \geq 2\alpha^{-1}$ , we get that for any  $\mathbf{p} = (\varepsilon, K, I)$  and each point  $y \in B(x, \delta)$ , the following implication holds:

$$x \in F_{[L^{\mathbf{p}}]} \Rightarrow \tilde{y} \in F_{[L]}.$$

Moreover, we also have  $|\langle e_i^*, \tilde{y} - y \rangle| \leq \alpha |\langle e_i^*, x - y \rangle|$  for all  $i \in I$ , so that

$$\|\tilde{y} - y\| \leq C\alpha \|x - y\|,$$

where  $C$  is the unconditionality constant of the basis  $(e_i)$ . If we now choose  $\alpha < C^{-1}\lambda$ , we conclude that if  $K \geq 2\alpha^{-1}$ , then  $F_{[L]}$  is  $\lambda$ -porous at no point  $x \in F_{[L^{\mathbf{p}}]}$  when  $\mathbf{p}$  has the form  $(\varepsilon, K, I)$ .  $\square$

It follows from the above claims that the family  $\mathcal{F} := (F_{[L]})_{L \in \mathfrak{L}_0}$  satisfies the hypotheses of Lemma 2.3. Thus, no set  $F_{[L]}$  is  $\sigma$ -porous ( $[L] \in \mathcal{L}_0$ ), and the proof of 2.1 is complete.  $\square$

### 3. A criterion for $\sigma$ -porosity

It is well-known that an operator  $T \in \mathcal{L}(X)$  is hypercyclic if and only if it is *topologically transitive*, which means that for each pair  $(U, V)$  of nonempty open subsets of  $X$ , one can find  $n \in \mathbb{N}$  such that  $T^n(U) \cap V \neq \emptyset$  (see [GE]). In this section, we show that if an operator is “topologically transitive with estimate”, then it is  $\sigma$ -porous hypercyclic. The following easy lemma will be needed.

**Lemma 3.1.** *Let  $(E, d)$  be a metric space, and let  $A \subset E$ . Assume there exist  $\delta_0 > 0$ , a dense set  $D \subset E$  and some constant  $c > 0$  such that: for all  $u \in D$  and every  $\delta \in (0, \delta_0)$ , one can find  $x \in E$  such that  $d(x, u) < \delta$  and  $B(x, c\delta) \cap A = \emptyset$ . Then  $A$  is porous.*

PROOF: Let  $a$  be any point of  $E$ . For any  $\delta \in (0, \delta_0)$ , one can find  $u \in D$  with  $d(u, a) < \frac{\delta}{2}$ , and then  $x \in E$  with  $d(x, u) < \frac{\delta}{2}$  and  $B(x, c\frac{\delta}{2}) \cap A = \emptyset$ . Then we

have  $x \in B(a, \delta)$  and  $B(x, \frac{c}{2}\delta) \cap A = \emptyset$ , which shows that  $A$  is  $\frac{c}{2}$ -porous at each point  $a \in E$  (actually *very porous* in the sense of [9]).  $\square$

Since this involves no additional complication, we formulate the announced criterion for an arbitrary sequence of continuous maps  $T_n : E \rightarrow E$  from a metric space  $(E, d)$  into itself. The sequence  $\mathbf{T} = (T_n)_{n \in \mathbb{N}}$  is said to be *universal* if there is some  $x \in E$  such that the set  $\{T_n(x); n \in \mathbb{N}\}$  is dense in  $E$ , and the set of universal points for  $\mathbf{T}$  is denoted by  $\text{Univ}(\mathbf{T})$ .

If  $T : (E, d) \rightarrow (E, d)$  is a continuous map, then, for each  $r > 0$ , we denote by  $\omega^{-1}(T, r)$  the largest number  $\delta \in [0, 1]$  such that  $d(x, y) < \delta \Rightarrow d(T(x), T(y)) < r$ .

**Theorem 3.2.** *Let  $(E, d)$  be a separable metric space, and let  $\mathbf{T} = (T_n)$  be a sequence of continuous maps,  $T_n : E \rightarrow E$ . Assume there exist a dense set  $\mathcal{D} \subset E$  and for each pair  $(v, r) \in \mathcal{D} \times (0, 1)$ , a dense set  $\mathcal{D}_{v,r} \subset E$  and positive real constants  $\delta_{v,r}, c_{v,r}$  such that the following holds true. For each  $u \in \mathcal{D}_{v,r}$  and every  $\delta \in (0, \delta_{v,r})$ , one can find  $x \in E$  and  $n \in \mathbb{N}$  such that*

- (a)  $d(x, u) < \delta$  and  $d(T_n(x), v) < r$ ;
- (b)  $\omega^{-1}(T_n, r) \geq c_{v,r}\delta$ .

Then  $X \setminus \text{Univ}(\mathbf{T})$  is  $\sigma$ -porous.

PROOF: It is enough to show that for each pair  $(v, r) \in \mathcal{D} \times (0, 1)$ , the set

$$A_{v,r} := \{x \in E; \forall n \in \mathbb{N} \ d(T^n(x), v) \geq 2r\}$$

is porous. Indeed,  $E \setminus \text{Univ}(\mathbf{T})$  is a countable union of such sets  $A_{v,r}$ , by separability of  $E$ . Now, it follows from the triangle inequality that for each  $u \in \mathcal{D}_{v,r}$  and every  $\delta \in (0, \delta_{v,r})$ , one can find  $x \in B(u, \delta)$  and  $n \in \mathbb{N}$  such that  $d(T_n(y), v) < 2r$  for all  $y \in B(x, c_{v,r}\delta)$ , hence  $B(x, c_{v,r}\delta) \cap A_{v,r} = \emptyset$ . Since  $\mathcal{D}_{v,r}$  is dense in  $X$ , this shows that  $A_{v,r}$  is porous by 3.1.  $\square$

In the linear setting, we get from 3.2 the following “universality criterion with estimate”, which is a natural variant of the well-known Hypercyclicity Criterion (see [6]).

**Corollary 3.3.** *Let  $X$  be a separable Banach space, and let  $\mathbf{T} = (T_n)$  be a sequence of continuous linear operators on  $X$ . Assume there exist a dense set  $\mathcal{D}^* \subset X$  and for each pair  $(v, r) \in \mathcal{D}^* \times (0, 1)$ , a dense set  $\mathcal{D}_{v,r}^* \subset X$  and positive real constants  $\delta_{v,r}^*, C_{v,r}$  such that the following holds true. For each  $u \in \mathcal{D}_{v,r}^*$  and every  $\delta \in (0, \delta_{v,r}^*)$ , one can find  $n \in \mathbb{N}$  and  $z \in X$  such that*

- (a1)  $\|T_n(u)\| < r$ ;
- (a2)  $\|z\| < \delta$  and  $\|T_n(z) - v\| < r$ ;
- (b)  $\|T_n\| \leq \frac{C_{v,r}}{\delta}$ .

Then  $X \setminus \text{Univ}(\mathbf{T})$  is  $\sigma$ -porous.

PROOF: One can apply 3.2 with  $\mathcal{D}_{v,r} := \mathcal{D}_{v,r/2}^*$ ,  $\delta_{v,r} := \delta_{v,r/2}^*$  and  $c_{v,r} := \frac{r}{C_{v,r/2}}$ . Given  $u \in \mathcal{D}_{v,r/2}^*$  and  $\delta \in (0, \delta_{v,r/2}^*)$ , choose  $n$  and  $z$  according to 3.3 and set  $x := u + z$ .  $\square$

*Remark.* It follows from the above proofs that, in 3.2 as well as in 3.3, the set  $X \setminus \text{Univ}(\mathbf{T})$  can in fact be covered by countably many closed sets which are  $\sigma$ -very porous in the sense of [9].

We now give two illustrations of 3.2. The first one is a recent unpublished result of D. Preiss, and the second one is a generalization of the second main result in [1]. We would like to thank D. Preiss for allowing us to include his example in this paper.

**Example 1** (Preiss). *There exist weighted backward shifts on  $X = c_0(\mathbb{N})$  or  $\ell^p(\mathbb{N})$  ( $1 \leq p < \infty$ ) which are  $\sigma$ -porous hypercyclic.*

PROOF: We apply Corollary 3.3 with  $\mathcal{D}^* = c_{00}$ , the space of all finitely supported vectors  $x \in X$ , and  $\mathcal{D}_{v,r}^* = c_{00}$  for all  $(v,r) \in c_{00} \times (0,1)$ . Let  $T_{\mathbf{w}}$  be a backward shift on  $X$  associated to some bounded sequence of positive numbers  $\mathbf{w} = (w_k)_{k \geq 1}$ , and let us see what properties of  $\mathbf{w}$  are needed. For  $p \leq q \in \mathbb{N}$ , we set  $w_{p,q} = \prod_{p \leq k \leq q} w_k$  (where we have put  $w_0 = 0$ ). Finally, we denote by  $(e_i)_{i \in \mathbb{N}}$  the canonical basis of  $X$ .

Let  $v = \sum_i v(i)e_i \in c_{00}$  be supported on some interval  $[0,p)$ . If  $n$  is any positive integer, then the vector

$$z_n := \sum_{i=0}^{p-1} \frac{v(i)}{w_{1+i,n+i}} e_{n+i}$$

satisfies  $T_{\mathbf{w}}^n(z_n) = v$ , and we have  $\|z_n\| \leq \|v\| \max\{(w_{1+i,n+i})^{-1}; 0 \leq i < p\}$ . Moreover, if  $u \in c_{00}$ , then  $T_{\mathbf{w}}^n(u) = 0$  if  $n$  is large enough. Finally, we have  $\|T_{\mathbf{w}}^n\| = \sup\{w_{1+i,n+i}; i \in \mathbb{N}\}$ . Thus, we see that  $T_{\mathbf{w}}$  will be  $\sigma$ -porous hypercyclic provided for each positive integer  $p$  the following holds for suitable constants  $M_p$  and  $C_p$ : for every  $M \geq M_p$ , one can find infinitely many integers  $n$  satisfying

- (a)  $w_{1+i,n+i} > M$  for all  $i \in \{0; \dots; p-1\}$ ;
- (b)  $w_{1+i,n+i} \leq C_p M$  for all  $i \in \mathbb{N}$ .

A weight sequence  $\mathbf{w}$  with that property can be obtained as follows. Let us denote by  $\mathbb{N}^*$  the set of all positive integers, and let  $(p_j, r_j)_{j \geq 1}$  be a sequence in  $\mathbb{N}^* \times (0, \infty)$  to be specified later. Then one can construct a sequence  $\mathbf{w} = (w_k)_{k \geq 1} \subset (0, 2)$  and an increasing sequence of positive integers  $(n_j)_{j \geq 1}$  such that the following properties hold for each  $j$ :

- (i)  $w_{1,k} = r_j$  for all  $k \in [n_j, n_j + p_j)$ ;
- (ii)  $w_{1+i,n_j+i} \leq 2$  for all  $i \geq p_j$ .

To do this, first choose  $n_1$  so that one can find  $w_1, \dots, w_{n_1} \in (0, 2)$  with  $w_{1,n_1} = r_1$ . Set  $w_k = 1$  for  $k \in (n_1, n_1 + p_1)$  in order to have (i) for  $j = 1$ . Then choose  $w_{n_1+p_1}, \dots, w_{2n_1-1+p_1} \in (0, 1)$  small enough to ensure  $w_{1+i,n_1+i} \leq 2$  for all  $i \in [p_1, n_1+p_1)$ . At this point, choose  $\varepsilon_1 > 0$  such that  $(1+\varepsilon_1)^{n_1} \leq 2$ , and find  $n_2$  large enough to ensure that one can construct  $w_{2n_1+p_1}, \dots, w_{n_2} \in (0, 1+\varepsilon_1)$  with  $w_{1,n_2} = r_2$ . Then (ii) is satisfied for  $j = 1$  and all  $i \in [p_1, n_2 - n_1]$ . Repeating the procedure, one gets the sequences  $(n_j)$  and  $(w_k)$ .

Now assume that the sequence  $(p_j, r_j)$  enumerates  $\mathbb{N}^* \times \mathbb{Q}^+$ , where  $\mathbb{Q}^+$  is the set of all positive rational numbers. Fixing  $p$  and setting  $\mathbf{N}_p := \{n_j; p_j = p\}$ , we show that (a) and (b) hold with suitable constants  $M_p$  and  $C_p$  and infinitely many  $n \in \mathbf{N}_p$ . If  $n = n_j \in \mathbf{N}_p$ , then, writing  $w_{1+i,n+i} = \frac{r_j}{w_{1,i}}$  if  $i < p$ , we see that property (i) and (ii) give

$$\begin{cases} a_p r_j \leq w_{1+i,n+i} \leq b_p r_j & \text{for all } i \in \{0; \dots; p-1\}, \\ w_{1+i,n+i} \leq 2 & \text{for all } i \geq p, \end{cases}$$

where  $a_p, b_p$  depend only on  $p$ . Thus, (a) and (b) are satisfied for a given  $M$  provided  $\frac{M}{a_p} < r_j \leq \frac{C_p M}{b_p}$  and  $C_p M \geq 2$ . Choosing  $C_p > \frac{b_p}{a_p}$ , this holds for infinitely many  $j$ 's whenever  $M \geq M_p := \frac{2}{C_p}$ .  $\square$

Our second illustration concerns translation operators on  $\mathcal{H}(\mathbb{C})$ . Since porosity makes sense only when a metric is given, we first have to choose some compatible metric on  $\mathcal{H}(\mathbb{C})$ . There are various reasonable ways of doing so. For example, to each sequence of positive numbers  $\bar{\varepsilon} = (\varepsilon_n)$  such that  $\sum_0^\infty \varepsilon_n < \infty$ , one may associate the metric  $d_{\bar{\varepsilon}}$  defined by

$$d_{\bar{\varepsilon}}(f, g) = \sum_0^\infty \varepsilon_n \min(1, \|f - g\|_{K_n}),$$

where  $K_n$  is the disk  $\bar{D}(0, n)$  and  $\|f\|_{K_n} = \sup\{|f(z)|; z \in K_n\}$ . With that kind of metrics, we have the following result.

**Example 2.** *Let  $\bar{\varepsilon} = (\varepsilon_n)$  be a summable sequence of positive numbers, and assume there exists some constant  $c > 0$  such that  $\sum_{k>n} \varepsilon_k \geq c \varepsilon_n$  for all  $n \in \mathbb{N}$ . If  $T \neq \text{id}$  is a translation operator on  $\mathcal{H}(\mathbb{C})$ , then  $T$  is  $\sigma$ -porous hypercyclic with respect to the metric  $d_{\bar{\varepsilon}}$ .*

PROOF: The operator  $T$  is defined by  $Tf(s) = f(s + \alpha)$ , where  $\alpha \in \mathbb{C} \setminus \{0\}$ . We check that the hypotheses of Theorem 3.2 are satisfied with  $\mathcal{D} = \mathcal{H}(\mathbb{C})$  and  $\mathcal{D}_{v,r} = \mathcal{H}(\mathbb{C})$ ,  $\delta_{v,r} = 1$  for all  $(v,r) \in \mathcal{H}(\mathbb{C}) \times (0,1)$ . Thus, we have to find some suitable constants  $c_{v,r}$ . Let us fix a pair  $(v,r) \in \mathcal{H}(\mathbb{C}) \times (0,1)$ , together with  $u \in \mathcal{H}(\mathbb{C})$ .

For simplicity, we will write  $d$  instead of  $d_{\bar{\varepsilon}}$  and  $\|\cdot\|_n$  instead of  $\|\cdot\|_{K_n}$ . We fix a positive integer  $p \geq |\alpha|$  and  $\eta > 0$  such that

$$\|f - g\|_p < \eta \Rightarrow d(f, g) < r.$$

Finally, we assume without loss of generality that  $\sum_0^\infty \varepsilon_n = 1$ .

Let  $\delta \in (0, 1)$ , and set  $N := \min\{n \in \mathbb{N}; \sum_{k>n} \varepsilon_k < \frac{\delta}{2}\}$ . To ensure property (a) in 3.2, it is enough to find some function  $x \in \mathcal{H}(\mathbb{C})$  and some integer  $n$  such that  $\|x - u\|_N < \frac{\delta}{2}$  and  $\|T^n(x) - v\|_p < \eta$ . In other words, we require  $|x(s) - u(s)| < \frac{\delta}{2}$  on  $K_N$  and  $|x(s) - v(s - n\alpha)| < \eta$  on  $n\alpha + K_p$ . Now, the two disks  $K_N$  and  $n\alpha + K_p$  are disjoint whenever  $n|\alpha| > N + p$ , and in that case Runge's Theorem provides an  $x \in \mathcal{H}(\mathbb{C})$  satisfying the required properties. Let  $n$  be the smallest integer satisfying  $n|\alpha| > N + p$ . Then (a) is satisfied with  $n$  and some  $x \in \mathcal{H}(\mathbb{C})$ , so it only remains to show that (b) holds for some suitable constant  $c_{v,r}$ .

By the choice of  $n$  and since  $p \geq |\alpha|$ , we have  $n|\alpha| \leq N + 2p$  and hence  $\|T^n(f) - T^n(g)\|_p \leq \|f - g\|_{N+3p}$  for all  $f, g \in \mathcal{H}(\mathbb{C})$ . By the choice of  $p$  and  $\eta$ , it is therefore enough to find some constant  $c$ , which may depend on  $v, r, p, \eta$  but must be independent of  $\delta$  (and hence of  $N$ ) such that

$$(1) \quad d(f, g) < c\delta \Rightarrow \|f - g\|_{N+3p} < \eta.$$

By assumption on  $\bar{\varepsilon}$ , there exists some constant  $c_p$  such that

$$\sum_{k \geq N+3p} \varepsilon_k \geq c_p \sum_{k \geq N} \varepsilon_k \geq c_p \frac{\delta}{2},$$

where the second inequality comes from the choice of  $N$ . By definition of the metric  $d_{\bar{\varepsilon}}$ , it follows that for any  $f, g \in \mathcal{H}(\mathbb{C})$ , we have

$$\frac{c_p}{2} \delta \min(1, \|f - g\|_{N+3p}) \leq d(f, g).$$

Therefore, (1) will be satisfied provided  $c < \frac{c_p}{2} \min(1, \eta)$ . This concludes the proof.  $\square$

*Remark 1.* We do not know what happens if the sequence  $(\varepsilon_n)$  tends very quickly to 0. We do not know either what can be said about the derivation operator  $D$ , another classical example of hypercyclic operator on  $\mathcal{H}(\mathbb{C})$ . In view of 2.1 and since  $D$  is a weighted backward shift with increasing weights, it seems reasonable in that case to “conjecture” that, at least for a certain class of metrics  $d_{\bar{\varepsilon}}$ , the operator  $D$  is not  $\sigma$ -porous hypercyclic.

*Remark 2.* Let  $X$  be a separable Fréchet space whose topology is generated by an increasing sequence of semi-norm  $(\rho_n)_{n \in \mathbb{N}}$ , and define a metric  $d$  on  $X$  by

$$d(x, y) = \sum_0^\infty \varepsilon_n \min(1, \rho_n(x - y)),$$

where  $(\varepsilon_n)$  is as in Example 2. Then one proves in exactly the same way that an operator  $T \in \mathcal{L}(X)$  is  $\sigma$ -porous hypercyclic provided it has the following property: given  $(u, v) \in X \times X$  and  $(N, p) \in \mathbb{N} \times \mathbb{N}$ , one can find for each  $\varepsilon \in (0, 1)$  a point  $x \in X$  and an integer  $n$  such that

- $\rho_N(x - u) < \varepsilon$  and  $\rho_p(T^n(x) - v) < \varepsilon$ ;
- $\rho_p(T^n(z)) \leq A_p \rho_{N+B_p}(z)$  for all  $z \in X$ , where  $A_p > 0$  and  $B_p \in \mathbb{N}$  depend only on  $p$ .

A similar property, called *Runge transitivity*, is introduced in [2].

#### 4. Haar-negligibility

In this section, we give some examples of hypercyclic operators which are *not* Haar-null hypercyclic. The main tool will be the following well-known and simple lemma (see [3]).

**Lemma 4.1.** *Let  $G$  be a Polish abelian group, and let  $A$  be a universally measurable subset of  $G$ . If  $A$  contains a translate of each compact set  $K \subset G$ , then  $A$  is not Haar-null.*

Our first result will be applied below to weighted shifts. Let us say that a sequence  $(f_i)_{i \in \mathbb{N}}$  in a Banach space  $X$  is *semi-basic* if there exists some finite constant  $C$  such that for all finitely supported sequences of scalars  $(\lambda_i)_{i \in \mathbb{N}}$  and each  $p \in \mathbb{N}$ , we have

$$|\lambda_p| \|f_p\| \leq C \sum_i \lambda_i f_i.$$

**Proposition 4.2.** *Let  $X$  be a Banach space with a Schauder basis  $(e_i)_{i \in \mathbb{N}}$ , and let  $T \in \mathcal{L}(X)$ . For each integer  $n \geq 1$ , set  $\theta_n := \limsup_{i \rightarrow \infty} \frac{\|T^n(e_i)\|}{\|e_i\|}$ . Assume the following properties hold true.*

- (a) *All sequences  $(T^n(e_i))_{i \in \mathbb{N}}$  are semi-basic, with uniformly bounded constants.*
- (b) *For each increasing sequence of natural numbers  $(p_n)_{n \geq 1}$ , the series  $\sum \frac{1}{\theta_n} e_{p_n}$  is convergent.*

*Then  $X \setminus HC(T)$  is not Haar-null.*

PROOF: Replacing  $e_i$  by  $\frac{e_i}{\|e_i\|}$ , we may assume that the Schauder basis  $(e_i)$  is normalized. It is enough to show that the set

$$F = \{x; \forall n \in \mathbb{N} \|T^n(x)\| \geq 1\}$$

is not Haar-null. We show that  $F$  contains a translate of each compact subset of  $X$ . If  $K \subset X$  is compact, then the sequence of coordinate functionals  $(e_i^*)$

tends to 0 uniformly on  $K$ , because  $\inf_i \|e_i\| > 0$ . Writing  $x_i$  instead of  $\langle e_i^*, x \rangle$ , it follows that one can choose an increasing sequence of integers  $(p_n)_{n \geq 1}$  such that

$$\begin{cases} \forall x \in K \quad |x_{p_n}| \leq \frac{1}{\theta_n} \\ \|T^n(e_{p_n})\| \geq \frac{1}{2}\theta_n. \end{cases}$$

Now, put  $z = \sum_{i=1}^{\infty} \frac{2}{\theta_n} e_{p_n}$ . For all  $x \in K$  and all  $n \geq 1$ , we have

$$\begin{aligned} \|T^n(x+z)\| &= \left\| \sum_{i=0}^{\infty} (z_i + x_i) T^n(e_i) \right\| \\ &\geq C^{-1} |z_{p_n} + x_{p_n}| \|T^n(e_{p_n})\| \\ &\geq \frac{\theta_n}{2C} \left( \frac{2}{\theta_n} - \frac{1}{\theta_n} \right) = \frac{1}{2C}, \end{aligned}$$

where  $C$  is a constant independent of  $n$  and  $K$ . It follows that  $K+z \subset (2C)^{-1}F$ . Since  $K$  is an arbitrary compact subset of  $X$ , this concludes the proof.  $\square$

From 4.2, we immediately get the following result, which says that weighted shifts with “large” weights are not Haar-null hypercyclic.

**Corollary 4.3.** *Let  $T$  be a weighted backward shift on  $X = c_0(\mathbb{N})$  or  $\ell^p(\mathbb{N})$  ( $1 \leq p < \infty$ ), with weight sequence  $(w_n)_{n \geq 1}$ . For each integer  $n \geq 1$ , set  $\theta_n := \limsup_{i \rightarrow \infty} \theta_{ni}$ , where  $\theta_{ni} = \prod_{i-n < j \leq i} w_j$ . If the sequence  $(1/\theta_n)_{n \geq 1}$  defines an element of  $X$  (i.e. if the series  $\sum \frac{1}{\theta_{i+1}} e_i$  is convergent in  $X$ ), then  $T$  is not Haar-null hypercyclic. This holds in particular if  $\inf_n w_n > 1$ .*

*Remark.* The hypothesis in 4.3 is stronger than the corresponding one in 2.2. Indeed, choosing some increasing sequence of integers  $(i_n)_{n \geq 1}$  with  $\theta_{ni_n} \geq \frac{1}{2}\theta_n$  for all  $n$  and setting  $x := \sum_{i=1}^{\infty} \frac{1}{\theta_{ni_n}} e_{i_n}$ , we have  $\|T^n(x)\| \geq 1$  for each positive integer  $n$ .

We now turn to operators on  $\mathcal{H}(\mathbb{C})$  which commute with translations. By a result of Godefroy and Shapiro ([5]), these are exactly the operators of the form  $T = \Phi(D)$ , where  $D$  is the derivation operator and  $\Phi$  is an entire function of exponential type. Moreover, we recall that such an operator is always hypercyclic, unless it is a multiple of the identity ([5]).

In what follows, we denote by  $c_k(f)$ ,  $k \in \mathbb{N}$ , the Taylor coefficients of a function  $f \in \mathcal{H}(\mathbb{C})$ . Let  $\mathfrak{E}$  be the class of all functions  $\Phi : \mathbb{C} \rightarrow \mathbb{C}$  satisfying the following properties:

- $\Phi$  is an entire function of exponential type;
- for all  $k, n \in \mathbb{N}$ , one can write  $c_k(\Phi^n) = a_k b_n p_{nk}$ , where  $p_{nk} \geq 0$ .

Clearly, the family  $\mathfrak{E}$  contains all entire functions of exponential type with non-negative coefficients, and all exponential functions  $e^{\alpha z}$ ,  $\alpha \in \mathbb{C}$ . More generally, it is easily checked that  $\mathfrak{E}$  contains all entire functions of exponential type  $\Phi$  such that  $c_k(\Phi) \in \alpha^k \mathbb{R}^+$ , for all  $k \in \mathbb{N}$  and some fixed complex number  $\alpha$ .

**Proposition 4.4.** *Let  $T$  be an operator on  $X = \mathcal{H}(\mathbb{C})$  of the form  $T = \Phi(D)$ , where  $D$  is the derivation operator and  $\Phi \in \mathfrak{E}$ . Assume there exists at least one  $T$ -orbit whose closure does not contain 0. Then  $X \setminus HC(T)$  is not Haar-null.*

In particular, we get

**Corollary 4.5.** *If  $T$  is the derivation operator or a translation operator on  $\mathcal{H}(\mathbb{C})$ , then  $T$  is not Haar-null hypercyclic.*

PROOF: In both cases, the operator  $T$  has the required form, and there exists a function  $f \in \mathcal{H}(\mathbb{C})$  whose orbit stays away from 0: if  $T$  is the derivation operator, one may take  $f(z) = e^z$ , and if  $T$  is a translation operator  $f = \mathbf{1}$ .  $\square$

The proof of 4.4 relies on the following lemma.

**Lemma 4.6.** *Let  $T$  be as in 4.4. Then there exists a sequence  $(a_n) \subset \mathbb{C}$  such that the following property holds true: for each compact set  $\mathcal{K} \subset X$ , one can find a single function  $\varphi \in X$  such that*

- $\forall n \in \mathbb{N} \quad T^n \varphi(0) \in \mathbb{R}^+ a_n$ ;
- $\forall f \in \mathcal{K} \quad \forall n \in \mathbb{N} \quad |T^n f(0)| \leq |T^n \varphi(0)|$ .

PROOF: Write  $c_k(\Phi^n) = a_n b_k p_{nk}$ , with  $p_{nk} \geq 0$ . We show that  $(a_n)$  does the job. Let  $\mathcal{K}$  be a compact subset of  $\mathcal{H}(\mathbb{C})$ , and for each  $k \in \mathbb{N}$ , put

$$c_k = \sup\{|c_k(f)|; f \in \mathcal{K}\}.$$

By Cauchy’s inequalities, we have  $\lim_{k \rightarrow \infty} c_k^{1/k} = 0$ . Thus, there exists an entire function  $\varphi$  such that  $b_k c_k(\varphi) = |b_k| c_k$  for all  $k \in \mathbb{N}$ . Since  $T^n f(0) = [\Phi^n(D)f](0) = a_n \sum_k p_{nk} k! b_k c_k(f)$  for each  $n \in \mathbb{N}$  and all  $f \in X$ , this function  $\varphi$  clearly works.  $\square$

PROOF OF 4.4: We fix a sequence  $(a_n)$  satisfying the conclusion of the previous lemma. By assumption, there exist some function  $f_0 \in X$  and some neighbourhood  $\mathcal{U}$  of 0 in  $X$  such that  $T^n f_0 \notin \mathcal{U}$  for all  $n \in \mathbb{N}$ . We may assume that  $\mathcal{U}$  has the form  $\{u \in X; \sup_{K_0} |u(z)| < \varepsilon_0\}$  for some compact set  $K_0 \subset \mathbb{C}$  and some  $\varepsilon_0 > 0$ ; and replacing  $f_0$  by  $f_0/\varepsilon_0$ , we may assume that  $\varepsilon_0 = 1$ . Thus, we have at hand some compact set  $K_0 \subset \mathbb{C}$  and some  $f_0 \in X$  such that  $\sup\{|T^n f_0(z)|; z \in K_0\} \geq 1$  for all  $n \in \mathbb{N}$ . Since  $T$  commutes with all translation operators, this means that  $\sup\{|T^n f(0)|; f \in \mathcal{K}_0\} \geq 1$  for all  $n$ , where  $\mathcal{K}_0 = \{\tau_z f_0; z \in K_0\}$ . Since  $\mathcal{K}_0$  is a compact subset of  $X$ , one can apply Lemma 4.6 to get  $\varphi \in X$  such that

$$\forall n \in \mathbb{N} \quad T^n \varphi(0) \in \mathbb{R}^+ a_n \quad \text{and} \quad |T^n \varphi(0)| \geq 1.$$

Now, let  $\mathcal{K}$  be any compact subset of  $X$ . By Lemma 4.6, one can find  $\psi \in X$  such that  $T^n\psi(0) \in \mathbb{R}^+a_n$  and  $|T^n f(0)| \leq |T^n\psi(0)|$ , for all  $f \in \mathcal{K}$  and each  $n \in \mathbb{N}$ . Putting  $h = \varphi + \psi$ , we have  $|T^n(h)(0)| = |T^n\varphi(0)| + |T^n\psi(0)| \geq 1 + |T^n\psi(0)|$  for each  $n \in \mathbb{N}$ , hence  $|T^n(f+h)(0)| \geq 1$  for all  $f \in \mathcal{K}$  and each  $n \in \mathbb{N}$ . In particular, it follows that  $\mathcal{K} + h \subset X \setminus HC(T)$ . Thus, we have proved that  $X \setminus HC(T)$  contains a translate of each compact subset of  $X$ .  $\square$

From the above propositions, the following questions obviously come to mind.

- Does there exist a weighted backward shift on  $\ell^2(\mathbb{N})$  which is Haar-null hypercyclic?
- Does there exist a nontrivial operator on  $\mathcal{H}(\mathbb{C})$  commuting with translations which is Haar-null hypercyclic?

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