SMOOTHNESS, ASYMPTOTIC SMOOTHNESS AND THE BLUM-HANSON PROPERTY

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Abstract. We isolate various sufficient conditions for a Banach space X to have the so-called Blum-Hanson property. In particular, we show that X has the Blum-Hanson property if either the modulus of asymptotic smoothness of X has an extremal behaviour at infinity, or if X is uniformly Gâteaux smooth and embeds isometrically into a Banach space with a 1-unconditional finite-dimensional decomposition.

1. Introduction

Let X be a Banach space, and let T be a power-bounded linear operator on X (i.e. $\sup_{n\in\mathbb{N}} \|T^n\| < \infty$). By the classical mean ergodic theorem (see e.g. [29]) if $x \in X$ and if the sequence of iterates $(T^n x)$ has a weakly convergent subsequence, then the arithmetic means

$$A_N(x) = \frac{1}{N}\sum_{n=1}^{N} T^n x$$

are *norm* convergent. In particular, if x has a weakly null T- orbit ($T^n x \xrightarrow{w} 0$), then $A_N(x) \xrightarrow{\|\,\|} 0$. When X is a Hilbert space and T is a contraction operator ($\|T\| \leqslant 1$), it turns out that a much stronger conclusion holds true: for any $x \in X$ with a weakly null T- orbit, the arithmetic means of $T^n x$ *along any increasing sequence of integers* (n_i) are norm convergent to 0. This was first proved by J. R. Blum and D. L. Hanson ([9]) for isometries induced by measure-preserving transformations, and later on in [2] and [24] for arbitrary contractions. For contractions on a general Banach space X, this strong conclusion may or may not hold true. When it does so (for every contraction operator on X), the space X is said to have the *Blum-Hanson property.* This property is the topic of the present paper.

To proceed further, let us fix some terminology. From now on, we consider real Banach spaces only. A sequence $(x_n)_{n\in\mathbb{N}} \subset X$ is a *Blum-Hanson sequence* if every subsequence of (x_n) is norm convergent to 0 in the Cesàro sense; that is, for any increasing sequence of integers (n_i), it holds that

$$\lim_{K\to\infty} \frac{1}{K}\left\| \sum_{i=1}^{K} x_{n_i} \right\| = 0\,.$$

Obviously, every norm null sequence is Blum-Hanson and every Blum-Hanson sequence is weakly null. In fact, it is shown in [36] that a sequence $(x_n) \subset X$ is Blum-Hanson if and only it is "uniformly weakly null", which means that for any $\varepsilon > 0$, there exists an integer N_ε such that $\forall x^* \in B_{X^*} \; : \; \sharp\{n \in \mathbb{N};\; |\langle x^*, x_n\rangle| \geqslant \varepsilon\} \leqslant N_\varepsilon$. (In the case where X is a Hilbert space, this was proved earlier in [8], where Blum-Hanson sequences are called *strongly mixing*).

An operator $T \in \mathcal{L}(X)$ satisfies the *Blum-Hanson dichotomy* at some point $x \in X$ if either the sequence $(T^n x)$ is not weakly null, or it is Blum-Hanson. We note that if $T \in \mathcal{L}(X)$ and if $z \in X$ has a weakly convergent T-orbit, then $\pi_T z := w\text{-}\lim T^n z$ is a fixed point of T and hence $T^n(z - \pi_T z) \xrightarrow{w} 0$. It follows that an operator satisfies the Blum-Hanson dichotomy at *all* points $x \in X$ if and only if the following holds: for any $z \in X$ with a weakly convergent T- orbit, every subsequence of $(T^n z)$ is norm convergent to $\pi_T z$ in the Cesàro sense.

Given a class of operators $\mathcal{C}$, we say that the Banach space X has the *Blum-Hanson property with respect to* $\mathcal{C}$ if every operator $T \in \mathcal{C} \cap \mathcal{L}(X)$ satisfies the Blum-Hanson dichotomy at all points $x \in X$. Thus, the Blum-Hanson property itself corresponds to the class $\mathcal{C}$ of all contraction operators. If one considers only those operators $T \in \mathcal{C}$ with weakly convergent orbits, one gets a formally weaker property, which we call the *conditional* Blum-Hanson property (with respect to $\mathcal{C}$).

Few results can be found in the literature regarding the Blum-Hanson property. In the "positive" direction and apart from Hilbert spaces, the most notable ones seem to be the following: ℓ_p, $1 \leqslant p < \infty$ has the Blum-Hanson property ([40]); L_1 has the conditional Blum-Hanson property ([2]); L_p has the conditional Blum-Hanson property with respect to isometries induced by measure-preserving transformations ([9]), and with respect to positive contractions ([3]); any positive contraction on L_p satisfies the Blum-Hanson dichotomy at all positive $f \in L_p$ ([7]); the same is true for Orlicz function spaces endowed with the Orlicz norm, provided that this norm is uniformly smooth ([38]). As for "negative" results, we mention the following: the space $\mathcal{C}(\mathbb{T}^2)$ does not have the conditional Blum-Hanson property ([1]); and ℓ_p, $1 < p < \infty$ does not have the conditional Blum-Hanson property with respect to power-bounded operators ([40]). (This last result shows in particular that the Blum-Hanson property is not preserved under renormings; in other words, this is not an *isomorphic* property of the space). The most exciting question is arguably whether L_p has the Blum-Hanson property.

In this note, our aim is to show that some of the above positive results, as well as some new ones, can be derived in a unified way from a general and rather simple theorem (Theorem 2.1) involving a certain "modulus" similar to the well known *modulus of asymptotic smoothness* of the given Banach space X. (See section 2 for the definition).

To be a little bit more precise, it follows from our main result that an "extremal" behaviour of the modulus of asymptotic smoothness *at infinity* entails the Blum-Hanson property for X. This is rather unexpected since, as far as we know, the behaviour of this modulus at infinity has never been considered. It also follows immediately from Theorem 2.1 that Banach spaces satisfying Kalton-Werner's *property* (m_p) for some $p \in (1, \infty]$ have the Blum-Hanson property. Finally, with little extra work we deduce from Theorem 2.1 that if the duality mapping of X has a certain weak continuity property, then X has the Blum-Hanson property; it follows in particular that *uniform Gâteaux differentiability* of the norm implies Blum-Hanson when combined with a suitable "approximation-like" property. As a concrete class of examples, we consider Orlicz spaces endowed with the *Luxemburg* norm: we show that asymptotically uniformly smooth small Orlicz sequence spaces have the Blum-Hanson property, and that any positive contraction on a Gâteaux smooth Orlicz function space L_θ satisfies the Blum-Hanson dichotomy at all positive $f \in L_\theta$.

The paper is organized as follows. Our main result is stated in section 2, and two typical examples are given immediately. Theorem 2.1 is proved in section 3. Results involving differentiability of the norm are collected in section 4. Section 5 is devoted to Orlicz spaces. Section 6 contains some remarks about very classical spaces (Hilbert, $\mathcal{C}(K)$ and L_p). In particular, we give a "new" proof of the Blum-Hanson property for Hilbert spaces, and we observe that $\mathcal{C}(K)$ fails the conditional Blum-Hanson property for any uncountable compact metric space K. Finally, section 7 contains some additional remarks and ends up with a few natural questions.

2. Main result, and two examples

Our main result (Theorem 2.1) is about sequences $(x_n) \subset X$ which are not necessarily of the form $x_n = T^n x$ for some contraction $T \in \mathcal{L}(X)$. We shall "only" assume that (x_n) is *shift-monotone*, in the following sense: for every finite increasing sequence of integers $n_1 < \cdots < n_k$, it holds that

$$\|x_{1+n_1} + \cdots x_{1+n_k}\| \leqslant \|x_{n_1} + \cdots + x_{n_k}\| .$$

This is indeed more general than assuming that (x_n) is an orbit of some contraction operator; see [47, Example 3.3.]. A similar property, called *convex shift-boundedness*, is considered in [47]. It is shown there that a convex shift-bounded sequence (x_n) is *weakly mixing to* 0 (i.e. $\frac{1}{N}\sum_{n=1}^N |\langle x^*, x_n\rangle| \to 0$ for every $x^* \in X^*$) if and only if the arithmetic means of (x_n) along any increasing sequence of integers with positive lower density are norm convergent to 0. For sequences of the form $x_n = T^n x$ where T is a power-bounded operator, this was proved earlier in [25].

Theorem 2.1 will be formulated using a "modulus" associated with a given convex cone $\mathbf{C} \subset X$ (i.e. a nonempty convex set which is closed under multiplication by nonnegative scalars). For any set $A \subset X$, let us denote by $\mathrm{WN}(A)$ the family of all weakly nul sequences $(y_n) \subset X$ with $y_n \in A$ for all n. Then, for any $x \in X$ and $t > 0$, we put

$$r_{\mathbf{C}}(t, x) = \sup_{(y_n)\in \mathrm{WN}(S_X \cap \mathbf{C})} \limsup_{n\to\infty} \|x + t\, y_n\| .$$

(Here and elsewhere, S_X is the unit sphere of X).

The trivial case $\mathrm{WN}(S_X \cap \mathbf{C}) = \varnothing$ is allowed: $\sup \varnothing$ declared to be $-\infty$. For example, $r_{\mathbf{C}}(t, x) \equiv -\infty$ if the Banach space X has the *Schur property*, i.e. when every weakly null sequence is in fact norm null.

The modulus r_X has already been used by many authors, see e.g [20], [21], [22], [34], [35], [42]. There is a simple connection with the modulus of asymptotic smoothness. The latter is one of the many moduli introduced by V. D. Milman in [39]. With the notation of [26], it is the function $\bar{\rho}_X : \mathbb{R}^+ \times S_X \to \mathbb{R}^+$ defined as follows:

$$\bar{\rho}_X(t,x) = \inf_E \sup_{y\in B_E} \|x+ty\| - 1\,,$$

where the infinimum $\inf_E$ is taken over all finite-codimensional subspaces $E \subset X$ (and B_E is the unit ball of E). The connection between the two moduli is the following: for any $x \in S_X$,

$$r_X(t,x) \leqslant \bar{\rho}_X(t,x) + 1\,. \tag{1}$$

This is fairly easy to check, using the fact that if (y_n) is a weakly null sequence in X then $\operatorname{dist}(y_n, E) \to 0$ for every finite-codimensional subspace $E \subset X$. Moreover, it is shown in [34] that equality holds in (1) as soon as X embeds isometrically into a Banach space with a shrinking Markushevich basis (for example, a reflexive Banach space).

We note that if $\mathrm{WN}(S_X \cap \mathbf{C}) \neq \varnothing$, then $r_{\mathbf{C}}(t,x) \geqslant t - \|x\|$ for all t. Moreover, since $r_{\mathbf{C}}(t,x)$ is obviously 1-Lipschitz with respect to t, the map $t \mapsto r_{\mathbf{C}}(t,x) - t$ is non-increasing. Hence, $r_{\mathbf{C}}(t,x) - t$ always has a limit $l_{\mathbf{C}}(x)$ as $t \to \infty$, and $l_{\mathbf{C}}(x) \geqslant -\|x\|$ in the nontrivial case $\mathrm{WN}(S_X \cap \mathbf{C}) \neq \varnothing$. (Actually, if the cone $\mathbf{C}$ is symmetric, then $r_{\mathbf{C}}(t,x) \geqslant t$ for all t and hence $l_{\mathbf{C}}(x) \geqslant 0$: this is because $t = \|ty\| \leqslant \frac{\|x+ty\|+\|x-ty\|}{2}$ for any $y \in S_X \cap \mathbf{C}$).

We can now state

Theorem 2.1. *Let X be a Banach space, and let $\mathbf{C} \subset X$ be a nonempty convex cone. Let also $(x_n)_{n\in\mathbb{Z}^+}$ be a shift-monotone, weakly null sequence in $\mathbf{C}$. If the initial point $x := x_0$ satisfies*

$$\lim_{t\to\infty} \Big(r_{\mathbf{C}}(t,x) - t \Big) \leqslant 0\,, \tag{$*$}$$

then (x_n) is a Blum-Hanson sequence.

Let us say that an operator $T \in \mathcal{L}(X)$ is $\mathbf{C}$-*positive* if it maps the cone $\mathbf{C}$ into itself. As an immediate consequence of Theorem 2.1, we get

Corollary 2.2. *Assume that $(*)$ holds for some $x \in \mathbf{C}$. Then, any $\mathbf{C}$-positive contraction on X satisfies the Blum-Hanson dichotomy at all $\xi \in \mathbb{R}^+ x$.*

Proof. Let $T \in \mathcal{L}(X)$ be a $\mathbf{C}$-positive contraction, and assume that $T^n\xi \xrightarrow{w} 0$ for some $\xi = \lambda x$ with $\lambda \geqslant 0$. To show that $(T^n\xi)$ is a Blum-Hanson sequence, we may obviously assume that $\xi \neq 0$. Then $\lambda \neq 0$ and $r_{\mathbf{C}}(t,\xi) = \lambda\, r_{\mathbf{C}}(\frac{t}{\lambda}, x)$ for all $t \in \mathbb{R}^+$, so $(*)$ is satisfied for ξ and the result follows by applying Theorem 2.1 with $x_n = T^n\xi$. □

Remark. Assume additionally that $\mathbf{C} - \mathbf{C} = X$. Then, the following equivalence holds for every $\mathbf{C}$-positive contraction T: all T-orbits are weakly null iff they are all Blum-Hanson. However, it does not follow directly from Corollary 2.2 that X has the conditional Blum-Hanson property with respect to $\mathbf{C}$-positive contractions. The point is that if a contraction T with weakly convergent orbits satisfies $T^n x \xrightarrow{w} 0$ for some $x \in X$ and if we write $x = u - v$ with $u, v \in \mathbf{C}$, then the sequences $(T^n u)$ and

$(T^n v)$ have no reason for being both weakly null even though they are both weakly convergent. When $X = L_p$ and $\mathbf{C} = L_p^+$, one can get round this difficulty with some extra work; see [3], paragraph (2.1).

For future reference, it is convenient to introduce the following terminology.

Definition 2.3. We shall say that a Banach space X has *extremal asymptotic smoothness at infinity* if the modulus r_X satisfies $\lim_{t\to\infty}(r_X(t,x) - t) \leqslant 0$ for all $x \in X$, and that X has extremal *uniform* asymptotic smoothness at infinity if $\lim_{t\to\infty}(r_X(t) - t) \leqslant 0$, where $r_X(t) = \sup_{x\in S_X} r_X(t,x)$.

Thanks to (1), we see that X has extremal asymptotic smoothness at infinity as soon as its modulus of asymptotic smoothness satisfies (for all $x \in S_X$)

$$(**) \qquad \lim_{t\to\infty} (\bar\rho_X(t,x) + 1 - t) = 0\,.$$

Note also that Banach spaces with the Schur property, for example the space ℓ_1, trivially have extremal (uniform) asymptotic smoothness at infinity. This makes the terminology perhaps confusing because ℓ_1 is usually considered as the "less smooth" of all Banach spaces (indeed, it has the "worst possible" modulus of asymptotic smoothness). But we prefer to use the modulus r_X rather than ρ_X because it leads to more general results, and yet we want to emphasize asymptotic smoothness.

Note that extremal asymptotic smoothness at infinity is a *hereditary* property, i.e. inherited by subspaces. Thus, we may state

Corollary 2.4. *If the Banach space X has extremal asymptotic smoothness at infinity, then every subspace of X has the Blum-Hanson property. In particular, X has Blum-Hanson if $(**)$ holds for all $x \in X$.*

The "in particular" part is rather unexpected, since sually what matters about the modulus of asymptotic smoothness is the behaviour or $\bar\rho_X(t,x)$ *as t goes to* 0. Indeed, the main property captured by the modulus $\bar\rho_X$ is the following: the Banach space X is said to be *asymptotically uniformly smooth* if

$$\lim_{t\to 0} \frac{\bar\rho_X(t)}{t} = 0\,,$$

where $\bar\rho_X(t) = \sup_{x\in S_X} \bar\rho(t,x)$.

Theorem 2.1 can also be applied when the given norm on X is smooth in a more usual sense, under a certain assumption on the duality mapping. We state the result right now in order to illustrate it with positive contractions on L_p, but the proof is postponed to section 4 (see Proposition 4.1).

Corollary 2.5. *Assume that the norm of X is uniformly Gâteaux differentiable on the unit sphere S_X, and denote by $J(y)$ the Gâteaux derivative of the norm at $y \in S_X$. Let $x \in \mathbf{C}$ be given. Assume that whenever (y_n) is a weakly null sequence in $S_X \cap \mathbf{C}$, it holds that $\langle J(y_n), x\rangle \to 0$. Then, any $\mathbf{C}$-positive contraction on X satisfies the Blum-Hanson dichotomy at x.*

We now give two hopefully illustrative examples.

The first one is about the so-called *properties* (m_p) introduced by N. Kalton and D. Werner in [28]. A Banach X has property (m_p), $1 \leqslant p \leqslant \infty$ if, for any $x \in X$ and every weakly null sequence $(x_n) \subset X$, it holds that

$$\limsup_{n\to\infty} \|x + x_n\| = (\|x\|^p + \limsup \|x_n\|^p)^{1/p}. \tag{2}$$

For $p = \infty$ the right-hand side is of course to be interpreted as $\max(\|x\|, \limsup \|x_n\|)$. We shall say that X has property *sub* - (m_p) if (2) holds with "=" replaced with "$\leqslant$"; equivalently, if $r_X(t,x) \leqslant (1+t^p)^{1/p}$ for all $x \in S_X$.

For example, ℓ_p has property (m_p) and c_0 has property (m_∞); any *Lorentz sequence space* $d(w,p)$ different from ℓ_∞ has property sub - (m_p) (see [32] for the definition); the Bergman space $B_p(\mathbb{D})$ on the unit disk has property (m_p); and for any continuous weight $w : [0,1] \to \mathbb{R}^+$ such that $w(r) = 0$ only at $r = 1$, the space β_w consisting of all functions f holomorphic on $\mathbb{D}$ such that $w(|z|)f(z) \to 0$ as $|z| \to 1$, with its natural norm, has property (m_∞) (see [28, pp. 163–164]). Note also that any Banach space has property sub - (m_1) and that, just like extremal asymptotic smoothness, (m_p) and sub - (m_p) are hereditary properties, i.e. inherited by subspaces.

Example 1. *For any $p \in (1,\infty]$, property sub - (m_p) implies extremal uniform asymptotic smoothness at infinity, and hence the Blum-Hanson property. In particular, any subspace of an ℓ_p or c_0 direct sum of Banach spaces with the Schur property has the Blum-Hanson property.*

Proof. If X has property sub - (m_p) then $r_X(t) \leqslant (1+t^p)^{1/p}$ if $p < \infty$, and $r_X(t) \leqslant \max(1,t)$ if $p = \infty$; so the first part is clear. For the second part, it is enough to show that any ℓ_p (resp. c_0) sum of Banach spaces with the Schur property has property (m_p) (resp. (m_∞)). But this is clear since if $X = \oplus_k E_k$ is such a space then (by the Schur property of each E_k) a sequence $(x_n) = (\oplus_k x_{n,k}) \subset X$ is weakly null if and only if it is bounded and $\|x_{n,k}\|_{E_k} \to 0$ as $n \to \infty$, for every $k \in \mathbb{N}$.

□

Remark 1. The ℓ_p case is a slight generalization of a result of Y. Tomilov and V. Müller [40]. Somewhat surprisingly, the c_0 case appears to be new. (That $X = c_0$ itself has the Blum-Hanson property was observed independently in [5]).

Remark 2. It is shown in [28] that a separable Banach space X not containing ℓ_1 has property (m_p), $1 < p < \infty$ if and only if it is almost isometric to a subspace of an ℓ_p direct sum of finite-dimensional spaces, and that X has property (m_∞) iff it is almost isometric to a subspace of c_0. Hence, the special case quoted above is in fact rather general.

Our second example is a result due to A. Bellow [7] (already mentioned in the introduction).

Example 2. *Any positive contraction on L_p, $1 < p < \infty$ satisfies the Blum-Hanson dichotomy at all $f \in L_p^+$ (the positive cone of L_p).*

Proof. The space L_p is uniformly (Fréchet) smooth, and the first key step in [7] is to show that for any $\varepsilon > 0$, one can find a constant C_ε such that the following inequality holds for every $f, g \in S_{L_p} \cap \in L_p^+$:

$$\int f\, J(g) \leqslant \varepsilon + C_\varepsilon \int g\, J(f)\,. \tag{3}$$

Now, the new thing is that the proof is already finished. Indeed, it follows at once from (3) that if (g_n) is a weakly null sequence in $S_{L_p} \cap L_p^+$, then $\langle J(g_n), f\rangle \to 0$ for every $f \in S_{L_p} \cap L_p^+$. Hence, we may apply Corollary 2.5.

For completeness and since the same idea will be used in section 5, we include a proof of (3) (not with the best constant $C(\varepsilon)$). Recall that the duality mapping $J : S_{L_p} \to S_{L_q}$ is given by

$$J(f) = |f|^{p-2} f\,;$$

so $J(f) = f^{p-1}$ if $f \in S_{L_p} \cap \in L_p^+$.

Let us fix $\varepsilon > 0$, and let $\eta > 0$ to be chosen later. If $f, g \in S_{L_p} \cap L_p^+$ then

$$\begin{aligned} \int f\, J(g) &= \int f g^{p-1} \\ &\leqslant \int_{\{f<\eta g\}} (\eta g) g^{p-1} + \int_{\{f>\eta^{-1} g\}} f(\eta f)^{p-1} + \int_{\{\eta g \leqslant f \leqslant \eta^{-1} g\}} (\eta^{-1} g)(\eta^{-1} f)^{p-1} \\ &\leqslant 2\eta^{p-1} + \eta^{-p} \int g\, J(f)\,, \end{aligned}$$

and the result follows by taking $\eta = (\varepsilon/2)^{1/p-1}$. ☐

3. Proof of theorem 2.1

The proof of Theorem 2.1 relies on the following simple lemma. Here and afterwards, for any $d, s \in \mathbb{N}$ we denote by $\mathrm{FIN}(s,d)$ the set of all finite sets $A \subset \mathbb{N}$ with cardinality $|A| = s$ and "gaps" of length at least d, i.e. $|n - n'| \geqslant d$ for any $n \neq n'$ in A.

Lemma 3.1. *Let $(x_n)_{n\in\mathbb{Z}^+}$ be a bounded sequence in X. For any $s \in \mathbb{N}$, set*

$$F(s) := \inf_{d\in\mathbb{N}} \sup_{A\in \mathrm{FIN}(s,d)} \left\| \sum_{n\in A} x_n \right\|.$$

Then (x_n) is a Blum-Hanson sequence if and only if $\displaystyle\lim_{s\to\infty} \frac{F(s)}{s} = 0$.

Proof. It is easy to see that if (x_n) is Blum-Hanson, then in fact

$$\lim_{|A|\to\infty} \frac{1}{|A|} \left\| \sum_{n\in A} x_n \right\| = 0\,.$$

Indeed, if this does not hold then one can find $\varepsilon > 0$ and a sequence of finite sets $(A_k)_{k\in\mathbb{N}}$ such that $|A_k| \to \infty$ and $\left\|\sum_{n\in A_k} x_n\right\| \geqslant \varepsilon\, |A_k|$ for all k. If $|A_k|$ is sufficiently fast increasing, then the sets $B_k := A_k \cap (\max A_{k-1}, \infty)$ satisfy $\max B_k < \min(B_{k+1})$

and $\left\|\sum_{n\in B_k} x_n\right\| \geqslant (\varepsilon/2)\,|B_k|$ for all k, and hence (x_n) is not Blum-Hanson (consider the increasing enumeration (n_i) of the set $\bigcup_k B_k$).

Conversely, assume that $\frac{F(s)}{s} \to 0$ as $s\to\infty$. Let $(n_i)_{i\geqslant 1}$ be an increasing sequence of integers, and let us fix $\varepsilon>0$. We have to find $K_0\in\mathbb{N}$ such that

$$\forall K\geqslant K_0\ :\ \left\|\frac{1}{K}\sum_{i=1}^K x_{n_i}\right\|\leqslant\varepsilon\,.$$

By assumption, one may pick $d,s\in\mathbb{N}$ such that

$$\forall A\in \mathrm{FIN}(s,d)\ :\ \left\|\sum_{n\in A} x_n\right\|\leqslant\varepsilon s\,.$$

Let K_0 be a large integer to be chosen later. Let also $K\geqslant K_0$, and let $k\in\mathbb{N}$ satisfy $ksd\leqslant K<(k+1)sd$.

One can partition the interval $[1,K]$ as

$$[1,K]=\bigcup_{l=1}^d\bigcup_{j=1}^k B_{l,j}\cup B\,,$$

where each $B_{l,j}$ is an arithmetic progression with cardinality s and "ratio" d, and $|B|<sd$. Explicitely:

$$B_{l,j}=\{b_{l,j}, b_{l,j}+d,\dots,b_{l,j}+(s-1)d\}\,,$$

where $b_{l,j}=(j-1)sd+l$. Putting $A_{l,j}:=\{n_i;\ i\in B_{l,j}\}$ and $A:=\{n_i;\ i\in B\}$, we then have $A_{l,j}\in\mathrm{FIN}(s,d)$ and $|A|<sd$. Hence, $\left\|\sum_{i=1}^K x_{n_i}\right\|$ can be estimated as follows:

$$\begin{aligned}\left\|\sum_{i=1}^K x_{n_i}\right\| &\leqslant \sum_{l=1}^d\sum_{j=1}^k\left\|\sum_{n\in A_{l,j}} x_n\right\|+\left\|\sum_{n\in A} x_n\right\|\\ &\leqslant kd\times\varepsilon s+C\,sd\,,\end{aligned}$$

where $C=\sup_n\|x_n\|$. Dividing by K and since $K\geqslant\max(ksd,K_0)$, we get

$$\left\|\frac{1}{K}\sum_{i=1}^K x_{n_i}\right\|\leqslant\varepsilon+\frac{Csd}{K_0}\,,$$

for every $K\geqslant K_0$. If we choose now $K_0\geqslant\frac{Csd}{\varepsilon}$ and replace ε with $\varepsilon/2$, this gives the required result.

□

The following observation will also be useful, mainly because it allows to replace $r_{\mathbf{C}}(t,x)$ with a modulus which is non-decreasing with respect to t. (The corresponding fact for the modulus of asymptotic smoothness can be found e.g. in [23]). From now on, we fix a convex cone $\mathbf{C}\subset X$.

Remark 3.2. *Assume that* $\mathrm{WN}(S_X \cap \mathbf{C}) \neq \varnothing$*. For* $x \in X$ *and* $t \geqslant 0$*, define*

$$\bar{r}_{\mathbf{C}}(t,x) = \sup_{(z_n)\in \mathrm{WN}(B_X\cap \mathbf{C})} \limsup_{n\to\infty} \|x + t\, z_n\| \,.$$

(In other words, $\bar{r}_{\mathbf{C}}(t,x)$ *is defined exactly as* $r_{\mathbf{C}}(t,x)$ *with the unit ball* B_X *in place of the unit sphere* S_X*).Then* $r_{\mathbf{C}}(t,x) = \bar{r}_{\mathbf{C}}(t,x)$ *whenever* $t > 2\|x\|$*. If* $\mathbf{C}$ *is symmetric, this holds for every* $t \geqslant 0$*.*

Proof. Let us fix $x \in X$. We have to show that $\limsup_{n\to\infty} \|x + t\, z_n\| \leqslant r_{\mathbf{C}}(t,x)$ for any weakly null sequence $(z_n) \subset B_X \cap \mathbf{C}$; and upon replacing (z_n) by a suitable subsequence, we may assume that both $\lim \|x + t\, z_n\|$ and $\lim \|z_n\|$ exist.

Choose $\varepsilon \in (0,1)$ such that $2\|x\| + \varepsilon t \leqslant t$. If $\lim \|z_n\| \leqslant \varepsilon$, then $\lim \|x + tz_n\| \leqslant \|x\| + \varepsilon t \leqslant t - \|x\| \leqslant r_{\mathbf{C}}(t,x)$. Otherwise, we may assume that $\|z_n\| > \varepsilon$ for all n. Then $y_n := \frac{z_n}{\|z_n\|} \xrightarrow{w} 0$, and $x + tz_n$ is a convex combination of $x + ty_n$ and $x + t\varepsilon y_n$. Since $y_n \in S_X$ and $\|\varepsilon y_n\| = \varepsilon$, it follows from the first case that $\lim \|x + tz_n\| \leqslant \max\left(\limsup \|x + ty_n\|, \limsup \|x + t\varepsilon y_n\|\right) \leqslant r_{\mathbf{C}}(t,x)$.

If $\mathbf{C}$ is symmetric, then $r_{\mathbf{C}}(t,x) \geqslant \|x\|$ because $\|x\| \leqslant \frac{\|x+ty\|+\|x-ty\|}{2}$ for every $y \in S_X \cap \mathbf{C}$. Then the proof splits into two parts as above according to whether $\lim \|z_n\|$ is 0 or > 0, expressing $x+tz_n$ as a convex combination of $x+ty_n$ and $x-ty_n$ in the second case.

□

Finally, we note the following trivial yet essential fact: for any $x \in X$ and every weakly null sequence $(z_d) \subset \mathbf{C}$,

$$\limsup_{d\to\infty} \|x + z_d\| \leqslant \bar{r}_{\mathbf{C}}\big(\limsup \|z_d\|, x\big) \,. \tag{4}$$

We can now give the

Proof of Theorem 2.1. We assume from the beginning that $\mathrm{WN}(S_X \cap \mathbf{C}) \neq \varnothing$, since otherwise we already know that every weakly null sequence $(x_n) \subset \mathbf{C}$ is norm null and hence Blum-Hanson.

Let $(x_n)_{n\in\mathbb{Z}^+} \subset \mathbf{C}$ be a shift-monotone, weakly null sequence such that

$$\lim_{t\to\infty} \left(r_{\mathbf{C}}(t,x_0) - t\right) \leqslant 0 \,.$$

Then $\lim_{t\to\infty} \left(\bar{r}_{\mathbf{C}}(t,x_0) - t\right) \leqslant 0$ as well by Remark 3.2. For notational simplicity we will just write $\bar{r}(t)$ instead of $\bar{r}_{\mathbf{C}}(t,x_0)$.

Let $F : \mathbb{N} \to \mathbb{R}^+$ be the function introduced in Lemma 3.1:

$$F(s) = \inf_{d\in\mathbb{N}} F_d(s) = \lim_{d\to\infty} F_d(s) \,,$$

where

$$F_d(s) = \sup_{A\in \mathrm{FIN}(s,d)} \left\| \sum_{n\in A} x_n \right\| \,.$$

(Since $F_d(s)$ is non-increasing with respect to d, the infimum $\inf_d$ is indeed a true limit).

The key point is the following

Fact. The function F satisfies the inductive inequality $F(s+1) \leqslant \bar{r}(F(s))$.

Proof of Fact. Let us fix $s \in \mathbb{N}$. By the definition of $F(s+1)$, one can choose a sequence $(A_d)_{d\in\mathbb{N}}$, where each A_d is a finite subset of $\mathbb{N}$ with cardinality $s+1$ and gaps at least d, such that

$$\lim_{d\to\infty} \Big\| \sum_{n\in A_d} x_n \Big\| = F(s+1)\,.$$

Write $A_d = \{n_{1,d}, \dots, n_{s+1,d}\}$, with $n_{1,d} < \cdots < n_{s+1,d}$. Since the sequence (x_n) is shift-monotone, we have

$$\begin{aligned} \Big\| \sum_{n\in A_d} x_n \Big\| &= \|x_{n_{1,d}} + x_{n_{2,d}} + \cdots + x_{n_{s+1,d}}\| \\ &\leqslant \|x_0 + \big(x_{n_{2,d}-n_{1,d}} + \cdots + x_{n_{s+1,d}-n_{1,d}}\big)\| \\ &:= \|x_0 + z_d\| \end{aligned}$$

for every $d \in \mathbb{N}$.

Since $n_{i,d} - n_{1,d} \geqslant d$ for every $i \in \{2, \dots, s+1\}$ and $x_n \xrightarrow{w} 0$, the sequence (z_d) is weakly null; and $z_d \in \mathbf{C}$ because $\mathbf{C}$ is a convex cone. By (4), it follows that

$$\limsup_{d\to\infty} \|x_0 + z_d\| \leqslant \bar{r}_{\mathbf{C}}(\limsup \|z_d\|, x_0) = \bar{r}(\limsup \|z_d\|)\,.$$

Moreover, z_d has the form $\sum_{m\in B_d} x_m$, for some set $B_d \subset \mathbb{N}$ with cardinality s and gaps at least d, i.e. $B_d \in \mathrm{FIN}(s,d)$. Hence,

$$\|z_d\| \leqslant F_d(s)$$

for all $d \in \mathbb{N}$; and since $\bar{r}(t)$ is non-decreasing with respect to t, it follows that $\bar{r}(\limsup \|z_d\|) \leqslant \bar{r}(\limsup F_d(s)) = \bar{r}(F(s))$. Altogether, we get

$$F(s+1) = \lim \|x_0 + z_d\| \leqslant \bar{r}(F(s))\,.$$

□

It is now easy to conclude the proof. By Lemma 3.1, we have to show that $F(s)/s \to 0$ as $s \to \infty$. Put $\bar{F}(s) = \max(F(1), \dots, F(s))$. Then $\bar{F}$ is non-decreasing and satisfies the same inductive inequality as F, i.e. $\bar{F}(s+1) \leqslant \bar{r}(\bar{F}(s))$ (again because $\bar{r}(t)$ is non-decreasing with respect to t). If $\bar{F}(s)$ has a finite limit as $s \to \infty$ then of course $\lim_{s\to\infty} \bar{F}(s)/s = 0$, and hence $\lim_{s\to\infty} F(s)/s = 0$. Otherwise, since $\lim_{t\to\infty}(\bar{r}(t) - t) \leqslant 0$, it follows from the inductive inequality that

$$\limsup_{s\to\infty} (\bar{F}(s+1) - \bar{F}(s)) \leqslant 0\,.$$

By Cesàro's theorem, we conclude that $\bar{F}(s)/s \to 0$ in this case as well.

□

Remark. An examination of the above proof reveals that assumption $(*)$ in Theorem 2.1 is a little bit too strong. In fact, to conclude that (x_n) is Blum-Hanson it is enough to assume the following:

$$(*') \qquad \inf_{k\in\mathbb{Z}^+} \lim_{t\to\infty} (r_{\mathbf{C}}(t,x_k)-t) \leqslant 0\,.$$

The proof is the same as the one given above, with minor adjustments. First, in the definition of $\mathrm{FIN}(s,d)$ one adds the condition "$\min A \geqslant d$". Then Lemma 3.1 remains true, with obvious changes in the proof (partition $[d,K]$ rather than $[1,K]$). Next, in the proof of Theorem 2.1, put $\bar r_k(t) = \bar r_{\mathbf{C}}(t,x_k)$. The key fact now reads as follows: *for any $k\in\mathbb{Z}^+$, the inductive inequality $F(s+1)\leqslant \bar r_k(F(s))$ holds.* This is proved in the same way as before, with the following changes: since $A_d\in \mathrm{FIN}(s,d)$ we know that $n_{1,d}=\min A_d\geqslant d$ for all d; so, for any fixed $k\in\mathbb{Z}^+$ and all $d>k$ we may write $\|x_{n_{1,d}}+x_{n_{2,d}}+\cdots+x_{n_{s+1,d}}\|\leqslant \|x_k+z_d\|$, where $z_d=x_{k+n_{2,d}-n_{1,d}}+\cdots+x_{k+n_{s+1,d}-n_{1,d}}$; then the proof of the fact can proceed as above (note that the set B_d is indeed in $\mathrm{FIN}(s,d)$ since $\min B_d\geqslant d$). Finally (assuming that $\bar F(s)\to\infty$) it follows from the modified fact and $(*')$ that $\limsup(\bar F(s+1)-\bar F(s))\leqslant \inf_k \lim_{t\to\infty}(\bar r_k(t)-t)\leqslant 0$, which gives the result.

4. Blum-Hanson and the duality mapping

In this section, we give several sufficient conditions for a Banach space X to have extremal asymptotic smoothness at infinity (Definition 2.3). In particular, we prove Corollary 2.5 and some related results where the smoothness of the norm is involved.

4.1. **Definitions.** Let us recall some standard definitions and notation.

For any $y\in X\backslash\{0\}$, we denote by $J(y)$ the set of all norming functionals for y,

$$J(y)=\{\phi\in X^*;\ \|\phi\|=1 \text{ and } \langle\phi,y\rangle=\|y\|\}\,.$$

The Banach space X is said to be *Gâteaux smooth* if the norm of X is Gâteaux differentiable at each point of the unit sphere of X. By the classical Šmulyan's criterion (see [16]), this holds if and only if the duality mapping is single-valued, i.e. $J(y)$ is a single point (also denoted by $J(y)$) for every $y\in S_X$. In this case, we have

$$(5) \qquad \|y+\varepsilon h\| = 1+\varepsilon\,\langle J(y),h\rangle + o(\varepsilon) \ \text{ as } \varepsilon\to 0\,,$$

for every fixed $y\in S_X$ and $h\neq 0$.

The space X is *uniformly* Gâteaux smooth if its norm is uniformly Gâteaux differentiable on the unit sphere, i.e. the duality mapping is single-valued and the "little o" in (5) is uniform with respect to $y\in S_X$, for every fixed $h\neq 0$. This a much weaker property than uniform *Fréchet* smoothnes: for example, uniformly Fréchet smooth Banach spaces are super-reflexive, but any separable Banach space has a uniformly Gâteaux smooth renorming (see [16]).

Finally, recall that X is said to be an *Asplund space* if every separable subspace of X has separable dual (this is the more convenient definition as far as the present paper is concerned). For example, X is Asplund as soon as it admits a Fréchet smooth renorming, and the converse is true if X is separable (see [16]).

4.2. **Vanishing duality mapping.** The next result says essentially that the definition of extremal asymptotic smoothness at infinity can be rephrased in terms of the duality mapping of X.

For convenience, we introduce the following ad hoc terminology. We shall say that a set-valued map $\Theta : A \to 2^{X^*}$ defined on a subset A of X is *vanishing along weakly null nets in* A at some point $x \in X$ if, whenever (z_α) is a weakly null net in A and $\phi_\alpha \in \Theta(z_\alpha)$, it follows that $\langle \phi_\alpha, x\rangle \to 0$. Vanishing along weakly null *sequences* is defined in the same way.

Recall also the notation of Theorem 2.1: given a convex cone $\mathbf{C} \subset X$, we say that condition $(*)$ holds for some $x \in X$ if

$$\lim_{t\to\infty} (r_{\mathbf{C}}(t,x) - t) \leqslant 0\,.$$

Proposition 4.1. *Let* $\mathbf{C}$ *be a convex cone in* X*. If the duality map* J *is vanishing along weakly null nets in* $S_X \cap \mathbf{C}$ *at some point* $x \in \mathbf{C}$ *then* $(*)$ *holds for* x*, and hence any* $\mathbf{C}$*-positive contraction on* X *satisfies the Blum-Hanson dichotomy at* x*. If either* X *is uniformly Gâteaux smooth or an Asplund space, it is enough to assume that* J *is vanishing along weakly null sequences.*

Proof. Towards a contradiction, assume that $(*)$ does *not* hold for x. Then one can find a sequence (t^k) tending to ∞ and, for each $k \in \mathbb{N}$, a weakly null sequence $(y_n^k)_{n\in\mathbb{N}} \subset S_X \cap \mathbf{C}$ such that

$$\lim_{n\to\infty} \|x + t^k y_n^k\| - t^k > c$$

for all k and some $c > 0$.

Dividing by t^k and putting $\varepsilon_k := 1/t^k$, we get $\lim_{n\to\infty} \|\varepsilon_k x + y_n^k\| - 1 > c\,\varepsilon_k$. It follows that one can find a weakly null net $(y_\alpha)_{\alpha\in\mathcal{A}} \subset S_X \cap \mathbf{C}$ and a net $(\varepsilon_\alpha)_{\alpha\in\mathcal{A}} \subset (0,\infty)$ tending to 0 such that $\|\varepsilon_\alpha x + y_\alpha\| - 1 > c\,\varepsilon_\alpha$ for every $\alpha \in \mathcal{A}$. (For example, one may proceed as follows. Let $\mathcal{A}$ be the set of all pairs (k,V) where $k \in \mathbb{N}$ and V is a weak neighbourhood of 0 in X, with the product ordering, i.e. $(k,V) \preceq (k',V')$ iff $k \leqslant k'$ and $V \supseteq V'$. For any $\alpha = (k,V) \in \mathcal{A}$, put $\varepsilon_\alpha := \varepsilon_k$, and $y_\alpha := y_n^k$, where n is the smallest integer such that $y_n^k \in V$ and $\|\varepsilon_k x + y_n^k\| - 1 > c\,\varepsilon_k$).

Put $z_\alpha := \frac{\varepsilon_\alpha x + y_\alpha}{\|\varepsilon_\alpha x + y_\alpha\|}\cdot$ Then $z_\alpha \in S_X \cap \mathbf{C}$, and the net (z_α) is weakly null because $\varepsilon_\alpha \to 0$; hence $\langle \phi_\alpha, x\rangle \to 0$ for any choice of $\phi_\alpha \in J(z_\alpha)$. Now, the map $\Phi(\varepsilon) = \|\varepsilon x + y_\alpha\|$ is convex and its right derivative is given by $\Phi'_d(\varepsilon) = \varepsilon\,\langle \phi(\varepsilon), x\rangle$, where $\phi(\varepsilon)$ is a norming functional for $\varepsilon x + y_\alpha$. The functional $\phi(\varepsilon)$ is of course also norming for $z_\alpha := \frac{\varepsilon x + y_\alpha}{\|\varepsilon x + y_\alpha\|}\cdot$ Hence, taking $\varepsilon = \varepsilon_\alpha$ we get $\phi_\alpha \in J(z_\alpha)$ such that

$$\|\varepsilon_\alpha x + y_\alpha\| - 1 = \Phi(\varepsilon_\alpha) - \Phi(0) \leqslant \varepsilon_\alpha\,\langle \phi_\alpha, x\rangle\,.$$

Thus, we see that $\|\varepsilon_\alpha x + y_\alpha\| - 1 = o(\varepsilon_\alpha)$, a contradiction since $\|\varepsilon_\alpha x + y_\alpha\| - 1 > c\,\varepsilon_\alpha$ for every $\alpha \in \mathcal{A}$.

If X is Asplund then the weak topology of any separable subspace of X is metrizable on bounded sets. Since in the above proof everything takes place in the separable subspace $\overline{\rm span}\left(\{x\} \cup \{y_n^k;\ n,k \in \mathbb{N}\}\right)$, it follows that one can replace nets by sequences in this case.

Finally, assume that X is uniformly Gâteaux smooth and, without loss of generality, that $WN(S_X \cap \mathbf{C}) \neq \varnothing$. If $t \to \infty$ then, by uniform smoothness, we have for any sequence $(y_n) \subset S_X$:

$$\begin{aligned} \|x + ty_n\| &= \frac{\|t^{-1}x + y_n\|}{t^{-1}} \\ &= \frac{1 + t^{-1}\langle J(y_n), x\rangle + o(t^{-1})}{t^{-1}} \\ &= t + \langle J(y_n), x\rangle + o(1)\,, \end{aligned}$$

where the "little o" is uniform with respect to (y_n). If J is vanishing at x along weakly null sequences in S_X, it follows immediately that

$$r_{\mathbf{C}}(t,x) = t + o(1)\,.$$

□

Remark 1. Assume that $\mathrm{WN}(S_X \cap \mathbf{C}) \neq \varnothing$ and that X is uniformly Gâteaux smooth. An examination of the above proof reveals that for a given $x \in \mathbf{C}$, the condition $\lim_{t\to\infty}(r_{\mathbf{C}}(t,x) - t) = 0$ is actually equivalent to the requirement that J should be vanishing at x along weakly null sequences in $S_X \cap \mathbf{C}$.

Remark 2. It follows from the proof that if X is uniformly *Fréchet* smooth and J is vanishing along weakly null sequences in $S_X \cap \mathbf{C}$ at all $x \in \mathbf{C}$, then $(*)$ holds uniformly with respect to $x \in S_X \cap \mathbf{C}$.

Remark 3. The proof of Proposition 4.1 is similar to that of [14, Theorem 5].

Remark 4. The space X is said to *have a weakly continuous duality mapping* if X is Gâteaux smooth and there exists a continuous increasing function $\mu : \mathbb{R}^+ \to \mathbb{R}^+$ with $\mu(0) = 0$ such that the map $J_\mu(y) = \mu(\|y\|)J(y)$ is w-w^* continuous on X (setting $J_\mu(0) = 0$). This property has proved to be quite important in fixed point theory since the classical work of F. Browder [10]. Obviously, it implies vanishing of the duality mapping along weakly null nets in S_X at all $x \in X$, and hence that $\lim_{t\to\infty}(r_X(t,x) - t) \leqslant 0$ for all x. In fact, one can prove directly that $\lim_{t\to\infty}(r_X(t,x) - t) = 0$ uniformly on S_X, because the modulus r_X can be computed explicitely. Indeed it is shown in [35] that if X has a weakly continuous duality mapping with "gauge" function μ and if we put $M(t) = \int_0^t \mu(s)\,ds$ then

$$\limsup \|x + tx_n\| = M^{-1}\Big(M(\|x\|) + M(t)\Big) \tag{6}$$

for all $x \in X$ and every weakly null sequence $(x_n) \subset S_X$; in particular, $r_X(t,x)$ is the right-hand side of (6). Now, it is not hard to see that $\mu(t^{-1}) = c\mu(t)^{-1}$, where $c = \mu(1)^2$ (see below). In particular, $\mu(t) \to \infty$ as $t \to \infty$ and hence (6) does imply that $r_X(t,x) - t \to 0$ uniformly on S_X. (To show that $\mu(t^{-1}) = c\mu(t)^{-1}$, note that for any $t > 1$ one can find a net $(y_\alpha) \subset tS_X$ converging weakly to some $y \in S_X$. Then $\mu(t)J(y_\alpha) \xrightarrow{w^*} \mu(1)J(y)$; but since $\frac{y_\alpha}{t} \xrightarrow{w} \frac{y}{t}$ and $J(\frac{z}{t}) = J(z)$ for any $z \in X\backslash\{0\}$, we also know that $\mu(1)J(y_\alpha) \xrightarrow{w^*} \mu(t^{-1})J(y)$, and the result follows).

4.3. **An appproximation-like property.** We now use Proposition 4.1 to isolate one reasonably general class of Banach spaces having extremal asymptotic smoothness at infinity.

To formulate the result, we introduce an "approximation-like" property for which we have not tried to find a name to avoid pedantry (see however the remark at the end of this sub-section). We shall say that a Banach space Z has *property* (?) if the following holds: for any $z \in Z$, one can find a sequence of compact operators $(\pi_K) \subset \mathcal{L}(Z)$ such that $\pi_K z \to z$ and $\limsup_K \|I - \pi_K\| \leqslant 1$. (Equivalently, one may require that $\|I - \pi_K\| \leqslant 1$ for all K). One example to keep in mind is the following: property (?) is satisfied if Z has a *reverse monotone* Schauder basis, i.e. a basis $(f_k)_{k\in\mathbb{N}}$ such that $\|I - \pi_K\| = 1$ for all K, where π_K is the canonical projection onto $\operatorname{span}\{f_1, \dots, f_K\}$. (For example, any 1-unconditional basis is reverse monotone). More generally, it is enough to assume that Z has a reverse monotone finite-dimensional Schauder decomposition.

Proposition 4.2. *If the Banach space X is uniformly Gâteaux smooth and embeds isometrically into a Banach space with property* (?)*, then X has extremal asymptotic smoothness at infinity (and hence the Blum-Hanson property).*

Proof. By Proposition 4.1 (with $\mathbf{C} = X$) it is enough to show that the duality mapping of X is vanishing along weakly null sequences in S_X at all $x \in S_X$. So let us fix a weakly null sequence $(y_n) \subset S_X$.

Let Z be a Banach space with property (?) such that X embeds isometrically into Z. Considering X as a subspace of Z, we still denote by $J(y_n)$ any Hahn-Banach extension of $J(y_n)$, $n \in \mathbb{N}$. Also, for any $\Phi \in Z^*$ we denote by $\|\Phi\|_{X^*}$ the norm of Φ viewed as a linear functional on X; that is $\|\Phi\|_{X^*} = \|\Phi_{|X}\|$. Finally, if (Φ_K) is a sequence in Z^*, we write "$\Phi_K \xrightarrow{w^*} 0$ in X^*" if the sequence $((\Phi_K)_{|X}) \subset X^*$ is w^*-null.

Let $x \in X$ be arbitrary, and let (π_K) be a sequence of compact operators on Z such that $\pi_K x \to x$ and $\limsup_K \|I - \pi_K\| \leqslant 1$. Since the sequence (y_n) is weakly null, we know that $\|\pi_K y_n\| \to 0$ for every fixed $K \in \mathbb{N}$. Hence, we can find a subsequence (y_{n_K}) of (y_n) such that $\|\pi_K y_{n_K}\| \to 0$ as $K \to \infty$. Then

$$\langle (I - \pi_K^*)J(y_{n_K}), y_{n_K}\rangle = 1 - \langle J(y_{n_K}), \pi_K y_{n_K}\rangle \xrightarrow{K\to\infty} 1.$$

Since $\limsup \|(I - \pi_K^*)J(y_{n_K})\|_{X^*} \leqslant \limsup \|(I - \pi_K^*)J(y_{n_K})\| \leqslant 1$ and, moreover, $y_{n_K} \in S_X$ and $\langle J(y_{n_K}), y_{n_K}\rangle = 1$, it follows that $\lim \|(I - \pi_K^*)J(y_{n_K})\|_{X^*} = 1 = \lim \|J(y_{n_K})\|_{X^*}$ and $\lim \|(I - \pi_K^*)J(y_{n_K}) + J(y_{n_K})\|_{X^*} = 2$. By the uniform Gâteaux smoothness of X, this implies (see [16, Theorem 6.7 and Proposition 6.2]) that

$$(I - \pi_K^*)J(y_{n_K}) - J(y_{n_K}) \xrightarrow{w^*} 0 \quad \text{in } X^*,$$

i.e. $\pi_K^* J(y_{n_K}) \xrightarrow{w^*} 0$ in X^*. In particular, $\langle J(y_{n_K}), \pi_K x\rangle \to 0$ and hence $\langle J(y_{n_K}), x\rangle \to 0$ since $\pi_K x \to x$.

Thus, we have shown that for every $x \in X$, one can find a subsequence (y_{n_K}) of (y_n) such that $\langle J(y_{n_K}), x\rangle \to 0$. Since this can be done starting with any subsequence of (y_n), this shows that $\langle J(y_n), x\rangle \to 0$ for all $x \in X$, as required.

□

Corollary 4.3. *If X is uniformly Gâteaux smooth and embeds isometrically into a Banach space with a reverse monotone (e.g.* 1*-unconditional) FDD, then X has the Blum-Hanson property.*

Remark. There are lots of well identified approximation properties in Banach space theory; see e.g. [11] or [12]. The one that seems closest to (?) is the so-called *Reverse Monotone Compact Approximation Property.* A Banach space Z has (RMCAP) if one can find a sequence of compact operators $(\pi_K) \subset \mathcal{L}(Z)$ such that $\pi_K z \to z$ for all $z \in Z$ and $\|I - \pi_K\| \to 1$. This is formally a much stronger property than (?), because in (?) the π_K's are allowed to depend on z. In view of the existing terminology property (?) could consistently be called the "Reverse Monotone Compact Point Approximation Property"; which is not a very exciting name. Incidentally, it is well known that L_p does not have (RMCAP) if $p \neq 2$. (A much stronger result is proved in [41]).

4.4. **Almost isometric embeddings.** Recall that a Banach space X is said to embed *almost isometrically* into another Banach space Z if it can be $(1+\varepsilon)$-embedded into Z for any $\varepsilon > 0$, i.e. one can find an operator $j : X \to Z$ such that

$$(1+\varepsilon)^{-1}\|x\| \leqslant \|jx\| \leqslant (1+\varepsilon)\|x\|$$

for all $x \in X$. Almost isometric embeddings are relevant in our matters because of the following remark: *extremal uniform asymptotic smoothness at infinity is preserved under almost isometric embeddings*; that is, X has extremal uniform asymptotic smoothness at infinity as soon as it embeds almost isometrically into a Banach space with this property. Indeed, it is not hard to check that if X embeds almost isometrically into Z then $r_X(t) \leqslant r_Z(t)$ for all $t \in \mathbb{R}^+$. (Recall that $r_X(t) = \sup_{x \in S_X} r_X(t,x)$). The corresponding fact for the modulus of asymptotic smoothness $\bar{\rho}_X$ is proved e.g. in [15, Lemma 2.1]

The following result is a "Fréchet" analogue of Proposition 4.2.

Proposition 4.4. *If the Banach space X embeds almost isometrically into a uniformly Fréchet smooth Banach space with property* (?)*, then X has extremal extremal uniform asymptotic smoothness at infinity (and hence the Blum-Hanson property).*

Proof. By the proof of Proposition 4.2 and Remark 2 after the proof of Proposition 4.1, any uniformly Fréchet smooth space Z with property (?) has extremal uniform asymptotic smoothness at infinity. Since the latter is preserved under almost isometric embeddings, the result follows.

□

In view of this result and of Proposition 4.2, it is natural to ask whether a uniformly *Gâteaux* smooth space X has extremal (not uniform) asymptotic smoothness at infinity as soon as it embeds almost isometrically into a Banach space with property (?). We now show that this does hold true (and in fact without any smoothness assumption) if (?) is replaced with a stronger property.

Given a function $c:\mathbb{R}^+\times\mathbb{R}^+\to\mathbb{R}^+$ such that $c(s,t)\geqslant s$ for any (s,t), let us say that a Banach space Z has *property* $(?)_c$ if the following holds: for any $z\in B_Z$, one can find a sequence of compact operators $(\pi_K)\subset\mathcal{L}(Z)$ such that $\pi_K z\to z$ and

$$\forall K\ \forall\Phi\in Z^*\ :\ c\left(\|(I-\pi_K^*)\Phi\|,|\pi_K^*\Phi(z)|\right)\leqslant\|\Phi\|\,. \tag{7}$$

So one requires $\|I-\pi_K\|\leqslant 1$ for all K, with a quantitative estimate provided by the function c. For example, ℓ_p, $1\leqslant p<\infty$ has property $(?)_c$ with $c(s,t)=(s^q+t^q)^{1/q}$.

Proposition 4.5. *Assume that there exists a continuous function $c:\mathbb{R}^+\times\mathbb{R}^+\to\mathbb{R}^+$ satisfying $c(s',t)\geqslant c(s,t)>s$ whenever $s'\geqslant s$ and $t>0$, such that for any $\varepsilon>0$, X can be $(1+\varepsilon)$-embedded into a Banach space with property $(?)_c$. Then X has extremal asymptotic smoothness at infinity (and hence the Blum-Hanson property).*

Proof. We show that the duality mapping of X is vanishing along weakly null nets in S_X at all $x\in S_X$. So let us fix a weakly null net $(y_\alpha)\subset S_X$, linear functionals $\phi_\alpha\in J(y_\alpha)$, and a point $x\in S_X$. Let also $\varepsilon\in(0,1]$ be arbitrary.

By assumption, there exists a Banach space space with property $(?)_c$ such that X can be $(1+\varepsilon)$-embedded into Z. Without loss of generality, we may assume that $X\subset Z$ as a set and $(1+\varepsilon)^{-1}\|\xi\|_Z\leqslant\|\xi\|_X\leqslant(1+\varepsilon)\|\xi\|_Z$ for every $\xi\in X$. Let us choose sequence of compact operators $\pi_K:Z\to Z$, such that $\pi_K x\to x$ and (7) holds with $z=x$.

Considering each ϕ_α as linear functional on $(X,\|\cdot\|_Z)$, choose any Hahn-Banach extension $\Phi_\alpha\in Z^*$. Then $\Phi_\alpha=\phi_\alpha$ on X (by definition) and $\|\Phi_\alpha\|\leqslant 1+\varepsilon\leqslant 2$ because $\|\phi_\alpha\|_{X^*}=1$. We claim that

$$\liminf_\alpha\|(I-\pi_K^*)\Phi_\alpha\|\geqslant(1+\varepsilon)^{-1} \tag{8}$$

for every $K\in\mathbb{N}$. Indeed, since $\|y_\alpha\|_Z\leqslant(1+\varepsilon)$ we have

$$\begin{aligned}\|(I-\pi_K^*)\Phi_\alpha\| &\geqslant (1+\varepsilon)^{-1}|\langle\Phi_\alpha,(I-\pi_K)y_\alpha\rangle|\\ &= (1+\varepsilon)^{-1}|1-\langle\Phi_\alpha,\pi_K y_\alpha\rangle|\,,\end{aligned}$$

because $\langle\Phi_\alpha,y_\alpha\rangle=\langle\phi_\alpha,y_\alpha\rangle=1$. Since $\|\pi_K y_\alpha\|_Z\to 0$ (because (y_α) is a bounded weakly null net and π_K is compact) and (Φ_α) is bounded, this gives (8).

By (7) and since $\|\Phi_\alpha\|\leqslant 1+\varepsilon$ for every α, it follows that

$$\limsup_\alpha c\left((1+\varepsilon)^{-1},|\pi_K^*\Phi_\alpha(x)|\right)\leqslant 1+\varepsilon$$

for every $K\in\mathbb{N}$. Since $c(s,t)>s$ for $t>0$ and since $\pi_K^*\Phi_\alpha(x)$ is uniformly bounded with respect to α and K, this implies that

$$\limsup_\alpha|\pi_K^*\Phi_\alpha(x)|\leqslant\delta(\varepsilon,x)\,,$$

where $\delta(\varepsilon,x)$ does not depend on $K\in\mathbb{N}$ and $\delta(\varepsilon,x)\to 0$ as $\varepsilon\to 0$.

Now, let us choose $K\in\mathbb{N}$ such that $\|(I-\pi_K)x\|_Z<\varepsilon$. Writing $\langle\phi_\alpha,x\rangle=\langle\Phi_\alpha,\pi_K x\rangle+\langle\Phi_\alpha,(I-\pi_K)x\rangle$, we get $|\langle\phi_\alpha,x\rangle|\leqslant|\langle\pi_K^*\Phi_\alpha,x\rangle|+2\varepsilon$ for all $n\in\mathbb{N}$, and hence

$$\limsup_\alpha|\langle\phi_\alpha,x\rangle|\leqslant\delta(\varepsilon,x)+2\varepsilon\,.$$

Since $\varepsilon \in (0,1]$ is arbitrary, we conclude that $\langle \phi_\alpha, x\rangle \to 0$ for every $x \in X$, as required.

□

4.5. **WORTH.** To conclude this section, we prove one more result of the type "smoothness plus something implies Blum-Hanson".

The "something" involved here is the so-called *Weak Orthogonality Property*, usually abbreviated as *WORTH.* A Banach space has WORTH if

$$\lim_{n\to\infty}\Big(\|x+y_n\| - \|x-y_n\|\Big) = 0$$

for every weakly null sequence $(y_n) \subset X$. This property has been considered a number of times in fixed point theory (see e.g. [45]). If weakly null sequences are replaced by weakly null bounded nets, one obtains the so-called *property* (au), which have been thoroughly studied recently by S. R. Cowell and N. Kalton [13], together with its dual version (au^*) (the latter was introduced in [30] under the name "(wM^*)"). Though perhaps innocent looking at first sight, these properties are in fact very strong. For example, it is shown in [13] that a separable reflexive Banach space has WORTH if and only if it can be $(1+\varepsilon)$-embedded into a Banach space with a shrinking 1-unconditional basis, for any $\varepsilon > 0$.

Proposition 4.6. *If X is uniformly Gâteaux smooth with property* WORTH*, then it has extremal asymptotic smothness at infinity (and hence the Blum-Hanson property).*

Proof. We may assume that X does not have the Schur property. By WORTH, the modulus r_X can be re-written as follows :

$$r_X(t,x) = \sup_{(y_n)\in \mathrm{WN}(S_X)} \limsup_{n\to\infty} \frac{\|x+ty_n\| + \|x-ty_n\|}{2}\cdot$$

Moreover, by uniform Gâteaux smoothness we have (as $t\to\infty$)

$$\begin{aligned}\|x+ty_n\| + \|x-ty_n\| &= \frac{\|t^{-1}x+y_n\| + \|t^{-1}x-y_n\|}{t^{-1}}\\ &= \frac{(1+t^{-1}\langle J(y_n),x\rangle + o(t^{-1})) + (1-t^{-1}\langle J(y_n),x\rangle + o(t^{-1}))}{t^{-1}}\\ &= 2t+o(1)\end{aligned}$$

where the "little o" is uniform with respect to $(y_n)\in \mathrm{WN}(S_X)$. Hence, we get $\lim_{t\to\infty}(r_X(t,x)-t) = 0$ for every $x\in X$. □

Remark. A strong form of WORTH is the important *property* (M) introduced by N. Kalton in [27]. A Banach space X has property (M) if

$$\limsup \|u+x_n\| = \limsup \|v+x_n\|$$

whenever $u,v\in X$ satisfy $\|u\| = \|v\|$ and (x_n) is a weakly null sequence in X. Obviously, property (M) is weaker than (m_p), for any $p\in(1,\infty]$. Since we saw in section 2 that (m_p) implies Blum-Hanson, it makes sense to ask whether (M) implies the Blum-Hanson property. By [28, Corollary 4.5], this is true for subspaces of L_p,

$1 < p < \infty$ and for subspaces of L_1 not containing ℓ_1, because any such space has property (m_r) for some $r > 1$. More generally, this is true for separable Banach spaces not containing ℓ_1 which are *stable* in the sense of [33]; see the proof of [27, Theorem 3.10].

5. Orlicz spaces

In this section, we apply the previous general results to the specific setting of Orlicz spaces. Not unexpectedly, the situation is similar to that of ℓ_p and L_p spaces (as far as the Blum-Hanson property is concerned).

Let $\theta : [0,\infty) \to [0,\infty)$ be an *Orlicz N-function*, i.e. an increasing convex function such that $\lim_{t\to\infty} \theta(t)/t = \infty$ and $\lim_{t\to 0} \theta(t)/t = 0$. Given any measure space (Ω,μ), the *Orlicz space* $L_\theta(\Omega,\mu)$ is the space of all (equivalence classes of) measurable functions $f : \Omega \to \mathbb{R}$ such that $\int_\Omega \theta(c\,|f|)\, d\mu < \infty$ for some $c > 0$. We equip $L_\theta(\Omega,\mu)$ with one of its two "natural" norms, the so-called *Luxemburg norm*:

$$\|f\| = \inf\left\{\lambda > 0;\ \int_\Omega \theta\left(\frac{|f|}{\lambda}\right) d\mu < \infty\right\}.$$

When $\Omega = \mathbb{Z}^+$ equipped with the counting measure, we denote the Orlicz space by ℓ_θ; and when $\Omega = (0,1)$ with Lebesgue measure, we simply write L_θ.

The *small* Orlicz space $M_\theta(\Omega,\mu)$ (also called the *Morse-Transue* space) is the subspace of $L_\theta(\Omega,\mu)$ consisting of all f such that $\int_\Omega \theta(c\,|f|)\, d\mu < \infty$ for *every* $c > 0$. We write m_θ when $\Omega = \mathbb{N}$ and M_θ when $\Omega = (0,1)$.

It is well known (see e.g. [43] and/or [32]) that $m_\theta = \ell_\theta$ if and only if θ satisfies the so-called Δ_2 *condition at* 0, i.e. $\limsup_{t\to 0} \theta(2t)/\theta(t) < \infty$, and that $M_\theta = L_\theta$ iff θ satisfies the Δ_2 condition *at* ∞, i.e. $\limsup_{t\to\infty} \theta(2t)/\theta(t) < \infty$. By the duality theory of Orlicz spaces, it follows that ℓ_θ is reflexive iff both θ and the conjugate Orlicz function θ^* satisfy the Δ_2 condition at 0, and that L_θ is reflexive iff θ and θ^* satisfy the Δ_2 condition at ∞.

We quote the following more "specialized" results:

- m_θ is asymptotically uniformly smooth if and only if θ^* satisfies the Δ_2 condition at 0 ([15]);
- L_θ is Gâteaux smooth iff θ is $\mathcal{C}^1$ and satisfies the Δ_2 condition at ∞ (see [44, Theorem X.4.3]);

We can now state our results about the Blum-Hanson property for Orlicz spaces. For the sake of "immediate applicability", we formulate the assumptions directly in terms of the Orlicz functions θ and θ^*; but this should of course be translated into properties of the Orlicz spaces (using the just mentioned results).

Proposition 5.1. *Let θ be an Orlicz N-function.*

1. *If θ^* satisfies the Δ_2 condition at 0, then every subspace of m_θ has the Blum-Hanson property.*
2. *If θ is $\mathcal{C}^1$-smooth and satisfies the Δ_2 condition at ∞ then, any positive contraction on L_θ satisfies the Blum-Hanson dichotomy at all $f \in L_\theta^+$ (the positive cone of L_θ).*

Proof. (1) By [15], the Δ_2 condition for θ^* means that $X = m_\theta$ is asymptotically uniformly smooth. Moreover, it is also shown in [15] that in this case the modulus of asymptotic smoothness of X behaves very nicely: one can find some constant $\alpha > 1$ such that

$$\bar{\rho}_X(t,x) \leqslant (1+t^\alpha)^{1/\alpha} - 1$$

for every $x \in S_X$ and all $t \geqslant 0$. (This is stated only for $t \in [0,1]$ in [15], but the proof works for any $t \geqslant 0$). This shows that m_θ has extremal (uniform) asymptotic smoothness at infinity, hence (1).

(2) Here, the assumption mean that L_θ is Gâteaux smooth. It is enough to show that the duality mapping $J = S_{L_\theta} \to S_{L_\theta^*}$ is vanishing along weakly null nets in $S_{L_\theta} \cap L_\theta^+$ at all $f \in S_{L_\theta} \cap L_\theta^+$.

We shall use the following known fact (see [43, Theorem VII.2.3]): if $f \in S_{L_\theta} \cap L_\theta^+$ then $J(f)$ is given by the formula

$$\langle J(f), g\rangle = \frac{1}{\int f\theta'(f)} \int g\,\theta'(f)\,. \tag{9}$$

This formula makes sense thanks to the Δ_2 condition. Indeed, if $h \in L_\theta$, then $\theta'(|h|) \in L_{\theta^*}$ by Δ_2 (see [43, proof of proposition III.4.8]) and hence, by Hölder's inequality (see e.g [43, Proposition III.3.1]), $k\,\theta'(|h|)$ is integrable for any $k, h \in L_\theta$.

We also need the following inequality :

$$\forall f \in L_\theta^+ \cap S_{L_\theta} \;:\; \int f\theta'(f) \geqslant 1\,. \tag{10}$$

To prove this, note that $\int \theta(|f|) = 1$ for every $f \in S_{L_\theta}$, by Δ_2 (see e.g [43, Proposition III.4.6]). Since θ' is non-decreasing and $\theta(0) = 0$, it follows that $t\theta'(t) \geqslant \theta(t)$ for all $t \geqslant 0$ and hence $\int f\theta'(f) \geqslant \int \theta(f)$ for any $f \in L_\theta^+$.

Now, let us fix a weakly null net $(f_\alpha) \subset L_\theta^+ \cap S_{L_\theta}$ and a function $g \in L_\theta^+$. We show that $\langle J(f_\alpha), g\rangle \to 0$.

Let $\varepsilon > 0$ be arbitrary. Since the function $g\,\theta'(g)$ is integrable (see the remark just after (9)), we may first choose $\eta > 0$ so that $\int g\,\theta'(\eta g) < \varepsilon$. Then, proceed as in the proof of Bellow's inequality (3) for L_p:

$$\begin{aligned}\int g\,\theta'(f_\alpha) &= \int_{f_\alpha < \eta g} + \int_{g < \eta f_\alpha} + \int_{\eta g \leqslant f_\alpha \leqslant \eta^{-1} g} \\ &\leqslant \int g\,\theta'(\eta g) + \eta \int f_\alpha\,\theta'(f_\alpha) + \eta^{-1} \int f_\alpha\,\theta'(\eta^{-1} g)\,.\end{aligned}$$

Using (9), (10) and assuming (as we may) that $\eta < \varepsilon$, we get

$$\langle J(f_\alpha), g\rangle \leqslant 2\varepsilon + \eta^{-1} \int f_\alpha\,\theta'(\eta^{-1} g)\,,$$

for every α. Since $\theta'(\eta^{-1} g) \in L_{\theta^*} = (L_\theta)^*$ by Δ_2 and since (f_α) is weakly null, it follows that $\limsup \langle J(f_\alpha), g\rangle \leqslant 2\varepsilon$, which concludes the proof.

□

Corollary 5.2. *Any subspace a reflexive Orlicz sequence space has the Blum-Hanson property.*

Remark. As mentioned in the introduction, the Blum-Hanson property for Orlicz function spaces endowed with the *Orlicz norm* has been studied in [38]. It is shown there (Theorem 7.7) that if L_θ is uniformly Fréchet smooth when endowed with the Orlicz norm, then it has the Blum-Hanson property with repect to positive contractions. The proof also makes use of a Bellow-like inequality (Lemma 7.2). Exactly as in the L_p case, it could be shortened by applying Proposition 4.1.

6. Very classical spaces

6.1. **Hilbert spaces.** We include here a superficially new proof of the Blum-Hanson property for complex Hilbert spaces. This is merely a rewriting of the one that can be found in [29]. However, we find it worth mentioning for two reasons: it is extremely simple (though not elementary), and it suggests than one could possibly prove the Blum-Hanson property for other spaces by using "functional calculus" arguments.

Let T be a contraction operator on a complex Hilbert space H, and assume that $T^n x \xrightarrow{w} 0$ for some $x \in H$.

Let σ_x be the *spectral measure* for T associated with x, i.e. the positive (finite) measure on $\mathbb{T}$ whose Fourier coefficients are given by

$$\widehat{\sigma_x}(n) = \begin{cases} \langle T^n x, x\rangle & n \geqslant 0 \\ \langle T^{*|n|} x, x\rangle & n \leqslant 0 \end{cases}$$

The assumption $T^n x \xrightarrow{w} 0$ means that σ_x is a Rajchman measure: $\widehat{\sigma_x}(n) \to 0$ as $|n| \to \infty$. Moreover, the following von Neumann–type inequality holds for every polynomial P (see e.g. [46, Proposition 1.1.2]):

$$\|P(T)\, x\|^2 \leqslant \int_{\mathbb{T}} |P|^2 \, d\sigma_x \,. \tag{11}$$

Now, let $(n_i)_{i\geqslant 1}$ be an increasing sequence of integers. Applying (11) with $P(z) = z^{n_1} + \cdots + z^{n_K}$, we get

$$\left\| \sum_{i=1}^{K} T^{n_i} x \right\|^2 \leqslant \sum_{i,j=1}^{K} \widehat{\sigma_x}(n_i - n_j) \tag{12}$$

for any $K \in \mathbb{N}$. Moreover, since $\widehat{\sigma_x}(n) \to 0$ as $|n| \to \infty$, it is a simple exercise to show that

$$\alpha_K = \frac{1}{K^2} \sum_{i,j=1}^{K} |\widehat{\sigma_x}(n_i - n_j)| \to 0$$

as $K \to \infty$. (Indeed, we have $\sharp \left\{(i,j) \in [\![1,K]\!]^2;\ |n_i - n_j| \leqslant N\right\} \leqslant 2KN$, so that $\limsup \alpha_K \leqslant \sup\{|\widehat{\sigma_x}(n)|;\ |n| > N\}$ for any $N \in \mathbb{N}$). By (12), it follows that the sequence $(T^n x)$ is Blum-Hanson.

6.2. $\mathcal{C}(K)$ **spaces.** As mentioned in the introduction, it is shown in [1] that the space $\mathcal{C}(\mathbb{T}^2)$ fails the conditional Blum-Hanson property. From this and known results about $\mathcal{C}(K)$ spaces, one can easily deduce

Proposition 6.1. *If K is an uncountable compact metrizable space, then $\mathcal{C}(K)$ fails the conditional Blum-Hanson property.*

Proof. This relies on the following trivial observation:

Fact. Let X be a Banach space, and let Z be a 1-complemented subspace of X. If Z fails the (conditional) Blum-Hanson property then so does X.

Proof of Fact. Let $\pi : X \to Z$ be a norm 1 projection from X onto Z, and let $j : Z \to X$ be the canonical embedding. If $T : Z \to Z$ is a contraction on Z, then $\widetilde{T} := jT\pi$ is a contraction on X extending T; and since $\widetilde{T}^n = jT^n\pi$ for all n, it has weakly convergent orbits as soon as T does. So the result is clear.

□

Now, we use the following facts, which are the key ingredients in the proof of Miljutin's theorem on the isomorphism of all $\mathcal{C}(K)$ for uncountable and metrizable K (see [4, p. 95]). Let $\Delta = \{0,1\}^{\mathbb{N}}$ be the usual Cantor space. Then, for every compact metrizable L the space $\mathcal{C}(L)$ is isometric to a 1-complemented subspace of $\mathcal{C}(\Delta)$; and if L is uncountable then the space $\mathcal{C}(\Delta)$ is isometric to a 1-complemented subspace of $\mathcal{C}(L)$. Applying this first with $L = \mathbb{T}^2$ we deduce that $\mathcal{C}(\Delta)$ fails the conditional Blum-Hanson property, and taking then $L = K$ we conclude that so does $\mathcal{C}(K)$.

□

Corollary 6.2. *The disk algebra $A(\mathbb{D})$ does not have the conditional Blum-Hanson property.*

Proof. Recall that the disk algebra is the space of all complex-valued functions which are continuous on the closed unit disk $\overline{\mathbb{D}} \subset \mathbb{C}$ and holomorphic on $\mathbb{D}$, endowed with the sup norm. Let K be an uncountable compact subset $\mathbb{T}$ with Lebesgue measure 0. By the Rudin-Carleson theorem, any continuous function $f : K \to \mathbb{C}$ can be extended to a function $\widetilde{f} \in A(\mathbb{D})$ with $\|\widetilde{f}\|_\infty = \|f\|_\infty$; and in fact, it was shown by A. Pełczyński that there is an isometric linear extension operator $E = \mathcal{C}(K) \to A(\mathbb{D})$ (see [37]). It follows at once that $A(\mathbb{D})$ has a 1-complemented subspace isometric to $\mathcal{C}(K)$ (namely $E\mathcal{C}(K)$), and hence that $A(\mathbb{D})$ cannot have the conditional Blum-Hanson property. □

Remark. It is quite plausible that no $\mathcal{C}(K)$ space (for infinite K) has the Blum-Hanson property. In any event, if K is an infinite compact (Hausdorff) space then $\mathcal{C}(K)$ does not have extremal asymptotic smoothness at infinity. Indeed, as in any infinite Hausdorff space one can find a countably infinite discrete D in K. Denoting by Ω the closure of D in K, the space $\mathcal{C}(K)$ contains an isometric copy of $\mathcal{C}(\Omega)$; so it is enough to show that $\mathcal{C}(\Omega)$ does not have extremal asymptotic smoothness at infinity. Write $D = \{d_n;\ n \in \mathbb{N}\}$. Since D is discrete, each $\{d_n\}$ is clopen in Ω, so the function $f_n = \mathbf{1}_{\{d_n\}}$ is in $\mathcal{C}(\Omega)$. Obviously, the sequence (f_n) is weakly null in

$X = \mathcal{C}(\Omega)$. Moreover, since $f_n \geqslant 0$ we have $\|\mathbf{1} + tf_n\|_\infty = 1 + t$ for every $t \geqslant 0$; so $r_X(t, \mathbf{1}) = 1 + t$ for all t.

In the case $K = \mathbb{T}^2$, the main result of [1] is in fact much more precise than Proposition 6.1: the space $\mathcal{C}(\mathbb{T}^2)$ fails the conditional Blum-Hanson property with respect to the very special class of *composition operators*, i.e. operators of the form $Tf = f \circ \varphi$. Interestingly enough, this does not hold for $K = [0, 1]$.

Proposition 6.3. *The space $\mathcal{C}([0, 1])$ has the conditional Blum-Hanson property with respect to composition operators.*

Indeed, let T be a composition operator ($Tf = f \circ \varphi$) on $\mathcal{C}([0, 1])$ induced by some continuous map $\varphi : [0, 1] \to [0, 1]$, and assume that T has weakly convergent orbits. This means exactly that the iterates φ^n converge pointwise on $[0, 1]$ to some *continuous* function $\alpha : [0, 1] \to [0, 1]$. Hence, it is enough to prove the following lemma. (This lemma is certainly well known but we couldn't locate a reference. The proof we give is due to D. Malicet, and we thank V. Munnier for explaining it).

Lemma 6.4. *Let $\varphi : [0, 1] \to [0, 1]$ be a continuous map. If $\varphi^n(x) \to \alpha(x)$ pointwise, where $\alpha : [0, 1] \to [0, 1]$ is continuous, then in fact $\varphi^n(x) \to \alpha(x)$* uniformly.

Proof. We note that the set of fixed points of φ is exactly the closed interval $I = \alpha([0, 1])$. If $I = [0, 1]$, there is nothing to prove. Otherwise, consider the space Λ obtained from $[0, 1]$ by identifying all the points of I, with the usual quotient topology. Then Λ is homeomorphic to $[0, 1]$, the map φ induces a continuous map $\widetilde{\varphi} : \Lambda \to \Lambda$ with a single fixed point $\widetilde{\alpha}$, and the iterates $\widetilde{\varphi}^n$ converge pointwise to $\widetilde{\alpha}$ on Λ. If we can show that $\widetilde{\varphi}^n \to \widetilde{\alpha}$ uniformly then we will get the result for φ. Therefore, all we need to do is to prove the following result : *If $\varphi : [0, 1] \to [0, 1]$ is a continuous map with a single fixed point α such that $\varphi^n(x) \to \alpha$ pointwise on $[0, 1]$, then the convergence is uniform.* To do this, the key point is the following

Fact. Let $J = [u, v]$ be a nontrivial compact interval of $\mathbb{R}$. If $\varphi : J \to J$ is continous and $\varphi^n(x) \to \alpha \in J$ pointwise on J, then φ cannot be onto.

Proof of Fact. If $\alpha = u$, then we must have $\varphi(x) < x$ for all $x \in]u, v]$, because $\varphi(x) - x$ has constant sign on $]u, v]$ by the intermediate value theorem ($\alpha = u$ is the only fixed point of φ) and $\varphi(v) \leqslant v$; in particular $\varphi(x) < v$ for all $x \in J$, which gives the result in this case. Likewise if $\alpha = v$. Now, assume that $\alpha \in]u, v[$ and that φ is onto. Then $\varphi(x) - x$ has constant sign on both intervals $[u, \alpha[$ and $]\alpha, v]$, and since $\varphi([u, v]) \subset [u, v]$ the only possible case is the following: $\varphi(x) > x$ on $[u, \alpha[$ and $\varphi(x) < x$ on $]\alpha, v]$. In particular, $\varphi(x) > u$ on $[u, \alpha]$ and $\varphi(x) < v$ on $[\alpha, v]$. Since φ is onto, we then have $v \in \varphi([u, \alpha])$ and $u \in \varphi([\alpha, v])$, whence $[\alpha, v] \subset \varphi([u, \alpha])$ and $[u, \alpha] \subset \varphi([\alpha, v])$. It follows that $[u, \alpha] \subset \varphi^2([u, \alpha])$; but this is a contradiction because φ^2 satisfies the same assumption as φ and hence $\varphi^2(x) > u$ on $[u, \alpha]$. □

Now, let $\varphi : [0, 1] \to [0, 1]$ be a continuous map such that $\varphi^n(x) \to \alpha$ pointwise on $[0, 1]$. Then $J = \bigcap_{n \geqslant 0} \varphi^n([0, 1])$ is compact interval containing α, and it is easily checked that $\varphi(J) = J$. By the above fact, it follows that $\bigcap_{n \geqslant 0} \varphi^n([0, 1]) = \{\alpha\}$; and from this it is not hard to deduce that $\varphi^n(x) \to \alpha$ uniformly.

□

6.3. **The space L_1.** In [2], the proof that $L_1 = L_1(0,1)$ has the conditional Blum-Hanson property proceeds roughly as follows. Using the so-called *linear modulus* associated with a given contraction T on L_1 and assuming that T has weakly convergent orbits, one breaks the underlying measure space into 2 pieces A and B such that T has norm null orbits on $L_1(A)$ and T is an *absolute* contraction on $L_1(B)$, i.e. a contraction on any L_p, $1 \leqslant p \leqslant \infty$. Then the absolutely contractive part is handled using the L_2 case. This seems to be very specific to L_1, and we see no way of using any kind of "smoothness" argument to shorten the proof.

6.4. **The space L_p.** If $L_p = L_p(0,1)$, $1 < p < \infty$ were to have the Blum-Hanson property, this could not be proved by a direct application of Theorem 2.1 with $\mathbf{C} = X = L_p$, except of course for $p = 2$. Indeed, L_p does not have extremal asymptotic smoothness at infinity.

One can see this somewhat indirectly by observing that the duality mapping of L_p is not vanishing along weakly null sequences (see Remark 1 just after Proposition 4.1). Indeed, let $\tau : (0,1) \to (0,1)$ be any strongly mixing transformation wrt Lebesgue measure, and let $Tf = f \circ \tau$ be the induced isometry on L_p. Since $p \neq 2$, one can find $g \in L_p$ such that $\int g = 0$ and $\int J(g) = c \neq 0$. Then $T^n g \xrightarrow{w} 0$ by the strong mixing property, but $J(T^n g)$ is not weakly null because $\int J(T^n g) \equiv c$. (This example is taken from [7]).

One may also check directly that L_p does not have extremal asymptotic smoothness at infinity. Consider a sequence (ξ_n) of independent random variables on the probability space $(\Omega, \mathbb{P}) = (0,1)$ with Lebesgue measure, such that $\mathbb{P}(\xi_n = a) = 1-\lambda$ and $\mathbb{P}(\xi_n = -b) = \lambda$, where $a \neq b$ (with $a, b > 0$) and λ are chosen in such a way that $\mathbb{E}(\xi_n) = 0$ and $\|\xi_n\|_{L_p} = 1$; explicitely, $(1-\lambda)a^p + \lambda b^p = 1$ and $(1-\lambda)a = \lambda b$. The sequence (ξ_n) is bounded in L_∞ and orthogonal in L_2, hence weakly null in L_p. On the other hand, $\|\mathbf{1} + t\,\xi_n\|_p = \big((1-\lambda)(1+ta)^p + \lambda(1-tb)^p\big)^{1/p}$ for all n, and it follows that

$$\frac{r_{L_p}(t,\mathbf{1})^p}{t^p} \geqslant (1-\lambda)a^p(1+a^{-1}t^{-1})^p + \lambda b^p(1-b^{-1}t^{-1})^p.$$

Since $(1-\lambda)a^p + \lambda b^p = 1$, the right-hand side is equivalent to $1 + ct^{-1}$ as $t \to \infty$, where $c = p\big((1-\lambda)a^{p-1} + \lambda b^{p-1}\big)$. Putting $\alpha = (1-\lambda)a = \lambda b$, we have $c = p\alpha(b^{p-2} - a^{p-2})$ and hence $c \neq 0$ if $p \neq 2$. Thus, taking $a < b$ if $p > 2$ and $a > b$ if $p < 2$, we see that $\lim_{t\to\infty}\big(r_{L_p}(t,\mathbf{1}) - t\big) \geqslant \frac{c}{p} > 0$. (This example is taken from [18]).

Incidentally, the sequence (ξ_n) above is Blum-Hanson. Indeed, by the Banach-Saks theorem the bounded sequence (ξ_n) has a subsequence whose arithmetic means are norm convergent, necessarily to $0 = w\text{-}\lim \xi_n$; and since (ξ_n) is *invariant under spreading* (i.e. $\|\sum_{n\in A} \xi_n\|_p$ depens only on the cardinality of the finite set $A \subset \mathbb{N}$), the same is in fact true for any subsequence of (ξ_n). One can also apply the mean ergodic theorem, as follows. Let X be the closed subspace of L_p generated by the ξ_n; then the shift map $\xi_n \mapsto \xi_{n+1}$ extends to an isometry S of X because the ξ_n are independent and identically distributed, and $\xi_n = S^n \xi_0$ by definition; by the mean ergodic theorem and the invariance under spreading, this gives the result. Finally, here is a more baroque proof using one of the results of the present paper: since the ξ_n are centred and independent, they form a bi-monotone Schauder basis of X (because $\|\xi + \xi'\|_p \geqslant \|\xi\|_p$ whenever ξ and ξ' are independent centred random variables); so the space X has the Blum-Hanson property by Proposition 4.2, and hence $(\xi_n) = (S^n\xi_0)$ is Blum-Hanson.

The last few lines suggest that there still might be some hope for showing that L_p has the Blum-Hanson property by applying something like Theorem 2.1. In this spirit, it is worth noting that for any finite measure space $(\Omega, \mathfrak{B}, \mu)$, the space $L_p(\Omega, \mu)$ satisfies a weak form of Kalton-Werner's property (m_p). Indeed, let us denote by τ the topology of convergence in measure (for measurable functions on Ω). It is not difficult to see that L_p has property (m_p) *with respect to the topology* τ; that is, if $f \in L_p(\Omega, \mu)$ and if $(f_n) \subset L_p(\Omega, \mu)$ is τ-convergent to 0, then

$$\limsup \|f + f_n\| = (\|f\|^p + \limsup \|f_n\|^p)^{1/p}.$$

It follows that any subspace of $L_p(\Omega, \mu)$ in which all weakly null sequences are τ-null has property (m_p), and hence the Blum-Hanson property. (This applies for example to the Bergman space $B_p(\mathbb{D})$, since weak convergence in $B_p(\mathbb{D})$ implies uniform convergence on compact sets). More generally, the proof of Theorem 2.1 yields the following result.

Proposition 6.5. *Let $(\Omega, \mathfrak{B}, \mu)$ be a finite measure space, and let T be a contraction on a subspace X of $L_p(\Omega, ,\mu)$. If $f \in X$ is such that $T^n f \xrightarrow{\tau} 0$, then the sequence $(T^n f)$ is Blum-Hanson.*

Hence, any subspace of L_p has the "τ–Blum-Hanson property". This leaves us certainly far from showing the Blum-Hanson property for L_p, but still this could be an interesting fact.

7. Concluding remarks, and some questions

7.1. **Sequences of contractions.** Using the same ideas as in the proof of Theorem 2.1, one can prove a more general result allowing to deal with sequences of contractions not necessarily of the form T^n for some T. We have no application, but this might be useful elsewhere.

Let $\mathcal{I}$ be the set of all finite intervals of positive integers, including the empty interval. We denote by $|\alpha|$ the length of an interval $\alpha \in \mathcal{I}$. We write $\alpha \prec \beta$ if $\alpha \subset \beta$ and $\min \alpha = \min \beta$. Finally, we say that a family of points $(x_\alpha)_{\alpha\in\mathcal{I}}$ in a Banach space X is *shift-monotone* if $\|x_{\alpha_1} + \cdots + x_{\alpha_k}\| \leqslant \|x_{\alpha_1\setminus\alpha_0} + \cdots + x_{\alpha_k\setminus\alpha_0}\|$ for every increasing sequence $\alpha_0 \prec \alpha_1 \prec \cdots \prec \alpha_k$ in $\mathcal{I}$. For example, if (x_n) is a shift-monotone *sequence* in X and $x_\alpha = x_{|\alpha|}$, then the family $(x_\alpha)_{\alpha\in\mathcal{I}}$ is shift-monotone.

Proposition 7.1. *Let $(x_\alpha)_{\alpha\in\mathcal{I}}$ be a shift-monotone family in a Banach space X. Assume that $x_\alpha \xrightarrow{w} 0$ as $|\alpha| \to \infty$, and that $\lim_{t\to\infty}(r_X(t, x_\varnothing) - t) \leqslant 0$. Then, for any infinite increasing sequence $\alpha_1 \prec \alpha_2 \prec \ldots$ in $\mathcal{I}$, the sequence (x_{α_n}) is Blum-Hanson.*

As an immediate consequence, we get

Corollary 7.2. *Let $(T_j)_{j\in\mathbb{N}}$ be a sequence of contractions on X, and let $x \in X$. Assume that $T_pT_{p+1}\cdots T_q\, x \xrightarrow{w} 0$ as $q-p \to +\infty$, and that $\lim_{t\to\infty}(r_X(t,x)-t) \leqslant 0$. Then the sequence $(T_1\cdots T_n x)_{n\in\mathbb{N}}$ is Blum-Hanson.*

Proof. Just apply Proposition 7.1 to the (shift-monotone) family $(x_\alpha)_{\alpha\in\mathcal{I}}$ defined by $x_\varnothing = x$ and $x_\alpha = T_p\cdots T_q\, x$ if $\alpha = [p,q]$. ☐

Proof of Proposition 7.1. For any $d, s \in \mathbb{N}$, let us denote by $\mathcal{F}(s,d)$ be the family of all finite sets $A \subset \mathcal{I}$ of the form $A = \{\alpha_1,\dots,\alpha_s\}$ with $\alpha_1 \prec \cdots \prec \alpha_s$ and $|\alpha_{i+1}\backslash\alpha_i| \geqslant d$ for all $i \in \{1,\dots,s-1\}$. Now define the function $F : \mathbb{N} \to \mathbb{R}_+$ in the obvious way:

$$F(s) = \inf_{d\in\mathbb{N}} \sup_{A\in\mathcal{F}(s,d)} \left\| \sum_{\alpha\in A} x_\alpha \right\| .$$

Then, one shows exactly as in the proof of Theorem 2.1 that $F(s)/s \to 0$ as $s \to \infty$; and the result follows.

☐

7.2. **Direct sums and sub-(m_p).** The following remarks show that properties sub-(m_p) are preserved under direct sums.

Proposition 7.3. *Let $(X_i)_{i\in I}$ be a family of Banach spaces.*

(1) *Let $p \in [1,\infty)$, and assume that each X_i has property sub-(m_{p_i}) for some $p_i \geqslant p$. Then the ℓ_p direct sum $\oplus_{\ell_p} X_i$ has property sub-(m_p).*
(2) *If all (X_i) have property sub-(m_∞), then $\oplus_{c_0} X_i$ has sub-(m_∞).*

Proof. (1) To avoid double subscripts, we write any vector in $X = \oplus_{\ell_p} X_i$ as $x = (x(i))_{i\in I}$. Moreover, we denote all norms involved (in X and in every space X_i) by the same symbol $\|\cdot\|$. Finally, we may assume that in fact $p_i = p$ for all i since sub-(m_q) obviously implies sub-(m_p) whenever $q \geqslant p$.

Let $x \in X$, and let (z_n) be any weakly null sequence in X. We have to show that

$$\limsup_{n\to\infty} \|x + z_n\|^p \leqslant \|x\|^p + \limsup \|z_n\|^p . \tag{13}$$

Since all z_n have countable support, we may assume (by a diagonal argument) that $\lim_n \|z_n(i)\|$ exists for all $i \in I$.

Let us fix $\varepsilon > 0$. By the definition of X, we may choose a finite set $I_\varepsilon \subset I$ such that

$$\sum_{i\notin I_\varepsilon} \|x(i)\|^p < \varepsilon^p .$$

Now, let $(\varepsilon_i)_{i\in I_\varepsilon}$ be positive numbers such that $\sum_i \varepsilon_i^p < \varepsilon^p$. Since each space X_i has property sub-(m_p) and all limits $\lim_n \|z_n(i)\|$ exist, one can find $N \in \mathbb{N}$ such that

$$\forall n \geqslant N\ \forall i \in I_\varepsilon\ :\ \|x(i) + z_n(i)\|^p \leqslant \|x(i)\|^p + \|z_n(i)\|^p + \varepsilon_i^p .$$

We then have for all $n \geqslant N$:

$$\begin{aligned}\|x+z_n\|^p &= \sum_{i\in I_\varepsilon}\|x(i)+z_n(i)\|^p+\sum_{i\notin I_\varepsilon}\|x(i)+z_n(i)\|^p\\ &\leqslant \sum_{i\in I_\varepsilon}\left(\|x(i)\|^p+\|z_n(i)\|^p+\varepsilon_i^p\right)+\sum_{i\notin I_\varepsilon}\|x(i)+z_n(i)\|^p\\ &\leqslant \varepsilon^p+\|x\|^p+\sum_{i\in I_\varepsilon}\|z_n(i)\|^p+\sum_{i\notin I_\varepsilon}\left(\|x(i)\|+\|z_n(i)\|\right)^p .\end{aligned}$$

By Minkowski's inequality for $\ell_p(I)$, it follows that

$$\begin{aligned}\|x+z_n\|^p &\leqslant \varepsilon^p+\|x\|^p+\left(\left(\sum_{i\in I}\|z_n(i)\|^p\right)^{\frac1p}+\left(\sum_{i\notin I_\varepsilon}\|x(i)\|^p\right)^{\frac1p}\right)^p\\ &\leqslant \varepsilon^p+\|x\|^p+\left(\|z_n\|+\varepsilon\right)^p\end{aligned}$$

for all $n \geqslant N$. Since ε is arbitray, this gives (13).

Part (2) is proved in the same way (the details are actually simpler).

□

Corollary 7.4. *Let I be an arbitrary index set. If X is a Banach space with property sub-(m_q) for some $q > 1$ then $\ell_p(I, X)$ has extremal asymptotic smoothness at infinity (and hence the Blum-Hanson property) for any $p \in (1, q]$. If X has property sub-(m_∞) then $c_0(I, X)$ has extremal asymptotic smoothness at infinity.*

Remark. Apart from trivial cases, ℓ_1 direct sums never have extremal asymptotic smoothness at infinity. In fact, if Z is a Banach space without the Schur property then, for any $Y \neq \{0\}$, the space $X = Y \oplus_{\ell_1} Z$ does not have extremal asymptotic smoothness at infinity. To see this, choose a weakly null sequence in $(z_n) \subset S_Z$ and observe that if $y \in S_Y$, then $\|(y, 0) + t(0, z_n)\| = 1 + t$ for every $t \geqslant 0$ and all $n \in \mathbb{N}$: this shows that $r_X(t, x) \equiv 1 + t$ for any $x \in S_X$ of the form $(y, 0)$. On the other hand, we don't know if a "nontrivial" ℓ_1 direct sum can ever have the Blum-Hanson property.

7.3. **A symmetric modulus.** For any Banach space X, consider the "symmetric" modulus $\widetilde{r}_X$ defined as follows:

$$\widetilde{r}_X(t,x)=\sup_{(y_n)\in\mathrm{WN}(S_X)}\ \limsup_{n\to\infty}\left(\frac{\|x+ty_n\|+\|x-ty_n\|}{2}\right).$$

Obviously $\widetilde{r}_X(t, x) \leqslant r_X(t, x)$. Moreover, the proof of Proposition 4.6 yields that if X is uniformly Gâteaux smooth (and does not have the Schur property) then $\lim_{t\to\infty}(\widetilde{r}_X(t, x) - t) = 0$ for every $x \in S_X$. That is, condition $(*)$ of Theorem 2.1 holds when r_X is replaced with $\widetilde{r}_X$.

From this, it is tempting to believe that a proof similar to that of Theorem 2.1 should yield the following result : *if T is a contraction on a uniformly Gâteaux smooth space X then, for any $x \in X$ with a weakly null orbit, one can find a choice of signs $(\varepsilon_n) \in \{-1, 1\}^{\mathbb{N}}$ such that the sequence $(\varepsilon_n T^n x)$ is Blum-Hanson.* However, this would in fact mean that uniformly Gâteaux smooth spaces have the Blum-Hanson property, since it is easily checked that a sequence (x_n) is Blum-Hanson if and only if $(\varepsilon_n x_n)$ is, for any choice of signs (ε_n).

To put this in perspective, it is worth recalling here that uniformly (Fréchet) smooth Banach spaces have the *Banach-Saks property* (se e.g. [17]); that is, any bounded sequence has a subsequence whose arithmetic means are norm convergent. By a well known result of P. Erdös and M. Magidor ([19], see also [6, II.6]), any bounded sequence in a space X with the Banach-Saks property has a subsequence all of whose further subsequences have norm convergent arithmetic means. In particular, if X has the Banach-Saks property then any weakly null sequence in X has a Blum-Hanson subsequence. (In fact, it is enough to assume that X has the *weak* Banach-Saks property, i.e. any weakly convergent sequence has a subsequence with norm convergent arithmetic means). Hence, if T is a contraction on X then, for any $x \in X$ with a weakly null orbit, one can find a (nontrivial) choice *of* 0*'s and* 1*'s* (ε_n) such that $(\varepsilon_n T^n x)$ is Blum-Hanson.

7.4. **How *not* to be Blum-Hanson.** Since asymptotic smoothness is "dual" to asymptotic *convexity*, it is natural to expect that an extremal behaviour of the modulus of asymptotic convexity should give rise to *non* Blum-Hanson sequences.

Recall that the modulus of asymptotic convexity of the Banach space X is the function $\bar{\delta}_X : \mathbb{R}^+ \times S_X \to \mathbb{R}^+$ defined by

$$\bar{\delta}_X(t,x) = \sup_E \inf_{y \in \breve{B}_E} \|x + ty\| - 1\,,$$

where the supremum is taken over all finite-codimensional subspaces E of X and $\breve{B}_E = \{y \in E;\ \|y\| \geqslant 1\}$. Obviously $\bar{\delta}_X(t,x) \geqslant 0$. The space X is said to be *asymptotically uniformly convex* if $\bar{\delta}_X(t) := \inf_{x\in S_X} \bar{\delta}_X(t,x) > 0$ for all $t > 0$. For example, ℓ_1 is asymptotically uniformly convex because $\bar{\delta}_X(t) = t$ for all t.

A closely related "modulus" is

$$d_X(t,x) = \inf_{(y_n)\in \mathrm{WN}(S_X)} \liminf \|x + ty_n\|\,.$$

(Again, the trivial case $\mathrm{WN}(S_X) = \varnothing$ is allowed: $\inf \varnothing$ is declared to be $+\infty$). In the terminology of [31], $t^{-1} \inf_{x\in S_X} d_X(t,x) - 1$ is the value of the *Opial modulus* of X at t^{-1}.

It is easy to check that $d_X(t,x) \geqslant 1 + \bar{\delta}(t,x) \geqslant t$ for all t (if $x \in S_X$), and that both $\bar{\delta}_X(t,x)$ and $d_X(t,x) - t$ have a (nonegative) limit as $t \to \infty$. The following result can now be proved along the same lines as Theorem 2.1.

Proposition 7.5. *Let* $(x_n)_{n\in\mathbb{Z}^+}$ *be a* reverse shift-monotone *sequence in* X*, i.e.* $\|x_{1+n_1} + \cdots + x_{1+n_k}\| \geqslant \|x_{n_1} + \cdots + x_{n_k}\|$ *for all finite increasing sequences* $n_1 < \cdots < n_k$*. Assume that the initial point* $x = x_0$ *satisfies*

$$\lim_{t\to\infty} \Big(d_X(s,x) - t\Big) > 0\,. \tag{14}$$

Then (x_n) *is not a Blum-Hanson sequence.*

As an immediate consequence, we get

Corollary 7.6. *Assume that* $\lim_{t\to\infty}(\bar\delta_X(t,x)+1-t)>0$ *for every* $x\in S_X$*. Then, no linear isometry on* X *can have any Blum-Hanson orbit (except* $\{0\}$*).*

To prove Proposition 7.5, one may obviously assume that the sequence (x_n) is weakly null. Then, the strategy is the same as for Theorem 2.1 (but reverting all the inequalities). The function F introduced in Lemma 3.1 is replaced with

$$G(s)=\sup_{d\in\mathbb{N}}\inf_{A\in\mathrm{FIN}_{s,d}}\Big\|\sum_{n\in A}x_n\Big\|\,,$$

and one shows that $\liminf\limits_{s\to\infty}\frac{G(s)}{s}>0$. To do this, one makes use of the inequality

$$G(s+1)\geqslant G(d_X(s,x_0))\,.$$

We shall not give any further detail, for a rather unpleasant reason: all the Banach spaces that we know for which $\lim_{t\to\infty}(\bar\delta_X(t,x)+1-t)>0$ for every $x\in S_X$ happen to have the Schur property; and for such spaces everything is trivial since Blum-Hanson sequences are norm null.

7.5. **Power-bounded operators.** As mentioned in the introduction, it is shown in [40] is that ℓ_p, $1<p<\infty$ does not have the conditional Blum-Hanson property with respect to power-bounded operators (in short, (CBHPB)). This has been extended by J. M. Augé [5]: any Banach$X$ space with a shrinking symmetric basis (e.g. $X=c_0$ or ℓ_p) fails (CBHPB). Since the property is easily seen to be inherited by complemented subspaces, it follows that any Banach space containing a complemented copy of c_0 or some ℓ^p, $1<p<\infty$ fails (CBHPB). For example, this holds for L_p, $1<p<\infty$ and for any separable Banach space containing a copy of c_0 (which is necessarily complemented by Sobczyk's theorem). Actually, we are aware of no example of a Banach space having the Blum-Hanson property with respect to power-bounded operators, apart from the trivial case of Banach spaces with the Schur property.

7.6. **Some questions.** To conclude the paper, we collect a few questions that appear to be quite natural.

(1) Does every uniformly Gâteaux smooth Banach space have the Blum-Hanson property?
(2) For which *countable* compact K does $\mathcal{C}(K)$ have the Blum-Hanson property?
(3) Does ℓ_∞ have the Blum-Hanson property?
(4) Let X be a Banach space with the Schur property, and let $(\Omega,\mathbb{P})$ be a probability space. Does $L_2(\Omega,\mathbb{P},X)$ have the Blum-Hanson property?
(5) Let X be a Banach space and assume that X has the Blum-Hanson property with respect to contractions with weakly null orbits. Does it follow that X has the conditional Blum-Hanson property (i.e. BH with respect to contractions with weakly convergent orbits)?
(6) Does L_1 have the full (not just conditional) Blum-Hanson property?

(7) Are the Blum-Hanson property and the conditional Blum-Hanson property equivalent?
(8) Does every subspace of L_1 have the (conditional) Blum-Hanson property?
(9) Does the Hardy space $H^p(\mathbb{D})$, $1 \leqslant p < \infty$ have property (?)?
(10) Does property (M) imply the Blum-Hanson property?
(11) Is there a Banach space with a 1-unconditional basis failing the Blum-Hanson property?
(12) Does the ℓ_1 direct sum $\ell_2 \oplus \ell_2$ have the Blum-Hanson property?
(13) Does L_1 has the (conditional) Blum-Hanson property with respect to power-bounded operators?
(14) Is there *any* Banach space X failing the Schur property but having the Blum-Hanson property with respect to power-bounded operators? Equivalently, is it true (or not) that if X is a Banach space without the Schur property, then X admit a renorming under which it fails the Blum-Hanson property?
(15) Which Banach spaces can be renormed to have the Blum-Hanson property?

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